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MINKOWSKI LACES

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Abstract. A family of curves generated as partial Minkowski combinations of curve segments in the Euclidean space is presented with some of their geometric properties and interesting shape features.

Keywords: Minkowski point set operations, modeling of curves

1 Introduction

A family of curves named "laces" is introduced and investigated, while some of their intrinsic geometric properties are derived and presented. Particular representatives of this two-parametric family of curves can be generated by means of Minkowski summative or Minkowski multiplicative combinations of two equally parameterised curve segments in the Euclidean space $\mathbf{E}^n$, as described in [3]. Minkowski operators represent a strong tool for modeling of complex shaped manifolds in the Euclidean space of dimention n, wich can be utilised in geometric modelling, in art and architecture, and in various applications of computer graphics design and rendering, see in [4]. Minkowski summative and multiplicative operators map pairs of smooth equally parameterised curves to smooth curves in the space. Various interesting forms of these space curves can be designed choosing specific curve segments $\boldsymbol{K}$ and $\boldsymbol{L}$ in different super-positions in plane $\mathbf{E}^2$, in space $\mathbf{E}^3$ or in the Euclidean spaces of higher dimensions, e.g. $\mathbf{E}^4$. Specific properties of curve segments generated as Minkowski combinations are described in the following sections.

2 Minkowski summative laces

Let the two curve segments $\boldsymbol{K}$ and $\boldsymbol{L}$ be determined by respective vector maps defined on the same interval $I \subset \mathbf{R}$

$$\boldsymbol{K}: \mathbf{rk}(t) = (xk_1(t), xk_2(t), \dots xk_n(t)), \tag{1}$$

$$\boldsymbol{L}: \mathbf{rl}(t) = (xl_1(t), xl_2(t), \dots xl_n(t)). \tag{2}$$

Based on the partial Minkowski sum of two point set one can generalized and define Minkowski summative combination of curves $\boldsymbol{K}$ and $\boldsymbol{L}$ as

$$\boldsymbol{S} = a\boldsymbol{K} \oplus b\boldsymbol{L},\ a, b \in \mathbf{R},\ a^2 + b^2 \neq 0. \tag{3}$$

Thus a family of curve segments in $\mathbf{E}^n$ can be derived parametrically represented on $I \subset \mathbf{R}$ by vector maps

$$\boldsymbol{S}: \mathbf{r}(t) = a.\mathbf{rk}(t) + b.\mathbf{rl}(t) = (xr_1(t), xr_2(t), \dots xr_n(t)), \tag{4}$$

where

$$xr_i(t) = a.\ xk_i(t) + b.\ xl_i(t), \text{ for } i = 1, 2, \dots, n. \tag{5}$$

Generated family of curves is called Minkowski summative laces.

Differential characteristics of this two-parametric family of curve segments in space $\mathbf{E}^2$ or $\mathbf{E}^3$ can be derived and represented by means of derivatives of vector representations of the two operand curves, while some properties tend to be inherited from the operand curves.

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Let the derivatives of the resulting curve ***S*** vector map (4) and (5) for i = 1, 2, 3 be defined on $I \subset \mathbf{R}$ in the form

$$\mathbf{r}'(t) = a.\mathbf{rk}'(t) + b.\mathbf{rl}'(t),\ \mathbf{r}''(t) = a.\mathbf{rk}''(t) + b.\mathbf{rl}''(t),\ \mathbf{r}'''(t) = a.\mathbf{rk}'''(t) + b.\mathbf{rl}'''(t) \qquad (6)$$

Then the formulas for the first and second curvature of curve segment ***S*** can be derived as

$$\kappa = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\left|\left(a.\mathbf{rk}'(t)+b.\mathbf{rl}'(t)\right)\times\left(a.\mathbf{rk}''(t)+b.\mathbf{rl}''(t)\right)\right|}{\left(\sqrt{\left(a.\mathbf{rk}'(t)+b.\mathbf{rl}'(t)\right).\left(a.\mathbf{rk}'(t)+b.\mathbf{rl}'(t)\right)}\right)^3} =$$

$$= \frac{\left|a^2\left(\mathbf{rk}'(t)\times\mathbf{rk}''(t)\right)+a.b\left(\mathbf{rk}'(t)\times\mathbf{rl}''(t)+\mathbf{rl}'(t)\times\mathbf{rk}''(t)\right)+b^2\left(\mathbf{rl}'(t)\times\mathbf{rl}''(t)\right)\right|}{\left(a^2\left|\mathbf{rk}'(t)\right|^2+2a.b.\mathbf{rk}'(t).\mathbf{rl}'(t)+b^2\left|\mathbf{rl}'(t)\right|^2\right)^{\frac{3}{2}}} \qquad (7)$$

$$\tau = \frac{[\mathbf{r}'(t),\mathbf{r}''(t),\mathbf{r}'''(t)]}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|^2} =$$

$$= \frac{\left[\left(a.\mathbf{rk}'(t)+b.\mathbf{rl}'(t)\right),\left(a.\mathbf{rk}''(t)+b.\mathbf{rl}''(t)\right),\left(a.\mathbf{rk}'''(t)+b.\mathbf{rl}'''(t)\right)\right]}{\left|\left(a.\mathbf{rk}'(t)+b.\mathbf{rl}'(t)\right)\times\left(a.\mathbf{rk}''(t)+b.\mathbf{rl}''(t)\right)\right|^2} \qquad (8)$$

For $a = b = 1$ the above formulas define curvatures of a curve segment generated as an ordinary Minkowski partial sum of curve segments ***K*** and ***L***, for $a = 1$ and $b = -1$, or $a = -1$ and $b = 1$ curvatures of curve segment that is the Minkowski partial difference of the two curve segments ***K*** and ***L***.

Minkowski linear combinations of various curves positioned into different planes in space are presented in fig 1.: on the left - circle and cissoid in perpendicular planes, in the middle – shamrock curve and leaf of Descartes positioned in parallel planes, while on the right are the same curve segments positioned in perpendicular planes.

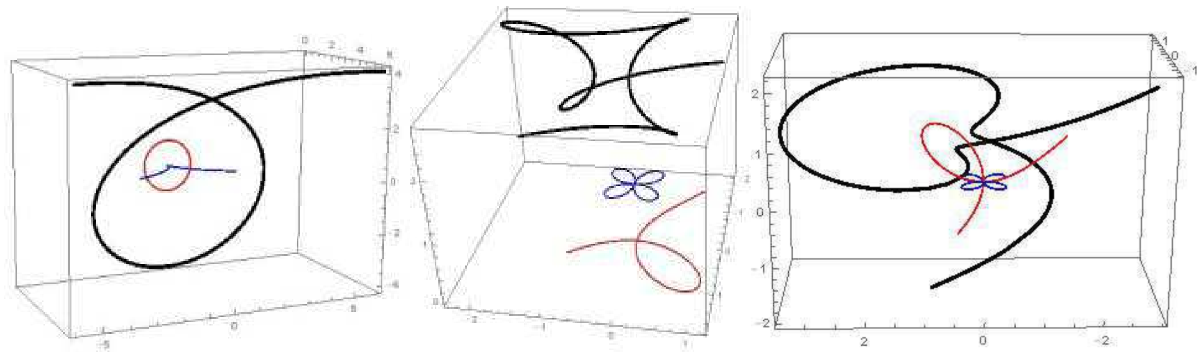

Figure 1: Minkovski summative combinations of two curve segments

3 Minkowski multiplicative laces

Partial Minkowski product of two curve segments is based on the definition of Minkowski multiplicative combination of curves ***K*** and ***L***

$$\boldsymbol{P} = a.\boldsymbol{K} \otimes b.\boldsymbol{L},\ a, b \in \mathbf{R},\ a, b \neq 0. \qquad (9)$$

This determines a family of curve segments in $\mathbf{E}^d$, $d = n(n-1)/2$, represented on $I \subset \mathbf{R}$ by vector maps

$$\boldsymbol{P}\colon \mathbf{p}(t) = a.\mathbf{rk}(t) \wedge b.\mathbf{rl}(t) = (xp_1(t), xp_2(t), \dots xp_d(t)), \qquad (10)$$

where the following relations hold for coordinate functions

$$xp_k(t) = a.b.(xk_i(t).\ xl_j(t) - xk_j(t).\ xl_i(t)), \text{ for } i, j = 1, 2, \dots, n, k = 1, 2, \dots, d. \qquad (11)$$

Differential characteristics of this two-parametric family of multiplicative laces can be also derived from derivatives of vector maps of the two respective operand curves.

Derivatives of the resulting curve segment $\boldsymbol{S}$ vector map (10) and (11) for $k = 1, 2, 3$ are defined on $I \subset \mathbf{R}$ in the form

$$\begin{aligned}
\mathbf{p}'(t) &= a.\mathbf{rk}'(t) \wedge b.\mathbf{rl}(t) + a.\mathbf{rk}(t) \wedge b.\mathbf{rl}'(t) = a.b.(\mathbf{rk}'(t) \wedge \mathbf{rl}(t) + \mathbf{rk}(t) \wedge \mathbf{rl}'(t)),\\
\mathbf{p}''(t) &= a.b.(\mathbf{rk}''(t) \wedge \mathbf{rl}(t) + 2\mathbf{rk}'(t) \wedge \mathbf{rl}'(t) + \mathbf{rk}(t) \wedge \mathbf{rl}''(t)),\\
\mathbf{p}'''(t) &= a.b.\left(\mathbf{rk}'''(t) \wedge \mathbf{rl}(t) + 3\mathbf{rk}''(t) \wedge \mathbf{rl}'(t) + 3\mathbf{rk}'(t) \wedge \mathbf{rl}''(t) + \mathbf{rk}(t) \wedge \mathbf{rl}'''(t)\right) \qquad (12)
\end{aligned}$$

Formulas for the first and second curvature of curve segment $\boldsymbol{S}$ can be determined in the forms

$$\begin{aligned}
\kappa &= \frac{|\mathbf{p}'(t) \times \mathbf{p}''(t)|}{|\mathbf{p}'(t)|^3} = \frac{a^2.b^2.\sqrt{\mathbf{u}(t).\mathbf{u}(t)}}{a^3.b^3\left(\sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}\right)^3} = \frac{1}{a.b.}\sqrt{\frac{\mathbf{u}(t).\mathbf{u}(t)}{(\mathbf{v}(t)\cdot\mathbf{v}(t))^3}}\\
\mathbf{u}(t) &= \left(\mathbf{rk}'(t) \wedge \mathbf{rl}(t) + \mathbf{rk}(t) \wedge \mathbf{rl}'(t)\right) \times \qquad (13)\\
&\quad \times \left(\mathbf{rk}''(t) \wedge \mathbf{rl}(t) + 2\mathbf{rk}'(t) \wedge \mathbf{rl}'(t) + \mathbf{rk}(t) \times \mathbf{rl}''(t)\right)\\
\mathbf{v}(t) &= \left(\mathbf{rk}'(t) \wedge \mathbf{rl}(t)\right)^2 + 2\left(\mathbf{rk}'(t) \wedge \mathbf{rl}(t)\right)\cdot\left(\mathbf{rk}(t) \wedge \mathbf{rl}'(t)\right) + \left(\mathbf{rk}(t) \wedge \mathbf{rl}'(t)\right)^2
\end{aligned}$$

$$\begin{aligned}
\tau &= \frac{[\mathbf{p}'(t), \mathbf{p}''(t), \mathbf{p}'''(t)]}{|\mathbf{p}'(t) \times \mathbf{p}''(t)|^2} = \frac{a^2.b^2.\mathbf{u}(t)\cdot\mathbf{p}'''(t)}{a^4.b^4.\mathbf{u}(t)\cdot\mathbf{u}(t)} = \frac{1}{a.b}.\frac{\mathbf{u}(t)\cdot\mathbf{w}(t)}{\mathbf{u}(t)\cdot\mathbf{u}(t)} \qquad (14)\\
\mathbf{w}(t) &= \mathbf{rk}'''(t) \wedge \mathbf{rl}(t) + 3\mathbf{rk}''(t) \wedge \mathbf{rl}'(t) + 3\mathbf{rk}'(t) \wedge \mathbf{rl}''(t) + \mathbf{rk}(t) \wedge \mathbf{rl}'''(t)
\end{aligned}$$

Real constants a and b appearing in the Minkowski multiplicative combinations of curve segments $\boldsymbol{K}$ and $\boldsymbol{L}$ serve as scaling coefficients of the resulting curve segment, resembled in the size of the resulting multiplicative lace curve, not in its shape. Thus the form, up to similarity, of Minkowski multiplicative laces is determined entirely by the two operand curves from the operation of Minkowski partial product. This property is also reflected in the above formulas of curvatures, which show that curvatures of any representative of the family of Minkowski multiplicative laces determined by non-zero coefficients a and b are just $\frac{1}{a.b}$ multiples of curvatures of ordinary Minkowski partial product of curves $\boldsymbol{K}$ and $\boldsymbol{L}$ for $a = b = 1$.

Minkowski multiplicative combinations of the same two curve segments that are used in fig. 1 to determine Minkowski summative combinations are presented in the fig. 2. Family of Minkowski multiplicative laces cherishes more sophisticated forms and shapes.

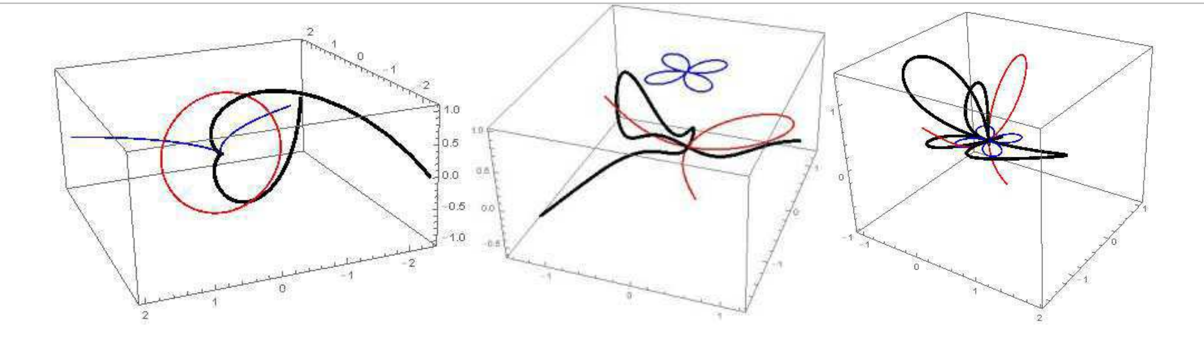

Figure 2: Minkowski multiplicative combinations of two curve segments

4 Conclusions

Minkowski summative and multiplicative combinations of two equally parameterised curve segments in the space $\mathbf{E}^n$ define a family of Minkowski lace curves, while combinations determined by pairs of differently parameterised curve segments $\boldsymbol{K}$ and $\boldsymbol{L}$ define smooth surfaces in $\mathbf{E}^n$. These flexible mathematical models of complex shaped objects in $\mathbf{E}^n$ enable dynamical modifications by means of shaping parameters, real numbers a and b. Thus they represent robust modelling tools that can be utilised by means of any dynamic mathematical software solution capable of drawing graphs of functions in more variables. Curves and surfaces can be smoothly shaped, and they are well adjustable with respect to their forms. Both mathematical models can be easily used in geometric modeling in computer graphics algorithms ([1, 2]).

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"SZNUROWADŁA" MINKOWSKIEGO

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