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Norm attaining bilinear forms on the plane with the octagonal norm

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Summary. For given unit vectors $x_1, \cdots, x_n$ of a real Banach space E, we define

$$NA\big(\mathcal{L}(^nE)\big)(x_1,\ldots,x_n) = \big\{T \in \mathcal{L}(^nE) : |T(x_1,\ldots,x_n)| = \|T\| = 1\big\},$$

where $\mathcal{L}(^nE)$ denotes the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup\{|T(x_1,\ldots,x_n)| : \|x_k\| = 1, 1 \leqslant k \leqslant n\}$. In this paper, we classify $NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((x_1,x_2),(y_1,y_2))$ for unit vectors $(x_1,x_2),(y_1,y_2) \in \mathbb{R}^2_{o(w)}$, where $\mathbb{R}^2_{o(w)} = \mathbb{R}^2$ with the octagonal norm with weight $0 < w < 1$.

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1. Introduction

Let us introduce a brief history of norm attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon–Nikodym property is sufficient

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for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon–Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jimenez-Sevilla and Paya [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}, n \geqslant 2$. We write S_E and B_E for the unit sphere and the closed unit ball of the real Banach space E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup\{|T(x_1, \ldots, x_n)| : \|x_k\| = 1, 1 \leqslant k \leqslant n\}$. The subspace of all continuous symmetric n-linear forms on E is denoted by $\mathcal{L}_s(^nE)$. A mapping $P: E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists $T \in \mathcal{L}(^nE)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup\{|P(x)| : \|x\| = 1\}$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

The elements $x_1, \ldots, x_n \in E$ is called *norming points* of $T \in \mathcal{L}(^nE)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$. In this case, T is called a *norm attaining* n-linear form at $x_1, \ldots, x_n$. Similarly, an element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. In this case, P is called a *norm attaining* n-homogeneous polynomial at x. Let $X = \mathcal{L}(^nE)$ or $\mathcal{L}_s(^nE)$. For $x, x_1, \ldots, x_n \in S_E$, we define

$$NA(X)(x_1, \ldots, x_n) = \left\{T \in X : |T(x_1, \ldots, x_n)| = \|T\| = 1\right\}$$

and

$$NA\big(\mathcal{P}(^nE)\big)(x) = \left\{P \in \mathcal{P}(^nE) : |P(x)| = \|P\| = 1\right\}.$$

Notice that

$$NA\big(\mathcal{L}(^nE)\big)(x_1, \ldots, x_n) = NA\big(\mathcal{L}(^nE)\big)(\pm x_1, \ldots, \pm x_n),$$
$$NA\big(\mathcal{L}_s(^nE)\big)(x_1, \ldots, x_n) = NA\big(\mathcal{L}_s(^nE)\big)(\pm x_{\sigma(1)}, \ldots, \pm x_{\sigma(n)})$$

and

$$NA\big(\mathcal{P}(^nE)\big)(x) = NA\big(\mathcal{P}(^nE)\big)(-x)$$

for all $x, x_1, \ldots, x_n \in S_E$ and for all permutation σ on $\{1, \ldots, n\}$.

It seems to be natural and interesting to study about

$$NA\big(\mathcal{L}(^nE)\big)(x_1, \ldots, x_n), NA\big(\mathcal{L}_s(^nE)\big)(x_1, \ldots, x_n) \text{ and } NA\big(\mathcal{P}(^nE)\big)(x)$$

for $x, x_1, \ldots, x_n \in S_E$. Kim [6] classified $NA(\mathcal{P}(^2l_p^2))((x_1, x_2))$ for $(x_1, x_2) \in S_{l_p^2}$ and $p = 1, 2, \infty$, where $l_p^2 = \mathbb{R}^2$ with the l_p-norm. Kim [8] classified $NA(\mathcal{L}(^2l_1^2)((x_1, x_2), (y_1, y_2))$ for $(x_1, x_2), (y_1, y_2) \in S_{l_1^2}$.

Let $\mathbb{R}^2_{o(w)}$ denote $\mathbb{R}^2$ with the octagonal norm with weight $0 < w < 1$

$$\|(x, y)\|_{o(w)} = \max\left\{|x| + w|y|, w|x| + |y|\right\}.$$

Notice that

$$\|(x,y)\|_{o(w)} = \|(\pm x, \pm y)\|_{o(w)} \text{ for } (x,y) \in \mathbb{R}^2_{o(w)}.$$

In this paper, we classify $NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((x_1,x_2),(y_1,y_2))$ for $(x_1,x_2),(y_1,y_2) \in S_{\mathbb{R}^2_{o(w)}}$.

2. Results

Let $T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ for some $a,b,c,d \in \mathbb{R}$. For simplicity, we denote $T = (a,b,c,d)$. Throughout this paper, we let $0 < w < 1$.

2.1. Theorem ([7]). *Let $T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ be such that $T((x_1,y_1),(x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 = (a,b,c,d)$ for some $a,b,c,d \in \mathbb{R}$. Then*

$$\|T\| = \max\Big\{|a|,|b|,|c|,|d|, \frac{|a|+|c|}{1+w}, \frac{|a|+|d|}{1+w}, \frac{|b|+|c|}{1+w}, \frac{|b|+|d|}{1+w}, \frac{|a+b|+|c+d|}{(1+w)^2}, \frac{|a-b|+|c-d|}{(1+w)^2}\Big\}.$$

2.2. Theorem. *Let $T = (a,b,c,d) \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$ for some $a,b,c,d \in \mathbb{R}$ and let (x_1,y_1), $(x_2,y_2) \in \mathbb{R}^2_{o(w)}$. The following are equivalent:*

(i) $T \in NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((x_1,y_1),(x_2,y_2))$;

(ii) $T_1 := (-a,b,-c,d) \in NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((-x_1,y_1),(x_2,y_2))$;

(iii) $T_2 := (-a,b,c,-d) \in NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((x_1,y_1),(-x_2,y_2))$;

(iv) $T_3 := (a,b,-c,-d) \in NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((-x_1,y_1),(-x_2,y_2))$;

(v) $T_4 := (-a,-b,-c,-d) \in NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((-x_1,-y_1),(x_2,y_2))$;

(vi) $T_5 := (a,-b,-c,d) \in NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((x_1,y_1),(x_2,-y_2))$.

Proof. It is obvious. □

2.3. Theorem. *Let $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2_{o(w)}$. Then*

$$\begin{aligned} NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((x_1,y_1),(x_2,y_2)) &= NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((-x_1,-y_1),(x_2,y_2)) \\ &= NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((x_1,y_1),(-x_2,-y_2)) \\ &= NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))((-x_1,-y_1),(-x_2,-y_2)). \end{aligned}$$

Proof. It is obvious. □

2.4. Lemma. *Let $x_j > 0$ and $\epsilon_j = \pm 1$ for $j = 1,\dots,n$. Suppose that $\sum_{1\leqslant j\leqslant n} x_j = 1$. Then $\left|\sum_{1\leqslant j\leqslant n} \epsilon_j x_j\right| = 1$ if and only if $(\epsilon_j = 1$ for every $j = 1,\dots,n)$ or $(\epsilon_j = -1$ for every $j = 1,\dots,n)$.*

Proof. $(\Leftarrow)$. It is obvious.

$(\Rightarrow)$. Use induction on n.

If $n = 1$, then it's ok. Suppose that $(\Rightarrow)$ is true for $n = k$ for some $k \in \mathbb{N}$. Suppose that

$$\sum_{1\leqslant j\leqslant k+1} x_j = 1 = \sum_{1\leqslant j\leqslant k+1} \epsilon_j x_j.$$

Let $y := \sum_{1\leqslant j\leqslant k} x_j > 0$.

Claim. $y = \left|\sum_{1\leqslant j\leqslant k} \epsilon_j x_j\right|$

It follows that

$$1 = \Big|\sum_{1\leqslant j\leqslant k+1} \epsilon_j x_j\Big| \leqslant \Big|\sum_{1\leqslant j\leqslant k} \epsilon_j x_j\Big| + x_{k+1} \leqslant y + x_{k+1} = 1,$$

which shows that

$$1 = \Big|\sum_{1\leqslant j\leqslant k} \epsilon_j x_j\Big| + x_{k+1} = y + x_{k+1}.$$

Hence, the claim holds. Notice that

$$1 = \Big|\sum_{1\leqslant j\leqslant k} \epsilon_j\Big(\frac{x_j}{y}\Big)\Big| = \sum_{1\leqslant j\leqslant k} \frac{x_j}{y}.$$

By the induction hypothesis, $(\epsilon_j = 1$ for every $j = 1, \ldots, k)$ or $(\epsilon_j = -1$ for every $j = 1, \ldots, k)$.

Case 1. $\epsilon_j = 1$ for every $j = 1, \ldots, k$.

We claim that $\epsilon_{k+1} = 1$. Assume that $\epsilon_{k+1} = -1$. Then

$$y + x_{k+1} = |y - x_{k+1}| = \pm(y - x_{k+1}).$$

If $y+x_{k+1} = y-x_{k+1}$, then $x_{k+1} = 0$, which is a contradiction. If $y+x_{k+1} = -(y-x_{k+1})$, then $y = 0$, which is a contradiction. Hence, $\epsilon_{k+1} = 1$. Therefore, $\epsilon_j = 1$ for every $j = 1, \ldots, k+1$.

Case 2. $\epsilon_j = -1$ for every $j = 1, \ldots, k$.

It follows that

$$y + x_{k+1} = |y - \epsilon_{k+1} x_{k+1}|.$$

By the proof of Case 1, $-\epsilon_{k+1} = 1$. Therefore, $\epsilon_j = -1$ for every $j = 1, \ldots, k+1$. We complete the proof. □

Notice that

$$\Big\{ \pm (0,1), \Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), \pm(1,0), \Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big)\Big\}$$

is the set of all extreme points of $B_{\mathbb{R}^2_{0(w)}}$. Let

$$\begin{aligned}
\Omega &= \Big\{(0,1), \Big(\frac{1}{1+w}, \frac{1}{1+w}\Big), (1,0), \Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big)\Big\},\\
B_1 &= \Big\{t(0,1) + (1-t)\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big) : 0 \leqslant t \leqslant 1\Big\},\\
B_2 &= \Big\{t\Big(\frac{1}{1+w}, \frac{1}{1+w}\Big) + (1-t)(1,0) : 0 \leqslant t \leqslant 1\Big\},\\
B_3 &= \Big\{t(1,0) + (1-t)\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big) : 0 \leqslant t \leqslant 1\Big\},\\
B_4 &= \Big\{t\Big(\frac{1}{1+w}, \frac{-1}{1+w}\Big) + (1-t)(0,-1) : 0 \leqslant t \leqslant 1\Big\},\\
A_{ij} &= B_i \times B_j \text{ for } i, j = 1, \dots, 4.
\end{aligned}$$

Notice that

$$S_{\mathbb{R}^2_{h(w)}} = \bigcup_{1 \leqslant k \leqslant 4} (\pm B_k).$$

We are in position to classify $NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X, Y)$ for every $X = (x_1, y_1)$, $Y = (x_2, y_2) \in S_{\mathbb{R}^2_{h(w)}}$. By Theorem 2.3, we may assume that $x_j \geqslant 0$ for every $j = 1, 2$.

2.5. Theorem. *Let $X, Y \in A_{ij}$ for some $i, j = 1, \dots, 4$. Write $X = (x_1, y_1)$, $Y = (x_2, y_2)$, $x_k \geqslant 0$ for every $k = 1, 2$. Then the following statements holds:*

(i) $(X, Y) \in (B_i \backslash \Omega) \times (B_j \backslash \Omega)$

$$\begin{aligned}
NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X, Y) = \big\{\pm T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)}) : 1 = T(U, U') &= T(U, V') = T(V, U')\\
&= T(V, V') \textit{ if } \operatorname{ext} B_i = \{U, V\},\\
&\quad \operatorname{ext} B_j = \{U', V'\}\big\},
\end{aligned}$$

where $\operatorname{ext} B_i$ *denotes the set of end points of* B_i. *Moreover,* $NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X, Y) = \{\pm T_0\}$ *for some* $T_0 \in \mathcal{L}(^2\mathbb{R}^2_{o(w)})$.

(ii) $(X, Y) \in (B_i \backslash \Omega) \times (B_j \cap \Omega)$

$$\begin{aligned}
NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)(X, Y) = \big\{ \pm T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)}) : 1 &= T(U, Y) = T(V, Y)\\
&\textit{if } \operatorname{ext} B_i = \{U, V\},\ |T(W_1, W_2)| \leqslant 1\\
&\textit{for all } W_k \in \operatorname{ext} B_{\mathbb{R}^2_{o(w)}},\ k = 1, 2\big\},
\end{aligned}$$

where $\operatorname{ext} B_{\mathbb{R}^2_{o(w)}}$ *denotes the set of all extreme points of* $B_{\mathbb{R}^2_{o(w)}}$.

(iii) $(X,Y)\in(B_i\cap\Omega)\times(B_j\backslash\Omega)$

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)(X,Y)=\Big\{\pm T\in\mathcal{L}(^2\mathbb{R}^2_{o(w)}):1=T(X,U')=T(X,V')\\ \textit{if}\ \text{ext}\,B_j=\{U',V'\},|T(W_1,W_2)|\leqslant 1\\ \textit{for all}\ W_k\in\text{ext}\,B_{\mathbb{R}^2_{o(w)}},\ k=1,2\Big\}.$$

(iv) $(X,Y)\in(B_i\cap\Omega)\times(B_j\cap\Omega)$

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)(X,Y)=\Big\{T\in\mathcal{L}(^2\mathbb{R}^2_{o(w)}):1=|T(X,Y)|,|T(W_1,W_2)|\leqslant 1\\ \textit{for all}\ W_k\in\text{ext}\,B_{\mathbb{R}^2_{o(w)}},\ k=1,2\Big\}.$$

Proof. Case 1. Let $X=tU+sV$ and $Y=t'U'+s'V'$ for some $0<t,t',s,s'<1, t+s=t'+s'=1$. Let

$$\mathcal{F}_1:=\Big\{\pm T\in\mathcal{L}(^2\mathbb{R}^2_{o(w)}):1=T(U,U')=T(U,V')=T(V,U')\\ =T(V,V')\text{ if ext}\,B_i=\{U,V\},\ \text{ext}\,B_j=\{U',V'\}\Big\}.$$

We claim that $NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X,Y)=\mathcal{F}_1$.
$(\subseteq)$: Suppose that $T\in NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X,Y)$.
Claim 1. $T(U,U')=T(U,V')=T(V,U')=T(V,V')=1$ or $T(U,U')=T(U,V')=T(V,U')=T(V,V')=-1$.

First, we will show that

$$\big|T(U,U')\big|=\big|T(U,V')\big|=\big|T(V,U')\big|=\big|T(V,V')\big|=1.$$

Assume the contrary. Without loss of generality we may assume that $|T(U,U')|<1$. It follows that

$$\begin{aligned}1&=\big|T(X,Y)\big|=\big|T(tU+sV,t'U'+s'V')\big|\\&=\big|tt'T(U,U')+ts'T(U,V')+t'sT(V,U')+ss'T(V,V')\big|\\&\leqslant tt'\big|T(U,U')\big|+ts'\big|T(U,V')\big|+t's\big|T(V,U')\big|+ss'\big|T(V,V')\big|\\&<tt'+ts'\big|T(U,V')\big|+t's\big|T(V,U')\big|+ss'\big|T(V,V')\big|\\&\leqslant(t+s)(t'+s')=1,\end{aligned}$$

which is a contradiction. Hence,

$$\big|T(U,U')\big|=\big|T(U,V')\big|=\big|T(V,U')\big|=\big|T(V,V')\big|=1.$$

Since $tt',ts',t's,ss'>0$ and

$$\big|tt'T(U,U')+ts'T(U,V')+t'sT(V,U')+ss'T(V,V')\big|=1,$$

by Lemma 2.4, the claim holds. Hence, $T \in \mathcal{F}_1$.

$(\supseteq)$: Let $T \in \mathcal{F}_1$.

Claim 2. $\|T\| = 1$

Suppose that $(X, Y) \in A_{11}$. Then $X = t(0,1) + s(w,1), Y = t'(0,1) + s'(w,1)$ and $T = \pm(w^2, 1, w, w)$. By Theorem 2.1, $\|T\| = 1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big) = \big\{\pm(w^2,1,w,w)\big\}.$$

Suppose that $(X, Y) \in A_{12}$. Then $X = t(0,1) + s(w,1), Y = t'(1,0) + s'(w,1)$ and $T = \pm(w, w, w^2, 1)$. By Theorem 2.1, $\|T\| = 1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big) = \big\{\pm(w,w,w^2,1)\big\}.$$

Suppose that $(X, Y) \in A_{13}$. Then $X = t(0,1) + s(w,1), Y = t'(1,0) + s'(w,-1)$ and $T = \pm(w, -w, -w^2, 1)$. By Theorem 2.1, $\|T\| = 1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big) = \big\{\pm(w,-w,-w^2,1)\big\}.$$

Suppose that $(X, Y) \in A_{14}$. Then $X = t(0,1) + s(w,1), Y = t'(0,-1) + s'(w,-1)$ and $T = \pm(w^2, -1, -w, w)$. By Theorem 2.1, $\|T\| = 1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big) = \big\{\pm(w^2,-1,-w,w)\big\}.$$

Suppose that $(X, Y) \in A_{21}$. Then $X = t(1,0) + s(w,1), Y = t'(0,1) + s'(w,1)$ and $T = \pm(w, w, 1, w^2)$. By Theorem 2.1, $\|T\| = 1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big) = \big\{\pm(w,w,1,w^2)\big\}.$$

Suppose that $(X, Y) \in A_{22}$. Then $(x_1, y_1) = t(1,0) + s(w,1), (x_2, y_2) = t'(1,0) + s'(w,1)$ and $T = \pm(1, w^2, w, w)$. By Theorem 2.1, $\|T\| = 1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big) = \big\{\pm(1,w^2,w,w)\big\}.$$

Suppose that $(X, Y) \in A_{23}$. Then $(x_1, y_1) = t(1,0) + s(w,1), (x_2, y_2) = t'(1,0) + s'(w,-1)$ and $T = \pm(1, -w^2, -w, w)$. By Theorem 2.1, $\|T\| = 1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big) = \big\{\pm(1,-w^2,-w,w)\big\}.$$

Suppose that $(X, Y) \in A_{24}$. Then $(x_1, y_1) = t(1,0) + s(w,1), (x_2, y_2) = t'(0,-1) + s'(w,-1)$ and $T = \pm(w, -w, -1, w^2)$. By Theorem 2.1, $\|T\| = 1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big) = \big\{\pm(w,-w,-1,w^2)\big\}.$$

Suppose that $(X,Y)\in A_{31}$. Then $(x_1,y_1)=t(1,0)+s(w,-1),(x_2,y_2)=t'(0,1)+s'(w,1)$ and $T=\pm(w,-w,1,-w^2)$. By Theorem 2.1, $\|T\|=1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big)=\big\{\pm(w,-w,1,-w^2)\big\}.$$

Suppose that $(X,Y)\in A_{32}$. Then $(x_1,y_1)=t(1,0)+s(w,-1),(x_2,y_2)=t'(1,0)+s'(w,1)$ and $T=\pm(1,-w^2,w,-w)$. By Theorem 2.1, $\|T\|=1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big)=\big\{\pm(1,-w^2,w,-w)\big\}.$$

Suppose that $(X,Y)\in A_{33}$. Then $(x_1,y_1)=t(1,0)+s(w,-1),(x_2,y_2)=t'(1,0)+s'(w,-1)$ and $T=\pm(1,w^2,-w,-w)$. By Theorem 2.1, $\|T\|=1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big)=\big\{\pm(1,w^2,-w,-w)\big\}.$$

Suppose that $(X,Y)\in A_{34}$. Then $(x_1,y_1)=t(1,0)+s(w,-1),(x_2,y_2)=t'(0,1)+s'(w,-1)$ and $T=\pm(w,w,-1,-w^2)$. By Theorem 2.1, $\|T\|=1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big)=\big\{\pm(w,w,-1,-w^2)\big\}.$$

Suppose that $(X,Y)\in A_{41}$. Then $(x_1,y_1)=t(0,-1)+s(w,-1),(x_2,y_2)=t'(0,1)+s'(w,1)$ and $T=\pm(w^2,-1,w,-w)$. By Theorem 2.1, $\|T\|=1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big)=\big\{\pm(w^2,-1,w,-w)\big\}.$$

Suppose that $(X,Y)\in A_{42}$. Then $(x_1,y_1)=t(0,-1)+s(w,-1),(x_2,y_2)=t'(1,0)+s'(w,1)$ and $T=\pm(w,-w,w^2,-1)$. By Theorem 2.1, $\|T\|=1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big)=\big\{\pm(w,-w,w^2,-1)\big\}.$$

Suppose that $(X,Y)\in A_{43}$. Then $(x_1,y_1)=t(0,-1)+s(w,-1),(x_2,y_2)=t'(1,0)+s'(w,-1)$ and $T=\pm(w,-w,w^2,-1)$. By Theorem 2.1, $\|T\|=1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big)=\big\{\pm(w,-w,w^2,-1)\big\}.$$

Suppose that $(X,Y)\in A_{44}$. Then $(x_1,y_1)=t(0,-1)+s(w,-1),(x_2,y_2)=t'(0,-1)+s'(w,-1)$ and $T=\pm(w^2,1,-w,-w)$. By Theorem 2.1, $\|T\|=1$. Hence,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)\big((x_1,x_2),(y_1,y_2)\big)=\big\{\pm(w^2,1,-w,-w)\big\}.$$

Hence, the claim 2 holds.

It follows that

$$\begin{aligned}\big|T(X,Y)\big|&=\big|T(tU+sV,t'U'+s'V')\big|\\&=\big|tt'T(U,U')+ts'T(U,V')+t'sT(V,U')+ss'T(V,V')\big|\\&=|tt'+ts'+t's+ss'|=(t+s)(t'+s')=1=\|T\|,\end{aligned}$$

which shows that $T \in NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X, Y)$.

Case 2. Let $X = tU + sV$ and $Y = t'U' + s'V'$ for some $0 < t, s < 1, t + s = 1, t' = 0$ or 1. Then $Y = U'$ or V'. Let

$$\mathcal{F}_2 = \Big\{\pm T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)}) : 1 = T(U, Y) = T(V, Y) \text{ if } \operatorname{ext} B_i = \{U, V\},\ |T(W_1, W_2)| \leqslant 1 \\ \text{for all } W_k \in \operatorname{ext} B_{\mathbb{R}^2_{o(w)}},\ k = 1, 2\Big\}.$$

We claim that $NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X, Y) = \mathcal{F}_2$.

$(\subseteq)$: Suppose that $T \in NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X, Y)$.

Claim 3. $T(U, Y) = T(V, Y) = 1$ or $T(U, Y) = T(V, Y) = -1$.

First, we will show that

$$\big|T(U, Y)\big| = \big|T(V, Y)\big| = 1.$$

Assume the contrary. Without loss of generality we may assume that $|T(V, Y)| < 1$. Let $t' = 0$. Then $Y = V'$. It follows that

$$\begin{aligned} 1 = \big|T(X, Y)\big| = \big|T(tU + sV, V')\big| = \big|tT(U, V') + sT(V, V')\big| \\ \leqslant t\big|T(U, Y)\big| + s\big|T(V, Y)\big| < t\big|T(U, Y)\big| + \big|T(V, Y)\big| \leqslant t + s = 1, \end{aligned}$$

which is a contradiction. Hence,

$$\big|T(U, Y)\big| = \big|T(V, Y)\big| = 1.$$

Since $t, s > 0$ and

$$\big|tT(U, Y) + sT(V, Y)\big| = 1,$$

by Lemma 2.4, the claim 3 holds.

Let $t' = 1$. Then $Y = U'$. It follows that

$$\begin{aligned} 1 = \big|T(X, Y)\big| = \big|T(tU + sV, U')\big| = \big|tT(U, U') + sT(V, U')\big| \\ \leqslant t\big|T(U, Y)\big| + s\big|T(V, Y)\big| < t\big|T(U, Y)\big| + \big|T(V, Y)\big| \leqslant t + s = 1, \end{aligned}$$

which is a contradiction. Hence,

$$\big|T(U, Y)\big| = \big|T(V, Y)\big| = 1.$$

Since $t, s > 0$ and

$$\big|tT(U, Y) + sT(V, Y)\big| = 1,$$

by Lemma 2.4, the claim 3 holds. Since

$$1 = \|T\| = \sup\Big\{|T(W_1, W_2)| : W_1, W_2 \in \operatorname{ext} B_{\mathbb{R}^2_{o(w)}}\Big\},$$

$T \in \mathcal{F}_2$. Hence, $NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X,Y) \subseteq \mathcal{F}_2$.

Notice that $\mathcal{F}_2 \subseteq NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X,Y)$ is obvious. Therefore,

$$NA\big(\mathcal{L}(^2\mathbb{R}^2_{o(w)})\big)(X,Y) = \mathcal{F}_2.$$

Case 3. Let $X = tU + sV$ and $Y = t'U' + s'V'$ for some $0 < t', s' < 1, t' + s' = 1, t = 0$ or 1. Then $X = U$ or V. Let

$$\mathcal{F}_3 = \big\{\pm T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)}) : 1 = T(X,U') = T(X,V') \text{ if } \operatorname{ext} B_j = \{U', V'\}, |T(W_1,W_2)| \leqslant 1 \\ \text{for all } W_k \in \operatorname{ext} B_{\mathbb{R}^2_{o(w)}},\ k = 1,2\big\}.$$

By analogous arguments as those of Case 2, $NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X,Y) = \mathcal{F}_3$.

Case 4. Notice that

$$(X,Y) = (U,U'), (U,V'), (V,U') \text{ or } (V,V').$$

Let

$$\mathcal{F}_4 = \big\{T \in \mathcal{L}(^2\mathbb{R}^2_{o(w)}) : 1 = |T(X,Y)|, |T(W_1,W_2)| \leqslant 1 \\ \text{for all } W_k \in \operatorname{ext} B_{\mathbb{R}^2_{o(w)}},\ k = 1,2\big\}.$$

It is obvious that $NA(\mathcal{L}(^2\mathbb{R}^2_{o(w)}))(X,Y) = \mathcal{F}_4$. Therefore, we complete the proof. □

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