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OSCILLATORY CRITERIA VIA LINEARIZATION OF HALF-LINEAR SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. In the paper, we study oscillation of the half-linear second order delay differential equations of the form

$$\big(r(t)(y'(t))^{\alpha}\big)' + p(t)y^{\alpha}(\tau(t)) = 0.$$

We introduce new monotonic properties of its nonoscillatory solutions and use them for linearization of considered equation which leads to new oscillatory criteria. The presented results essentially improve existing ones.

Keywords: second order differential equations, delay, monotonic properties, linearization, oscillation.

Mathematics Subject Classification: 34K11, 34C10.

1. INTRODUCTION

In this paper, we shall study the asymptotic and oscillation behavior of the solutions of half-linear second order delay differential equations

$$(r(t)(y'(t))^{\alpha})' + p(t)y^{\alpha}(\tau(t)) = 0. \tag{E}$$

We shall assume that

(H_1) $p, r \in C([t_0,\infty))$, $p(t) > 0$, $r(t) > 0$, α is the ratio of two positive odd integers,
(H_2) $\tau(t) \in C([t_0,\infty))$, $\tau(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$.

Moreover, it is assumed that

$$R(t) = \int_{t_0}^{t} r^{-1/\alpha}(s)\,\mathrm{d}s \to \infty \quad \text{as } t \to \infty. \tag{1.1}$$

By a solution of Eq. (E) we mean a function $y(t) \in C^1([T_y,\infty))$, $T_y \geq t_0$, such that has property $r(t)(y'(t))^\alpha \in C^1([T_y,\infty))$ and $y(t)$ satisfies Eq. (E) on $[T_y,\infty)$. We consider only those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_y,\infty)$ and otherwise it is called to be nonoscillatory. An equation itself is said to be oscillatory if all its solutions are oscillatory.

The problem of establishing oscillatory criteria for various types of differential equations has been a very active research area over the past decades. A large amount of papers have been devoted to this problem. We mention here several outstanding monographs by Agarwal *et al.* [1], Došly and Řehák [5], Erbe *et al.* [7], Kiguradze and Chanturia [10], Lade *et al.* [13] and Györi and Ladas [9] and papers [2–12].

Koplatadze *et al.* [11] presented very nice oscillatory criterion for

$$y''(t) + p(t)y(\tau(t)) = 0, \tag{1.2}$$

based on the following monotonic properties of positive solutions:

$$y(t)\uparrow, \qquad \frac{y(t)}{t}\downarrow. \tag{1.3}$$

The aim of this paper is to establish new comparison theorems for investigation of (E). Our first task is linearization of (E) in the sense that we would deduce oscillation of studied equation from that of its linear forms. To achieve this goal we provide new monotonic properties of possible nonoscillatory solutions of (E) which are new even for (1.2) and improves (1.3).

The second task is to provide new oscillatory criteria taking the linear forms of (E) into account. The third aim is to test the strength of general criteria derived via Euler differential equation.

2. PRELIMINARY RESULTS

We start with some useful lemmas concerning monotonic properties of nonoscillatory solutions for studied equations.

Lemma 2.1. *Let $y(t)$ be a positive solution of (E). Then $r(t)(y'(t))^\alpha > 0$ and $\frac{y(t)}{R(t)}$ is decreasing for $t \geq t_1 \geq t_0$. Moreover, if*

$$\int_{t_0}^{\infty} R^\alpha(\tau(s))p(s)\mathrm{d}s = \infty, \tag{2.1}$$

then

$$\lim_{t\to\infty} \frac{y(t)}{R(t)} = 0. \tag{2.2}$$

Proof. Assume that $y(t)$ is a positive solution of (E). Then $(r(t)(y'(t))^\alpha)' < 0$ and there exists $t_1 \geq t_0$ that $r(t)(y'(t))^\alpha$ has constant sign for $t \geq t_1$. Assume on the contrary that $r(t)(y'(t))^\alpha < 0$. Then there exists constant $k > 0$ such that $r(t)(y'(t))^\alpha < -k < 0$. Integrating the previous inequality from t_1 to t and using (1.1), we have

$$y(t) \leq y(t_1) - kR(t) \to -\infty \text{ as } t \to \infty.$$

This is a contradiction and we can conclude that $r(t)(y'(t))^\alpha > 0$. Employing the monotonic property of $r^{1/\alpha}(t)y'(t)$, we obtain

$$y(t) \geq \int_{t_1}^{t} \frac{r^{1/\alpha}(t)y'(t)}{r^{1/\alpha}(t)} \mathrm{d}s \geq r^{1/\alpha}(t)y'(t)R(t), \tag{2.3}$$

which implies $(\frac{y(t)}{R(t)})' < 0$. On the other hand, since $\frac{y(t)}{R(t)}$ is positive and decreasing there exists

$$\lim_{t\to\infty} \frac{y(t)}{R(t)} = \ell \geq 0.$$

Assume on the contrary that $\ell > 0$. Then $\frac{y(t)}{R(t)} \geq \ell$, $t \geq t_1$. Integrating (E) from t_1 to t, we obtain

$$r(t_1)(y'(t_1))^\alpha \geq \ell^\alpha \int_{t_1}^{t} p(s)R^\alpha(\tau(s))\mathrm{d}s,$$

which for $t \to \infty$ contradicts with (2.1). So that $\lim\limits_{t\to\infty} \frac{y(y)}{R(t)} = 0$. The proof is completed. □

Remark 2.2. The monotonic property $\frac{y(t)}{R(t)} \downarrow$ of (E) corresponds to $\frac{y(t)}{t} \downarrow$ for (1.2). The next considerations are intended to improve this property.

Since $R(t)$ is increasing, there exists $\lambda \geq 1$ such that

$$\frac{R(t)}{R(\tau(t))} \geq \lambda. \tag{2.4}$$

Theorem 2.3. *Let* (2.1) *hold and there exist a positive constant* β *such that*

$$\frac{1}{\alpha} R^\alpha(\tau(t))r^{1/\alpha}(t)R(t)p(t) \geq \beta \text{ for } t \geq t_0. \tag{2.5}$$

If $y(t)$ *is a positive solution of* (E), *then*

$$\frac{y(t)}{R^{1-\beta}(t)} \quad \text{is decreasing for } t \geq t_1, \tag{2.6}$$

$$\frac{y(t)}{R^{\beta_0}(t)} \quad \text{is increasing for } t \geq t_1, \text{ where } \beta_0 = \beta^{1/\alpha}\lambda^\beta. \tag{2.7}$$

Proof. Assume that $y(t)$ is a positive solution of (E). Note that (2.2) implies

$$\lim_{t\to\infty} r^{1/\alpha}(t)y'(t) = 0. \tag{2.8}$$

Therefore an integration of (E) yields

$$r^{1/\alpha}(t)y'(t) = \left[\int_t^\infty p(s)y^\alpha(\tau(s))\mathrm{d}s\right]^{1/\alpha}. \tag{2.9}$$

It is easy to see that

$$(r(t)(y'(t))^\alpha)' = \alpha\left(r^{1/\alpha}(t)y'(t)\right)^{\alpha-1}\left(r^{1/\alpha}(t)y'(t)\right)'.$$

Setting into (E), we have

$$\left(r^{1/\alpha}(t)y'(t)\right)' + \frac{1}{\alpha}\left(r^{1/\alpha}(t)y'(t)\right)^{1-\alpha} p(t)y^\alpha(\tau(t)) = 0.$$

Then $w(t) = r^{1/\alpha}(t)y'(t)$ is positive decreasing and satisfies

$$w'(t) + \frac{1}{\alpha}w^{1-\alpha}(t)p(t)y^\alpha(\tau(t)) = 0. \tag{2.10}$$

On the other hand, (2.3) implies

$$y(t) \geq r^{1/\alpha}(t)y'(t)R(t) = w(t)R(t)$$

and so

$$y^\alpha(\tau(t)) \geq w^\alpha(\tau(t))R^\alpha(\tau(t)) \geq w^\alpha(t)R^\alpha(\tau(t)).$$

Substituting the last inequality into (2.10), we get

$$w'(t) + \frac{1}{\alpha}p(t)R^\alpha(\tau(t))w(t) \leq 0$$

and

$$w'(t) + \frac{\beta}{R(t)r^{1/\alpha}(t)}w(t) \leq 0$$

which implies

$$-w'(t)R(t) \geq \frac{\beta}{r^{1/\alpha}(t)}w(t) = \beta y'(t).$$

We introduce the auxiliary function

$$f(t) = (1-\beta)y(t) - r^{1/\alpha}(t)y'(t)R(t). \tag{2.11}$$

Simple computation shows that

$$f'(t) = -\beta y'(t) - w'(t)R(t) \geq -\beta y'(t) + \beta y'(t) = 0.$$

So function $f(t)$ is increasing and has constant sign, eventually. First we admit that $f(t) \le 0$ for $t \ge t_1$. This implies that $y/R^{1-\beta}$ is increasing. Using this fact together with (2.9), we have

$$\begin{aligned}
r^{1/\alpha}(t)y'(t) &= \left[\int_t^\infty p(s)y^\alpha(\tau(s))\mathrm{d}s\right]^{1/\alpha} = \left[\int_t^\infty \frac{y^\alpha(\tau(s))}{R^\alpha(\tau(s))}\frac{\beta\alpha}{R(s)r^{1/\alpha}(s)}\mathrm{d}s\right]^{1/\alpha} \\
&\ge \left[\int_t^\infty \frac{\beta\alpha\, y^\alpha(s)}{R^{\alpha+1}(s)r^{1/\alpha}(s)}\mathrm{d}s\right]^{1/\alpha} = \left[\int_t^\infty \frac{\beta\alpha\, y^\alpha(s)}{R^{\alpha(1-\beta)}(s)}\frac{R^{-\alpha\beta-1}(s)}{r^{1/\alpha}(s)}\mathrm{d}s\right]^{1/\alpha} \\
&\ge \frac{y(t)}{R^{1-\beta}(t)}\left[\int_t^\infty \beta\alpha\frac{R^{-\alpha\beta-1}(s)}{r^{1/\alpha}(s)}\mathrm{d}s\right]^{1/\alpha} = \frac{y(t)}{R(t)}.
\end{aligned}$$

It follows from the last inequality that $\frac{y(t)}{R(t)}$ is increasing. This is a contradiction and we conclude that and $f(t) > 0$ which implies that $\frac{y(t)}{R^{1-\beta}(t)}$ is decreasing.

Now, we shall show that $\frac{y(t)}{R^{\beta_0}(t)}$ is increasing. Taking into account that $\frac{y(t)}{R^{1-\beta}(t)}$ is decreasing and $y(t)$ is increasing, it follows from (2.9) that

$$\begin{aligned}
r^{1/\alpha}(t)y'(t) &= \left[\int_t^\infty \beta\alpha\frac{y^\alpha(\tau(s))}{R^\alpha(\tau(s))}\frac{1}{R(s)r^{1/\alpha}(s)}\mathrm{d}s\right]^{1/\alpha} \\
&\ge \left[\int_t^\infty \beta\alpha\frac{y^\alpha(\tau(s))}{R^{\alpha(1-\beta)}(\tau(s))}\frac{R^{-\alpha\beta}(\tau(s))}{R(s)r^{1/\alpha}(s)}\mathrm{d}s\right]^{1/\alpha} \\
&\ge \left[\int_t^\infty \beta\alpha\frac{y^\alpha(s)}{R^{\alpha(1-\beta)}(s)}\frac{R^{-\alpha\beta}(\tau(s))}{R(s)r^{1/\alpha}(s)}\mathrm{d}s\right]^{1/\alpha} \\
&\ge y(t)\left[\int_t^\infty \beta\alpha\frac{R^{\alpha\beta}(s)}{R^{\alpha\beta}(\tau(s))}\frac{1}{R^{\alpha+1}(s)r^{1/\alpha}(s)}\mathrm{d}s\right]^{1/\alpha} \\
&\ge y(t)\left[\int_t^\infty \beta\alpha\lambda^{\alpha\beta}\frac{1}{R^{\alpha+1}(s)r^{1/\alpha}(s)}\mathrm{d}s\right]^{1/\alpha} \\
&= \frac{y(t)\beta^{1/\alpha}\lambda^\beta}{R(t)}.
\end{aligned}$$

The last inequality implies that $\left(\frac{y(t)}{R^{\beta_0}}\right)' > 0$. Hence, $\frac{y(t)}{R^{\beta_0}}$ is increasing and the proof is complete. □

Remark 2.4. The monotonic properties

$$\frac{y(t)}{R^{\beta_0}(t)} \uparrow, \qquad \frac{y(t)}{R^{1-\beta}(t)} \downarrow$$

essentially improves the normally used ones

$$y(t) \uparrow, \qquad \frac{y(t)}{R(t)} \downarrow .$$

and are new even for (1.2).

3. COMPARISON RESULTS

Now we are prepared to provide new comparison principles that significantly simplify the examination of half-linear differential equations. We separately discuss the cases $\alpha > 1$ and $0 < \alpha < 1$.

Theorem 3.1. *Let* $\alpha > 1$, *and* (2.1), (2.5) *hold. Then* (E) *is oscillatory provided that*

$$\left(r^{1/\alpha}(t)y'(t)\right)' + \frac{(1-\beta)^{1-\alpha}\lambda^{\beta(\alpha-1)}}{\alpha} R^{\alpha-1}(\tau(t))p(t)y(\tau(t)) = 0. \tag{L_1}$$

is oscillatory.

Proof. Assume on the contrary that $y(t)$ is a positive solution of (E). It is easy to see that

$$[r(t)(y'(t))^\alpha]' = \left[(r^{1/\alpha}(t)y'(t))^\alpha\right]' = \alpha\left(r^{1/\alpha}(t)y'(t)\right)^{\alpha-1}\left(r^{1/\alpha}(t)y'(t)\right)'.$$

Using the above relation in (E), we obtain

$$\left(r^{1/\alpha}(t)y'(t)\right)' + \frac{1}{\alpha}\left(r^{1/\alpha}(t)y'(t)\right)^{1-\alpha} p(t)y^\alpha(\tau(t)) = 0. \tag{3.1}$$

Since $\frac{y(t)}{R^{1-\beta}}(t)$ is decreasing, we are to the inequality

$$y(t) \geq \frac{r^{1/\alpha}(t)y'(t)}{1-\beta}R(t) \tag{3.2}$$

which for $\alpha > 1$ yields

$$y^{1-\alpha}(t) \leq \frac{(r^{1/\alpha}(t)y'(t))^{1-\alpha}}{(1-\beta)^{1-\alpha}}R^{1-\alpha}(t).$$

Hence

$$\left(r^{1/\alpha}(t)y'(t)\right)^{1-\alpha} \geq \frac{y^{1-\alpha}(t)}{R^{1-\alpha}(t)}(1-\beta)^{1-\alpha}. \tag{3.3}$$

Using again the monotonic property of $\frac{y(t)}{R^{1-\beta}}(t)$, we get

$$y^{1-\alpha}(t) \geq \frac{y^{1-\alpha}(\tau(t))}{R^{(1-\beta)(1-\alpha)}(\tau(t))} R^{(1-\beta)(1-\alpha)}(t). \tag{3.4}$$

Substituting (3.4) into (3.3), we have in view of (2.4) that

$$\begin{aligned}\left(r^{1/\alpha}(t)y'(t)\right)^{1-\alpha} &\geq \frac{(1-\beta)^{1-\alpha}R^{\beta(\alpha-1)}(t)}{R^{(1-\beta)(1-\alpha)}(\tau(t))} y^{1-\alpha}(\tau(t)) \\ &\geq \frac{(1-\beta)^{1-\alpha}\lambda^{\beta(\alpha-1)}}{R^{1-\alpha}(\tau(t))} y^{1-\alpha}(\tau(t)).\end{aligned} \tag{3.5}$$

Combining (3.1) and (3.5), we obtain that $y(t)$ obeys the linear differential inequality

$$\left(r^{1/\alpha}(t)y'(t)\right)' + \frac{(1-\beta)^{1-\alpha}\lambda^{\beta(\alpha-1)}}{\alpha} R^{\alpha-1}(\tau(t))p(t)y(\tau(t)) \leq 0. \tag{3.6}$$

On the other hand, Corollary 1 in [12] ensures that the corresponding differential equation (L_1) has a positive solution. This is a contradiction and the proof is complete now. □

Theorem 3.2. *Let* $0 < \alpha < 1$, *and* (2.1), (2.5) *hold. Then* (E) *is oscillatory provided that*

$$\left(r^{1/\alpha}(t)y'(t)\right)' + \frac{\beta^{\frac{1-\alpha}{\alpha}}\lambda^{1-\alpha}}{\alpha(1-\beta_0)^{\frac{1-\alpha}{\alpha}}} R^{\alpha-1}(t)y(\tau(t)) = 0. \tag{L_2}$$

is oscillatory.

Proof. Assume on the contrary that $y(t)$ is a positive solution of (E). Differentiation of (2.9) leads to the equation

$$(r^{1/\alpha}(t)y'(t))' + \frac{1}{\alpha}\left[\int_t^\infty p(s)y^\alpha(s)\mathrm{d}s\right]^{\frac{1-\alpha}{\alpha}} p(t)y^\alpha(\tau(t)) = 0.$$

Employing that $\frac{y(t)}{R^{\beta_0}}(t)$ is an increasing function, we have

$$(r^{1/\alpha}(t)y'(t))' + \frac{1}{\alpha}\frac{y^{1-\alpha}(\tau(t))}{R^{\beta_0(1-\alpha)}(\tau(t))}\left[\int_t^\infty p(s)R^{\alpha\beta_0}(\tau(s))\mathrm{d}s\right]^{\frac{1-\alpha}{\alpha}} p(t)y^\alpha(\tau(s)) \leq 0.$$

Therefore, $y(t)$ satisfies the linear differential inequality

$$(r^{1/\alpha}(t)y'(t))' + \frac{1}{\alpha}\frac{p(t)}{R^{\beta_0(1-\alpha)}(\tau(t))}\left[\int_t^\infty p(s)R^{\alpha\beta_0}(\tau(s))\mathrm{d}s\right]^{\frac{1-\alpha}{\alpha}} y(\tau(s)) \leq 0. \tag{3.7}$$

Moreover, using (2.4) and (2.5), we obtain

$$\int_t^\infty p(s)R^{\alpha\beta_0}(\tau(s))\mathrm{d}s \geq \alpha\beta\int_t^\infty \frac{R^{\alpha(\beta_0-1)}(\tau(s))}{r^{1/\alpha}(s)R(s)}\mathrm{d}s$$

$$\geq \alpha\beta\lambda^{\alpha(1-\beta_0)}\int_t^\infty \frac{R^{\alpha(\beta_0-1)-1}(s)}{r^{1/\alpha}(s)}\mathrm{d}s = \frac{\beta\lambda^{\alpha(1-\beta_0)}}{1-\beta_0}R^{\alpha(\beta_0-1)}(t).$$

Substituting into (3.7), we get

$$(r^{1/\alpha}(t)y'(t))' + \frac{\beta^{\frac{1-\alpha}{\alpha}}\lambda^{(1-\beta_0)(1-\alpha)}}{\alpha(1-\beta_0)^{\frac{1-\alpha}{\alpha}}}\frac{R^{(\beta_0-1)(1-\alpha)}(t)}{R^{\beta_0(1-\alpha)}(\tau(t))}p(t)y(\tau(t)) \leq 0.$$

which in view of (2.4) yields that $y(t)$ is a positive solution of the differential inequality

$$\left(r^{1/\alpha}(t)y'(t)\right)' + \frac{\beta^{\frac{1-\alpha}{\alpha}}\lambda^{1-\alpha}}{\alpha(1-\beta_0)^{\frac{1-\alpha}{\alpha}}}R^{\alpha-1}(t)y(\tau(t)) \leq 0.$$

By Corollary 1 in [12] the corresponding differential equation (L_2) has also a positive solution. This is a contradiction and the proof is complete now. □

Comparison results presented in Theorems 3.1 and 3.2 reduce the examination of oscillatory properties for (E) to that of linear equations (L_1) and (L_2).

4. OSCILLATORY CRITERIA

In this part we apply the results from the previous section for establishing new oscillatory criteria.

To simplify our notation let us denote

$$\kappa = \frac{(1-\beta)^{1-\alpha}\lambda^{\beta(\alpha-1)}}{\alpha}.$$

Theorem 4.1. *Let* $\alpha > 1$, *and* (2.1), (2.5) *hold. If*

$$\limsup_{t\to\infty}\left\{R^{\beta-1}(\tau(t))\int_{t_0}^{\tau(t)} p(s)R(s)R^{\alpha-\beta}(\tau(s))\mathrm{d}s + R^{\beta}(\tau(t))\int_{\tau(t)}^{t} p(s)R^{\alpha-\beta}(\tau(s))\mathrm{d}s + R^{1-\beta_0}(\tau(t))\int_t^\infty p(s)R^{\alpha+\beta_0-1}(\tau(s))\mathrm{d}s\right\} > \frac{1}{\kappa},$$

then (E) *is oscillatory.*

Proof. Assume on the contrary that (E) is not oscillatory. By Theorem 3.1, equation (L_1) is also nonoscillatory and we may assume that it possesses an eventually positive solution $y(t)$. An integration of (L_1) yields

$$y'(t) \geq \frac{\kappa}{r^{1/\alpha}(t)} \int_t^\infty p(s) R^{\alpha-1}(\tau(s)) y(\tau(s))\,\mathrm{d}s.$$

Integrating once more, one gets

$$\begin{aligned} y(t) &\geq \kappa \int_{t_1}^t \frac{1}{r^{1/\alpha}(u)} \int_u^\infty p(s) R^{\alpha-1}(\tau(s)) y(\tau(s))\,\mathrm{d}s\mathrm{d}u \\ &= \kappa \int_{t_1}^t \frac{1}{r^{1/\alpha}(u)} \int_u^t p(s) R^{\alpha-1}(\tau(s)) y(\tau(s))\,\mathrm{d}s\mathrm{d}u \\ &\quad + \kappa \int_{t_1}^t \frac{1}{r^{1/\alpha}(u)} \int_t^\infty p(s) R^{\alpha-1}(\tau(s)) y(\tau(s))\,\mathrm{d}s\mathrm{d}u. \end{aligned}$$

Changing the order of integration, we obtain

$$y(t) \geq \kappa \int_{t_1}^t p(s) R(s) R^{\alpha-1}(\tau(s)) y(\tau(s))\,\mathrm{d}s + \kappa R(t) \int_t^\infty p(s) R^{\alpha-1}(\tau(s)) y(\tau(s))\,\mathrm{d}s.$$

Hence

$$\begin{aligned} y(\tau(t)) \geq \kappa \int_{t_1}^{\tau(t)} & p(s) R(s) R^{\alpha-1}(\tau(s)) y(\tau(s))\,\mathrm{d}s \\ &+ \kappa R(\tau(t)) \int_{\tau(t)}^t p(s) R^{\alpha-1}(\tau(s)) y(\tau(s))\,\mathrm{d}s \\ &+ \kappa R(\tau(t)) \int_t^\infty p(s) R^{\alpha-1}(\tau(s)) y(\tau(s))\,\mathrm{d}s. \end{aligned}$$

Employing the fact that $y/R^{1-\beta}$ is decreasing and y/R^{β_0} is increasing the previous inequality implies

$$\begin{aligned} y(\tau(t)) \geq \kappa \frac{y(\tau(s))}{R^{1-\beta}(\tau(t))} \int_{t_1}^{\tau(t)} p(s)R(s)R^{\alpha-\beta}(\tau(s))\,\mathrm{d}s \\ + \kappa R(\tau(t))\frac{y(\tau(t))}{R^{1-\beta}(\tau(t))} \int_{\tau(t)}^{t} p(s)R^{\alpha-\beta}(\tau(s))\,\mathrm{d}s \\ + \kappa R(\tau(t))\frac{y(\tau(t))}{R^{\beta_0}(\tau(t))} \int_{t}^{\infty} p(s)R^{\alpha-1}(\tau(s))\,\mathrm{d}s. \end{aligned}$$

After simplification, one can see that

$$\begin{aligned} \Bigg\{ R^{(\beta-1)}(\tau(t)) \int_{t_1}^{\tau(t)} p(s)R(s)R^{\alpha-\beta}(\tau(s))\mathrm{d}s \\ + R^{\beta}(\tau(t)) \int_{\tau(t)}^{t} p(s)R^{\alpha-\beta}(\tau(s))\mathrm{d}s + R^{1-\beta_0}(\tau(t)) \int_{t}^{\infty} p(s)R^{\alpha+\beta_0-1}(\tau(s))\mathrm{d}s \Bigg\} \leq \frac{1}{\kappa}. \end{aligned}$$

This is a contradiction and the proof is complete now. □

We denote

$$\omega = \frac{\beta^{\frac{1-\alpha}{\alpha}}\lambda^{1-\alpha}}{\alpha(1-\beta_0)^{\frac{1-\alpha}{\alpha}}}.$$

Theorem 4.2. *Let* $0 < \alpha < 1$, *and* (2.1), (2.5) *hold. If*

$$\begin{aligned} \limsup_{t\to\infty} \Bigg\{ R^{\beta-1}(\tau(t)) \int_{t_1}^{\tau(t)} R^{\alpha}(s)R^{1-\beta}(\tau(s))p(s)\mathrm{d}s \\ + R^{\beta}(\tau(t)) \int_{\tau(t)}^{t} R^{\alpha-1}(s)R^{1-\beta}(\tau(s))p(s)\mathrm{d}s \\ + R^{1-\beta_0}(\tau(t)) \int_{t}^{\infty} R^{\alpha-1}(s)R^{\beta_0}(\tau(s))p(s)\mathrm{d}s \Bigg\} > \frac{1}{\omega}, \end{aligned}$$

then (E) *is oscillatory*

Proof. Assume on the contrary that (E) is not oscillatory. By Theorem 3.2, equation (L_2) is also nonoscillatory and we may assume that it possesses an eventually positive solution $y(t)$. An integration of (L_2) yields

$$y'(t) \geq \frac{\omega}{r^{1/\alpha}(t)} \int_t^\infty R^{\alpha-1}(s)p(s)y(\tau(s))\,\mathrm{d}s.$$

Then

$$\begin{aligned} y(t) &\geq \omega \int_{t_1}^t \frac{1}{r^{1/\alpha}(u)} \int_u^\infty R^{\alpha-1}(s)p(s)y(\tau(s))\,\mathrm{d}s\mathrm{d}u \\ &= \omega \int_{t_1}^t \frac{1}{r^{1/\alpha}(u)} \int_u^t R^{\alpha-1}(s)p(s)y(\tau(s))\,\mathrm{d}s\mathrm{d}u \\ &\quad + \omega \int_{t_1}^t \frac{1}{r^{1/\alpha}(u)} \int_t^\infty R^{\alpha-1}(s)p(s)y(\tau(s))\,\mathrm{d}s\mathrm{d}u. \end{aligned}$$

Consequently

$$y(t) \geq \omega \int_{t_1}^t R^{\alpha}(s)p(s)y(\tau(s))\,\mathrm{d}s + \omega R(t) \int_t^\infty R^{\alpha-1}(s)p(s)y(\tau(s))\,\mathrm{d}s$$

and so

$$\begin{aligned} y(\tau(t)) \geq \omega \int_{t_1}^{\tau(t)} R^{\alpha}(s)p(s)y(\tau(s))\mathrm{d}s + \omega R(\tau(t)) \int_{\tau(t)}^t R^{\alpha-1}(s)p(s)y(\tau(s))\,\mathrm{d}s \\ + \omega R(\tau(t)) \int_t^\infty R^{\alpha-1}(s)p(s)y(\tau(s))\,\mathrm{d}s. \end{aligned}$$

Since $y/R^{1-\beta}$ is decreasing and y/R^{β_0} is increasing the last inequality provides

$$\begin{aligned} y(\tau(t)) \geq \omega \frac{y(\tau(t))}{R^{1-\beta}(\tau(t))} &\int_{t_1}^{\tau(t)} R^{\alpha}(s)R^{1-\beta}(\tau(s))p(s)\,\mathrm{d}s \\ &+ \omega R(\tau(t)) \frac{y(\tau(t))}{R^{1-\beta}(\tau(t))} \int_{\tau(t)}^t R^{\alpha-1}(s)R^{1-\beta}(\tau(s))p(s)\,\mathrm{d}s \\ &+ \omega R(\tau(t)) \frac{y(\tau(t))}{R^{\beta_0}(\tau(t))} \int_t^\infty R^{\alpha-1}(s)R^{\beta_0}(\tau(s))p(s)\,\mathrm{d}s. \end{aligned}$$

Consequently,

$$\left\{R^{\beta-1}(\tau(t))\int_{t_1}^{\tau(t)} R^{\alpha}(s)R^{1-\beta}(\tau(s))p(s)\,\mathrm{d}s\right.$$
$$+\,R^{\beta}(\tau(t))\int_{\tau(t)}^{t} R^{\alpha-1}(s)R^{1-\beta}(\tau(s))p(s)\,\mathrm{d}s$$
$$\left.+\,R^{1-\beta_0}(\tau(t))\int_{t}^{\infty} R^{\alpha-1}(s)R^{\beta_0}(\tau(s))p(s)\,\mathrm{d}s\right\} \le \frac{1}{\omega}.$$

This is a contradiction and the proof is complete now. □

5. EXAMPLES

The quality of oscillatory criteria is usually tested via Euler differential equation.

Example 5.1. *We consider the general Euler differential equation*

$$(r(t)(y'(t))^{\alpha})' + \frac{a}{R^{\alpha+1}(t)r^{1/\alpha}(t)}y^{\alpha}(\tau(t)) = 0 \qquad (E_x)$$

with $a > 0$, $r(t) = t^{\gamma}$, $\tau(t) = bt$ *and* $0 < b < 1$. *Then*

$$\frac{R(t)}{R(\tau(t))} = \lambda = b^{\gamma/\alpha-1}, \qquad \beta = \frac{a}{\alpha}b^{\alpha-\gamma} \qquad \beta_0 = \left(\frac{a}{\alpha}\right)^{1/\alpha} b^{1-\gamma/\alpha+\beta(\gamma/\alpha-1)}.$$

$$\kappa = \frac{(1-\beta)^{1-\alpha}}{\alpha}\left(\frac{1}{b}\right)^{\beta(\alpha-1)(1-\gamma/\alpha)}$$

and

$$\omega = \frac{\beta^{\frac{1-\alpha}{\alpha}}}{\alpha(1-\beta_0)^{\frac{1-\alpha}{\alpha}}}\left(\frac{1}{b}\right)^{(1-\alpha)(1-\gamma/\alpha)}.$$

By Theorem 4.1, Eq. (E_x) *with* $\alpha > 1$ *is oscillatory provided that*

$$\frac{a}{1-\beta}b^{(1-\gamma/\alpha)(\alpha-\beta)} + \frac{a}{\beta}b^{(1-\gamma/\alpha)(\alpha-\beta)}\left(1 - b^{(1-\gamma/\alpha)\beta}\right) + \frac{a}{1-\beta_0}b^{(\alpha-\gamma)} > \frac{1}{\kappa}$$

and Theorem 4.2 implies that (E_x) *with* $0 < \alpha < 1$ *is oscillatory provided that*

$$\frac{a}{1-\beta}b^{(1-\gamma/\alpha)(1-\beta)} + \frac{a}{\beta}b^{(1-\gamma/\alpha)(1-\beta)}\left(1 - b^{(1-\gamma/\alpha)\beta}\right) + \frac{a}{1-\beta_0}b^{(1-\gamma/\alpha)} > \frac{1}{\omega}.$$

Setting values for α *and* γ, *the above criteria generated the corresponding oscillatory results for* (E_x).

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