DOI 10.2478/ama-2024-0018 acta mechanica et automatica, vol.18 no.1 (2024)

MODIFIED INTEGRAL TRANSFORM FOR SOLVING BENNEY-LUKE AND SINGULAR PSEUDO-HYPERBOLIC EQUATIONS

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received 8 February 2023, revised 9 August 2023, accepted 20 August 2023

Abstract: In this article, we propose a technique based on modified double integral transforms used to solve certain equations of materials science, namely Benney–Luke (BL) and singular pseudo-hyperbolic (SP-H) equations. We have established some analytical results. This method can provide accurate one-step solutions, although the equations used may exhibit a singularity in the initial conditions. Some numerical examples have been discussed for illustration and to show the effectiveness of the technique for certain types of equations. We have developed an exact solution in just one step, whereas other approaches require several stages to succeed in a particular solution, making the proposed strategy particularly successful and straightforward to apply to various varieties of the B–L and SP-H equations.

Key words: convergence, pseudo-hyperbolic equation, Benney–Luke equation

1. INTRODUCTION

Mathematical modelling of physical phenomena has attracted much attention from researchers in recent years [1–7]. Despite the development work of these models, we are still faced with equations that are difficult to deal with analytically and perhaps numerically. Accordingly, we must always look for simple and effective tools to solve these types of equations. For example, the non-linear evolution equations (NLEEs) would have a large amount of low recurrence because of their wide-ranging provisions

Since the appearance of solitary waves in natural sciences is intensifying day by day, it is important to seek exact traveling wave solutions to NLEEs.

"So far as the exact solutions to NLEEs are concerned, the data furnished in the present study pertaining to their structure will assist in preventing the structure from claiming changes attributable to unpredictable physical phenomena."

Among these endeavours are the strategy of homotopy examining, the strategy of sticking, the variable strategy of reiteration [8–12], the differential transforms method and Laplace transform [13–15] differential transforms method, Elzaki transform, double Laplace transform, the projected differential transform method[16–22] and Adomian decomposition (ADM) method [23].

Numerous explanations and numerical strategies have been proposed for non-linear PDE solutions with fractional derivatives, such as local partial variance replication [9], local Fourier fracture method, Yang–Fourier transform and Yang–Laplace transforms, and other methods. Two ways to replicate Laplace transform are currently proposed by Wu in the literature [17–20].

The Benney–Luke-type equations consist of a reduced potential flow water wave model based on the assumptions of a small amplitude parameter and a small dispersion parameter (defined by the square of the ratio of the typical depth over a horizontal length scale). The physical representation of the B–L is used to explore the dynamic behaviour of the solutions produced. A suitable selection of arbitrary constants and the parameters found in the solutions of the B–L equation are used to carry out the simulation.

The P-H equation is a well-known mathematical physics equation with numerous applications in domains such as longitudinal vibrations, thermoelasticity, plasma physics, nerve conduction and reaction diffusion, and a variety of other physical phenomena.

In this paper, we introduce an advanced method that relies on modified integral transform (MIT) [21], and one that uses simple techniques. Additionally, we contemplate the combination of MIT and the present technique to solve the B–L and SP-H equations [22]. Further, we demonstrate the means to select the right initial values for the solution in one step.

The single modified transform (SMT) is characterised by

$$\varepsilon[\Phi(v)] = T(\mu) = \mu^2 \int_0^\infty \Phi(\mu v)\, e^{-v} dv \ .$$

2. THE PRINCIPLE OF METHOD

In this segment, we present the MIT of the function, as well as its partial derivatives.

We moreover look at the convergence of this method.

Definition: Let $\Phi(\mathrm{w},\mathrm{v}),\ \mathrm{w},\mathrm{v} \in \mathrm{R}^+$ be a function that can be communicated as a convergent infinite series; at that point, the

DOI 10.2478/ama-2024-0018

MIT is given by:

$$\varepsilon_2[\Phi(w,v),\eta,\mu] = T(\eta,\mu) =$$

$$\eta\mu \int_0^\infty \int_0^\infty \Phi(w,v)\, e^{-\left(\frac{w}{\eta}+\frac{v}{\mu}\right)}\, dw dv, \quad w,v > 0 \tag{1}$$

where η and μ are complex values. To see the solutions of the B–L and SP-H equations, by the combination of MIT and the unused strategy, to begin with we start by taking after the MIT of partial derivatives:

$$\begin{aligned}
&\varepsilon_2\left[\frac{\partial\Phi}{\partial w}\right] = \frac{1}{\eta}T(\eta,\mu) - \eta T(0,\mu) \quad \varepsilon_2\left[\frac{\partial^2\Phi}{\partial w^2}\right] = \frac{1}{\eta^2}T(\eta,\mu) - T(0,\mu) - \eta\frac{\partial T(0,\mu)}{\partial w}\\
&\varepsilon_2\left[\frac{\partial\Phi}{\partial v}\right] = \frac{1}{\mu}T(\eta,\mu) - \mu T(\eta,0) \quad \varepsilon_2\left[\frac{\partial^2\Phi}{\partial v^2}\right] = \frac{1}{\mu^2}T(\eta,\mu) - T(\eta,0) - \mu\frac{\partial T(\eta,0)}{\partial v}\\
&\varepsilon_2\left[\frac{\partial^2\Phi}{\partial w\,\partial v}\right] = \frac{1}{\eta\mu}T(\eta,\mu) - \frac{\mu}{\eta}T(\eta,0) - \frac{\eta}{\mu}T(0,\mu) + \eta\mu T(0,0)
\end{aligned} \tag{2}$$

For more details, see Tarig et al. [24].

Here we ought to examinea few theorems of convergence of MIT.

Theorem 1: If:

$$\mu\int_0^\infty e^{-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}v$$

converges at $\mu = \mu_0$,

then:

$$\mu\int_0^\infty e^{-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}v$$

converges at $\mu < \mu_0$.

Proof: Let

$$p(w,v) = \mu_0\int_0^v e^{-\frac{u}{\mu_0}}\Phi(w,u)\mathrm{du}, \quad 0 < v < \infty;$$

then:

$$\text{(i) } \mathrm{p}(w,0) = 0 \quad \text{(ii) } \lim_{v\to\infty} \mathrm{p}(w,v) \;\; exist \quad \text{(iii) } \mathrm{p}_v(w,v) = \mu_0 e^{-\frac{v}{\mu_0}}\Phi(w,v)$$

Choosing ε_1 , R_1, such that $0 < \varepsilon_1 < R_1$, then we have:

$$\mu\int_{\varepsilon_1}^{R_1} e^{-\frac{v}{\mu}}\Phi(w,v)\mathrm{dt} = \mu\int_{\varepsilon_1}^{R_1}\frac{1}{\mu_0}e^{-\frac{v}{\mu}}p_v(w,v)e^{\frac{v}{\mu_0}}\mathrm{dt} =$$

$$\frac{\mu}{\mu_0}\int_{\varepsilon_1}^{R_1} e^{-\left(\frac{\mu_0-\mu}{\mu\mu_0}\right)v} p_v(w,v)\mathrm{d}v$$

The last integral becomes:

$$\frac{\mu}{\mu_0}\int_{\varepsilon_1}^{R_1} e^{-\left(\frac{\mu_0-\mu}{\mu\mu_0}\right)v} p_v(w,v)\mathrm{d}v =$$

$$\frac{\mu}{\mu_0}\left\{\begin{aligned}&e^{-\left(\frac{\mu_0-\mu}{\mu\mu_0}\right)R_1}p(w,\mathrm{R}_1) - e^{-\left(\frac{\mu_0-\mu}{\mu\mu_0}\right)\varepsilon_1}p(w,\varepsilon_1)\\ &+\left(\frac{\mu_0-\mu}{\mu\mu_0}\right)\int_{\varepsilon_1}^{R_1} e^{-\left(\frac{\mu_0-\mu}{\mu\mu_0}\right)v}p(w,v)\mathrm{d}v\end{aligned}\right\}.$$

Now take,$\varepsilon_1 \to 0$, $R_1 \to \infty$, and if $\mu < \mu_0$, then we have:

$$\mu\int_0^\infty e^{-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}v = \left(\frac{\mu_0-\mu}{\mu\mu_0}\right)\int_0^\infty e^{-\left(\frac{\mu_0-\mu}{\mu\mu_0}\right)v}p(w,v)\mathrm{d}v,$$

$$\mu < \mu_0, \tag{3}$$

Theorem 1 is demonstrated if the last integral converges.

Using the limits test for convergence to obtain,

$\lim_{v\to\infty} v^2 e^{-\left(\frac{\mu_0-\mu}{\mu\mu_0}\right)v}p(w,v) = 0$, finite, therefore,

$\mu\int_0^\infty e^{-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}v$ converges for $\mu < \mu_0$.

Theorem 2: If:

$$Q(w,\mu) = \mu\int_0^\infty e^{-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}v$$

converges to r$\mu < \mu_0$, and $\eta\int_0^\infty e^{-\frac{w}{\eta}}Q(w,\mu)\mathrm{d}w$

converges at $\eta = \eta_0$,

then: $\eta\int_0^\infty e^{-\frac{w}{\eta}}Q(w,\eta)\mathrm{d}w$

converges for $\eta < \eta_0$.

Proof: The verification of this theorem is at the same strategy as in Theorem 1.

Theorem3: If:

$\Phi(w,v)$

is continuous, and if:

$$\eta\mu\int_0^\infty\int_0^\infty e^{-\frac{w}{\eta}-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}w\mathrm{d}v$$

converges for

$\mu = \mu_0$, $\eta = \eta_0$,

then:

$$\eta\mu\int_0^\infty\int_0^\infty e^{-\frac{w}{\eta}-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}w\mathrm{d}v$$

converges for

$\eta < \eta_0$, $\mu < \mu_0$.

Proof:

$$\eta\mu\int_0^\infty\int_0^\infty e^{-\frac{w}{\eta}-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}w\mathrm{d}v =$$

$$\eta\int_0^\infty e^{-\frac{w}{\eta}}\left\{\mu\int_0^\infty e^{-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}v\right\}\mathrm{d}w = \eta\int_0^\infty e^{-\frac{w}{\eta}}Q(w,\mu)\mathrm{d}w$$

Where

$$Q(w,\mu) = \mu\int_0^\infty e^{-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}v$$

By using Theorems 1 and 2 we discover that:

$$\eta\mu\int_0^\infty\int_0^\infty e^{-\frac{w}{\eta}-\frac{v}{\mu}}\Phi(w,v)\mathrm{d}w\mathrm{d}v$$

converges for

$\eta < \eta_0\,,\ \ \mu < \mu_0.$

Here we will introduce the general equation that holds in the equations of B–L and SP-H in the following recipe:

$$U_{vv} - aU_{ww} + bU_{wwww} - cU_{wwvv} + dU_vU_{ww} + eU_wU_{wv} - g(w)\frac{\partial}{\partial w}\left(w\frac{\partial U}{\partial w}\right) - g(w)\frac{\partial^2}{\partial w\,\partial v}\left(w\frac{\partial U}{\partial w}\right) = f(w,v) \quad (4)$$

and given that a, b, c, d, e, are constants, we deliver:

If $g(w) = 0\,,\ e = 2\,,\ a = d = 1$, then Eq. (4) becomes,

$$U_{vv} - U_{ww} + bU_{wwww} - cU_{wwvv} + U_vU_{ww} + 2U_wU_{wv} = f(w,v) \quad (5)$$

Eq. (5) is referred to as the B–L equation, where

$c = \sigma - \frac{1}{3}$, σ is the bond number.

If $a = b = c = d = e = 0\,,\ g(w) = \frac{1}{w}$, then Eq. (4) becomes,

$$U_{vv} - \frac{1}{w}\frac{\partial}{\partial w}\left(w\frac{\partial U}{\partial w}\right) - \frac{1}{w}\frac{\partial^2}{\partial w\,\partial v}\left(w\frac{\partial U}{\partial w}\right) = f(w,v) \quad (6)$$

Eq. (6) is referred to as the SP-H equation.

3. THE NEW STRATEGY OF METHOD

To, clarify our strategy, we represent taking after the equation:

$$DU(w,v) + RU(w,v) + GU(w,v) + NU(w,v) = f(w,v) \quad (7)$$

where D maybe a second-order operator with respect to R that is a linear operator, N could be a non-linear operator and G could be a singular operator.

With the initial conditions,

$$U(w,0) = f_1(w)\ \ ,\ U_t(w,0) = f_2(w) \quad (8)$$

To reveal the solution of Esq.(7) and (8), we take the MITof Eq. (7), and the SMT of Eq. (8), and resultantly obtain,

$$\frac{1}{\mu^2}\varepsilon_2(U(w,v)) - K_1(\eta) - \mu K_2(\eta) = \varepsilon_2\begin{bmatrix} f(w,v) - RU(w,v) - \\ GU(w,v) - NU(w,v)\end{bmatrix} \quad (9)$$

where $K_1(\eta)$ and $K_2(\eta)$ are SNIT of $f_1(w)$ and $f_2(w)$, respectively.

We accept that the solution of Eq. (7) can be composed within the series form:

$$U(w,v)\ = \sum_{n=0}^{\infty} U_n(w,v)\,, \quad (10)$$

We take the inverse of the MITof Eq. (9), and utilising Eq. (10),we obtain,

$$\sum_{n=0}^{\infty} U_n(w,v) = F(w,v) + {\varepsilon_2}^{-1}\left\{\mu^2\varepsilon_2\begin{bmatrix} f(w,v) - RU(w,v) \\ -GU(w,v) - NU(w,v)\end{bmatrix}\right\} \quad (11)$$

This strategy depends on the means for choosing the beginning iteration $U_0(w,v)$, which goes to the exact solution by a few steps, for action in case we prefer, for the case on the off chance that we choose, for $U_0(w,v) = F(w,v)$.

Then, the solution $U(x,t)$can be recursively decided by utilising the relation,

$$U_{n+1}(w,v) = \varepsilon_2^{-1}\{\mu^2\varepsilon_2[f(w,v) - RU_n(w,v) - GU_n(w,v) - NU_n(w,v)]\}$$
$$U_0(w,v) = F(w,v)$$

From these equations, we can obtain:

$U_0(w,v), U_1(w,v), U_2(w,v)\,, \ldots\,,$

And after that, we will obtain the solution in a series form of Eq. (10).

When referring to a solution in physics, the word "exact" typically denotes one that fully resolves the physics and mathematics of the issue at hand as opposed to an approximation, perturbative, etc. solution. There is no requirement that exact solutions be closed-form where the non-linear equation has stable solutions known as soliton solutions and where non-linearity and dispersion are perfectly balanced.

4. APPLICATION

To demonstrate the productivity of this strategy in solving the B–L and SP-H equations by taking only one step, the following examples may be considered:

Example 1: In Eq. (5), we put,

$f(w,v) = 2v$

With the initial conditions,

$$U(w,0) = 1,\ U_v(w,0) = w, \quad (12)$$

We utilise the same steps in area 3 to induce,

$$\frac{1}{\mu^2}\varepsilon_2(U(w,v)) - \eta^2 - \mu\eta^3 = \varepsilon_2\begin{bmatrix} U_{ww} - bU_{wwww} + cU_{wwvv} \\ -U_vU_{ww} - 2U_wU_{wv} + 2v\end{bmatrix} \quad (13)$$

We apply the inverse MIT to Eq. (13), to obtain,

$$U(w,v) = 1 + wv + \varepsilon_2^{-1}\left\{\mu^2\varepsilon_2\begin{bmatrix} U_{ww} - bU_{wwww} + cU_{wwvv} \\ -U_vU_{ww} - 22U_wU_{wv} + 2v\end{bmatrix}\right\}$$

At that point the recursive relation is,

$$U_{n+1}(w,v) = \varepsilon_2^{-1}\left\{\mu^2\varepsilon_2\begin{bmatrix} (U_n)_{ww} - b(U_n)_{wwww} + c(U_n)_{wwvv} \\ -(U_n)_v(U_n)_{ww} - 2(U_n)_w(U_n)_{wv} + 2v\end{bmatrix}\right\}, \quad (14)$$
$$U_0(w,v) = 1 + wv,$$

The primary few components are given by,

$U_0(w,v) = 1 + wv\,,\quad U_1(w,v) = \varepsilon^{-1}\{\mu^2\varepsilon_2[0]\} = 0$

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Then the solution of Eq. (5) with $f(w,v) = 2v$ would be

$$U(w,v)\ = \sum_{n=0}^{\infty} U_n(w,v) = 1 + wv\,.$$

The exact solution was discovered using only one step when compared to the solutions in the study of Gündogdu and Gözükızıl [25], where two methods, namely the Laplace decomposition

DOI 10.2478/ama-2024-0018

method and the Adomian decomposition method, were employed. With these two methods, the exact solution was not reached directly; rather, it was obtained by deleting the noise term from the components.

Example 2: In this example, we put,

$f(w,v) = -(w^2\sin v + 4\sin v + 4\cos v)$ in Eq. (6),

with,

$$U(w,0) = 0\ ,\ U_v(w,0) = w^2 \tag{15}$$

Here, we use the same steps that we used earlier (in Example 1), to endure;

$$\varepsilon_2(U(w,v)) = \frac{2\mu^3\eta^4}{1+\mu^2} + \varepsilon_2\left[\begin{matrix}\frac{1}{w}\frac{\partial}{\partial w}\left(w\frac{\partial U}{\partial w}\right) + \frac{1}{w}\frac{\partial^2}{\partial w\,\partial v}\left(w\frac{\partial U}{\partial w}\right)\\ -4\sin v - 4\cos v\end{matrix}\right] \tag{16}$$

Utilising the inverse MIT of Eq. (16), we find that:

$$U(w,v) = \\ w^2\sin v + \varepsilon_2^{-1}\left\{\mu^2\varepsilon_2\left[\begin{matrix}\frac{1}{w}\frac{\partial}{\partial w}\left(w\frac{\partial U}{\partial w}\right) + \frac{1}{w}\frac{\partial^2}{\partial w\,\partial v}\left(w\frac{\partial U}{\partial w}\right)\\ -4\sin v - 4\cos v\end{matrix}\right]\right\}$$

At that point the recursive relations is,

$$U_{n+1}(w,v) = \varepsilon_2^{-1}\left\{\mu^2\varepsilon_2\left[\frac{1}{w}\frac{\partial}{\partial w}\left(w\frac{\partial U_n}{\partial w}\right) + \frac{1}{w}\frac{\partial^2}{\partial w\,\partial v}\left(w\frac{\partial U_n}{\partial w}\right) - 4\sin v - 4\cos v\right]\right\}$$
$$U_0(w,v) = w^2\sin v$$

As before, from that point onwards, we are able to discover the primary few components within the form,

$$U_0(w,v) = w^2\sin v\,, \qquad U_1(w,v) = \varepsilon_2^{-1}\{\mu^2\varepsilon_2[0]\} = 0$$

.
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Then, the solution of Eq. (6), with

$f(w,v) = -(w^2\sin v + 4\sin v + 4\cos v)$, is given by:

$$U(w,v) = \sum_{n=0}^{\infty} U_n(w,v) = w^2\sin v\,.$$

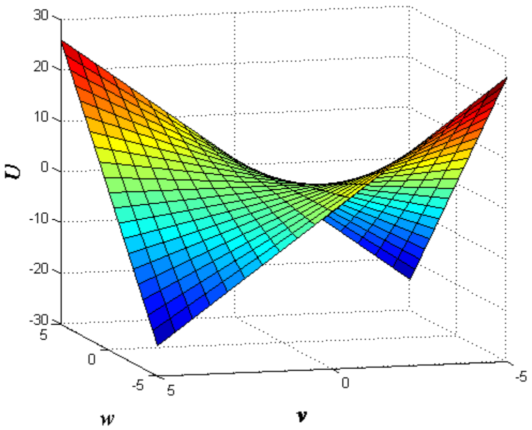

Fig. 1. Result obtained in example 1 of B–L equation

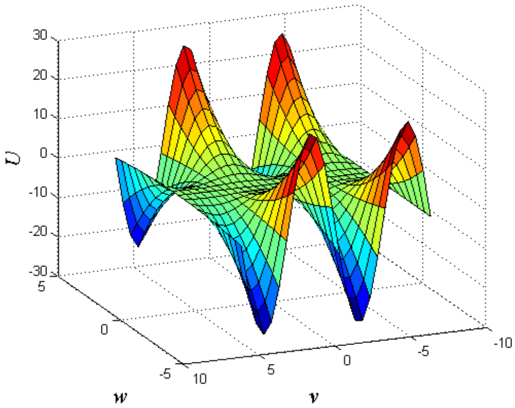

Fig. 1. Obtained solution in example 2 of SP-H equation

5. DISCUSSION

In Fig. 1, we have shown the solution to Example 1; comparing the same with Fig. 2, which represents the solution for Example 2, we notice that the solution in the latter example is characterised by more ripples, and this is probably due to the nature of the expression of the term f, which is characterised by the presence of sinusoidal functions. Figs.1 and 2 illustrate how the solutions physically move. Furthermore, the comprehension of physical processes, particularly in oceanography and geophysical science, may be aided by the proposed solutions.

6. CONCLUSION

In the present study, using the new MIT technique, we have dealt with a solution to the B–L and SP-H equations. We remark that the method can find a one-step solution despite the singularity that may appear in the initial conditions. Moreover, we have developed some analytical tools necessary for the convergence of the proposed method. We note that this strategy will be faster and less demanding with respect to the task of findingsolutions. Although they frequently appear in applied science and engineering, the N-dimensional hyperbolic equations of the B–L and SP-H equations still haven't been solved. Solving these equations in the near future could help us better understand the problems that represent these equations in real-world situations.

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Acknowledgement: The authors extend their appreciation to the Deanship of Scientific Research at University of Jeddah for funding this work though General Research Project under grant No. UJ-23-DR-157.

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