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A robust predictive actuator fault-tolerant control scheme for Takagi-Sugeno fuzzy systems

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Abstract. The paper deals with the problem of robust predictive fault-tolerant control for nonlinear discrete-time systems described by the Takagi-Sugeno models. The proposed approach is based on a triple stage procedure, i.e. it starts from fault estimation while the fault is compensated with a robust controller. The robust controller is designed without taking into account the input constraints related with the actuator saturation that may change due to its faulty behaviour. Thus, to check the compensation feasibility, the robust invariant set is developed, which takes into account the input constraints. If the current state does not belong to the robust invariant set, then suitable predictive control actions are performed in order to enhance the invariant set. This appealing phenomenon makes it possible to enlarge the domain of attraction, which makes the proposed approach an efficient solution for the fault-tolerant control. The final part of the paper shows an illustrative example regarding the application of the proposed approach to the twin-rotor system.

Key words: fault diagnosis, fault identification, robust control, robust invariant set, predictive control, fault-tolerant control.

1. Introduction

A permanent increase in the complexity, efficiency, and reliability of modern industrial systems necessitates a continuous development in control and fault diagnosis. A moderate combination of these two paradigms is intensively studied under the name of Fault-Tolerant Control (FTC). The Fault-Tolerant Control (FTC) systems are classified into two distinct classes [1]: passive and active. In the passive FTC [2], controllers are designed to be robust against a set of predefined faults, therefore there is no need for fault diagnosis, but such a design usually degrades the overall performance. In the contrast to the passive ones, active FTC schemes react to faults actively by reconfiguring control actions in such a way that the system stability and acceptable performance are maintained. To achieve that, the control system relies on the Fault Detection and Isolation (FDI) [3–7] as well as an accommodation technique [8]. Most of the existing works treat the FDI and FTC problems separately. Unfortunately, a perfect FDI and fault identification are impossible and hence there always is an inaccuracy related to this process. Thus, there is a need for integrated FDI and FTC schemes for both linear and nonlinear systems [9].

A number of books was published in the last decade on the emerging problem of the FTC. In particular, the book [10], which is mainly devoted to fault diagnosis and its applications provides some general rules for the hardware-redundancy-based FTC. On the contrary, the work [11] introduces the concepts of the active and passive FTC. It also investigates the problem of performance and stability of the FTC under imperfect fault diagnosis. In particular, the authors consider (under a chain of some, not necessary easy to satisfy assumptions) the effect of a delayed fault detection and an imperfect fault identification but the fault diagnosis [12, 13] scheme is treated separately during the design and no real integration of the fault diagnosis and the FTC is proposed. The FTC is also treated in a very interesting work [14] where the number of practical case studies of FTC is presented, i.e., a winding machine, a three-tank system, and an active suspension system. Unfortunately, in spite of the incontestable appeal of the proposed approaches neither the FTC integrated with the fault diagnosis nor a systematic approach to nonlinear systems are studied.

The proposed approach overcomes the above-mentioned difficulties and provides an elegant way of incorporating fault diagnosis (particularly fault identification) into the fault-tolerant control framework. The proposed approach is based on a triple stage procedure, i.e. it starts from fault estimation, then the fault is compensated with a robust controller. The robust controller is designed without taking into account the input constraints related with the actuator saturation. Thus, to check the compensation feasibility, the robust invariant set is developed, which takes into account the input constraints. If the current state does not belong to the robust invariant set, then suitable predictive control actions are performed in order to enhance the invariant set. This appealing phenomenon makes it possible to enlarge the domain of attraction, which makes the proposed approach an efficient solution. Indeed, the presented solution can be perceived as an extension of the recent developments in this area [15], which shows a fault estimation and compensation strategy for nonlinear systems. The novelty of the scheme boils down to:

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- introduction of robustness to exogenous disturbances, through the $\mathcal{H}_\infty$ approach,
- introduction of the triple stage procedure: fault estimation, fault compensation with robust controller, and predictive control enhancing the applicability of the approach,
- extension of the work of [16] to the case with exogenous disturbances,
- development of robust invariant set extending the usual framework proposed by [16].

The paper is organised as follows. Section 2 presents preliminaries regarding the problem being undertaken. Robust fault estimation and control approach is proposed in Sec. 3. Subsequently, Sec. 4 presents the development of a robust invariant set while Sec. 5 presents an efficient robust predictive fault-tolerant control strategy, which enhances the performance of the overall scheme. The final part of the paper contains a numerical example, which shows the performance of the proposed approach.

2. A general description of the fault-tolerant scheme

A nonlinear dynamic system can be described in a relatively simple way by a Takagi-Sugeno fuzzy model, which uses series of locally linearised models from the nonlinear system, parameter identification of an a priori given structure or transformation of a nonlinear model using the nonlinear sector approach (see, e.g. [3,17,18]). According to this model, a nonlinear dynamic systems can be linearised around a number of operating points. Each of these linear models represents the local system behaviour around the operating point. Thus, a fuzzy fusion of all linear model outputs describes the global system behaviour. A Takagi-Sugeno model is described by fuzzy IF-THEN rules. The presented structure may represent a nonlinear system with control-affine state equation. It has a rule base of M rules, each having p antecedents, where i-th rule is expressed as

$$\begin{aligned} R^i: \ \text{IF } s_k^1 \text{ is } F_1^i \text{ and } \dots \text{ and } s_k^p \text{ is } F_p^i,\\ \text{THEN } \boldsymbol{x}_{f,k+1} = \boldsymbol{A}^i\boldsymbol{x}_{f,k} + \boldsymbol{B}^i\boldsymbol{u}_{f,k} + \boldsymbol{B}^i\boldsymbol{f}_k + \boldsymbol{W}^i\boldsymbol{w}_k, \end{aligned} \tag{1}$$

in which $\boldsymbol{x}_{f,k} \in \mathbb{R}^n$ stands for the state and $\boldsymbol{u}_{f,k} \in \mathbb{R}^r$ denotes the nominal control input, $\boldsymbol{f}_k \in \mathbb{R}^r$ is the actuator fault, $i = 1,\dots,M$, F_j^i $(j = 1,\dots,p)$ are fuzzy sets and $\boldsymbol{s}_k = [s_k^1, s_k^2, \dots, s_k^p]$ is a known vector of premise variables [3,17]. It is of course possible to employ the approach that can be dedicated for the case when some of the premise variables are unmeasurable [19,20]. Similarly, the approach can be extended to the case when only some state variables are available, but this leads to the need for employing observer-based schemes [15]. However, the above mentioned strategies are beyond the scope of this paper. Additionally, $\boldsymbol{w}_k \in l_2$ is an exogenous disturbance vector, while:

$$l_2 = \left\{ \mathbf{w} \in \mathbb{R}^r |\ \|\mathbf{w}\|_{l_2} < +\infty \right\}, \tag{2}$$

$$\|\mathbf{w}\|_{l_2} = \left(\sum_{k=0}^{\infty} \|\boldsymbol{w}_k\|^2 \right)^{1/2}. \tag{3}$$

Given a pair of $(\boldsymbol{s}_k, \boldsymbol{u}_k)$ and a product inference engine, the final output of the normalized T-S fuzzy model can be inferred as:

$$\begin{aligned} \boldsymbol{x}_{f,k+1} &= \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)[\boldsymbol{A}^i\boldsymbol{x}_{f,k} + \boldsymbol{B}^i\boldsymbol{u}_k + \boldsymbol{B}^i\boldsymbol{f}_k + \boldsymbol{W}^i\boldsymbol{w}_k],\\ &= \boldsymbol{A}(h_k)\boldsymbol{x}_{f,k} + \boldsymbol{B}(h_k)\boldsymbol{u}_k + \boldsymbol{B}(h_k)\boldsymbol{f}_k + \boldsymbol{W}(h_k)\boldsymbol{w}_k, \end{aligned} \tag{4}$$

where $h_i(\boldsymbol{s}_k)$ are normalised rule firing strengths defined as

$$h_i(\boldsymbol{s}_k) = \frac{\mathop{\mathcal{T}}\limits_{j=1}^{p} \mu_{F_j^i}(s_k^j)}{\sum\limits_{i=1}^{M} (\mathop{\mathcal{T}}\limits_{j=1}^{p} \mu_{F_j^i}(s_k^j))} \tag{5}$$

and $\mathcal{T}$ denotes a t-norm (e.g., product). The term $\mu_{F_j^i}(s_k^j)$ is the grade of membership of the premise variable s_k^j. Moreover, the rule firing strengths $h_i(\boldsymbol{s}_k)$ $(i = 1,\dots,M)$ satisfy the following constraints

$$\begin{cases} \sum\limits_{i=1}^{M} h_i(\boldsymbol{s}_k) = 1, \\ 0 \le h_i(\boldsymbol{s}_k) \le 1, \qquad \forall i = 1,\dots,M. \end{cases} \tag{6}$$

For the purpose of further deliberations, the following set of assumptions is imposed:

Assumption 1: The matrix $\boldsymbol{B}^i$, $i = 1,\dots,M$ fulfils rank$(\boldsymbol{B}^i) = r$.

Assumption 2: The fault satisfies

$$\boldsymbol{f}_k = \boldsymbol{f}_{k-1} + \overline{\boldsymbol{v}}_k, \qquad \overline{\boldsymbol{v}}_k \in l_2. \tag{7}$$

Assumption 3: The control limits shaping $\mathbb{U}$ are given by

$$-\overline{\boldsymbol{u}}_i \le \boldsymbol{u}_{i,k} \le \overline{\boldsymbol{u}}_i, \qquad i = 1,\dots,r, \tag{8}$$

where $\overline{\boldsymbol{u}}_i > 0$ $(i = 1,\dots,r)$ are given control bounds and $\boldsymbol{u}_{i,k}$ stands for ith component of $\boldsymbol{u}_k$.

Assumption 4:

$$\boldsymbol{W}(h_k)\boldsymbol{w}_k \in \text{col}(\boldsymbol{B}(h_k)), \tag{9}$$

where $\text{col}(\boldsymbol{B}(h_k)) = \{\boldsymbol{\alpha} \in \mathbb{R}^n : \boldsymbol{\alpha} = \boldsymbol{B}(h_k)\boldsymbol{\beta}\}$ for some $\boldsymbol{\beta} \in \mathbb{R}^r$.

Due to the simplicity of presentation, these limits are symmetrical around zero but with an appropriate scaling it is relatively easy to introduce non-symmetrical ones. Note also that *Assumption 4* is a technical condition, which will be used in the subsequent part of this paper. It describes possible settings of disturbance distribution matrix $\boldsymbol{W}(h_k)$ were the simplest choice is $\boldsymbol{W}(h_k) = \boldsymbol{B}(h_k)$.

The main objective of the subsequent part of the paper is to design the control strategy in such a way that the system (4) will converge to the origin irrespective of the presence of the fault $\boldsymbol{f}_k$. The proposed control scheme is as follows:

$$\boldsymbol{u}_{f,k} = -\boldsymbol{K}(h_k)\boldsymbol{x}_k - \widehat{\boldsymbol{f}}_{k-1} + \boldsymbol{c}_k, \tag{10}$$

while the predicted future input is described by:

$$\boldsymbol{u}_{f,j} = \begin{cases} -\boldsymbol{K}(h_j)\boldsymbol{x}_j - \widehat{\boldsymbol{f}}_{k-1} + \boldsymbol{c}_j, & j = k,\dots,k+n_c-1, \\ -\boldsymbol{K}(h_j)\boldsymbol{x}_j - \widehat{\boldsymbol{f}}_{k-1}, & j \ge k+n_c. \end{cases} \tag{11}$$

where:

- n_c is the prediction horizon,
- $\boldsymbol{K}(h_k) = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k)\boldsymbol{K}^i$ is the $\mathcal{H}_\infty$ Parallel Distributed Compensation (PDC) controller designed to achieve robustness with respect to exogenous disturbances $\boldsymbol{w}_k$,
- $\widehat{\boldsymbol{f}}_{k-1}$ is the fault estimate, which compensates the effect of a fault,
- $\boldsymbol{c}_j$ is a vector introducing additional design freedom, which should be exploited when the fault compensation does not provide the expected results due to the actuator saturation.

Note that beyond the prediction horizon n_c, $\boldsymbol{c}_j$ is set to zero, which denotes the feasibility of the $\mathcal{H}_\infty$ control. Thus, the design of the proposed control strategy boils down to solving a set of problems:

- to design a robust PDC controller $\boldsymbol{K}(h_k)$ in such way that a prescribed disturbance attenuation level is achieved with respect to $\boldsymbol{x}_{f,k}$ while guaranteeing its convergence to the origin,
- to estimate the fault $\boldsymbol{f}_k$,
- to determine a set of states for which the robust PDC controller along with the fault compensation (under the control constraints) is feasible,
- to determine $\boldsymbol{c}_j$ in such a way as to enhance a set of states and, hence making the control problem feasible.

Since the general scheme is given, the remaining part of the paper is devoted to solving the above-mentioned design problems.

3. Fault estimation and robust control

In this section, the fault estimation technique is proposed, which along with the robust PDC controller $\boldsymbol{K}(h_k)$ is used to compensate the effect of a fault and feed the system in such a way that the state $\boldsymbol{x}_{f,k}$ goes to the origin. Note that the designs of the fault estimator and the robust PDC controller are realised for the unconstrained case. Moreover, the free control parameter $\boldsymbol{c}_j$ (cf. (11)) is set to zero. Following the seminal paper [16] along with further developments, the constraints are introduced during the development of the set of states, for which the robust PDC controller along with the fault compensation is feasible as well as during the computation of $\boldsymbol{c}_k$, which enhance a set of states and, hence making the control problem feasible.

Thus, following [6, 21], by computing

$$\boldsymbol{H}(h_k) = \boldsymbol{B}(h_k)^+ = \left[\boldsymbol{B}(h_k)^T\boldsymbol{B}(h_k)\right]^{-1}\boldsymbol{B}(h_k)^T, \tag{12}$$

and then multiplying (4) by $\boldsymbol{H}(h_k)$ along with extracting $\boldsymbol{f}_k$, it can be shown that:

$$\begin{aligned}\boldsymbol{f}_k &= \boldsymbol{H}(h_k)\boldsymbol{x}_{f,k+1} - \boldsymbol{H}(h_k)\boldsymbol{A}(h_k)\boldsymbol{x}_{f,k}\\ &\quad -\boldsymbol{u}_{f,k} - \boldsymbol{H}(h_k)\boldsymbol{W}(h_k)\boldsymbol{w}_k,\end{aligned} \tag{13}$$

while its estimate can be given as:

$$\widehat{\boldsymbol{f}}_k = \boldsymbol{H}(h_k)\boldsymbol{x}_{f,k+1} - \boldsymbol{H}(h_k)\boldsymbol{A}(h_k)\boldsymbol{x}_{f,k} - \boldsymbol{u}_{f,k}, \tag{14}$$

with the associated fault estimation error

$$\boldsymbol{\varepsilon}_{f,k} = \boldsymbol{f}_k - \widehat{\boldsymbol{f}}_k = -\boldsymbol{H}(h_k)\boldsymbol{W}(h_k)\boldsymbol{w}_k. \tag{15}$$

The main difficulty of the above approach is related with the existence of $\boldsymbol{H}(h_k)$, which boils down to checking the full rank property of all convex combinations of $\boldsymbol{B}^i, i = 1,\dots,M$. It is an obvious fact that, in a general case, $\boldsymbol{H}(h_k)$ is not a convex combination of $(\boldsymbol{B}^i)^+$, $i = 1,\dots,M$. Unfortunately, to the authors' knowledge there are no suitable conditions for checking this property in the control engineering–related literature. Let us define

$$\boldsymbol{Q}_{p,p} = (\boldsymbol{B}^p)^T\boldsymbol{B}^p, \qquad p = 1,\dots M, \tag{16}$$

$$\begin{aligned}\boldsymbol{Q}_{p,a} = (\boldsymbol{B}^p)^T\boldsymbol{B}^a + (\boldsymbol{B}^a)^T\boldsymbol{B}^p - (\boldsymbol{B}^a)^T\boldsymbol{B}^a - (\boldsymbol{B}^p)^T\boldsymbol{B}^p\\ \text{for} \quad p < a,\end{aligned} \tag{17}$$

$$\boldsymbol{R}_p = \boldsymbol{R}^p_{a,b} = \begin{cases} \boldsymbol{Q}_{p,p} & \text{if} \quad (a,b) = (1,1)\\ \boldsymbol{Q}_{b-1,p} & \text{if} \quad a = 1 \wedge b = 2,\dots,p\\ \boldsymbol{I} & \text{if} \quad a = b \wedge 1 < b < p\\ -\boldsymbol{I} & \text{if} \quad b = 1 \wedge a = p+1\\ \boldsymbol{0} & \text{otherwise}\end{cases} \tag{18}$$

Theorem 1. The following statements are equivalent

(a) All convex combinations of $\boldsymbol{B}^1,\dots,\boldsymbol{B}^M$ have full rank.
(b) $\boldsymbol{B}^M$ has full row rank and the $(M-1)Mn$-by-$(M-1)Mn$ matrix

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{R}_1\boldsymbol{R}_M^{-1} & (\boldsymbol{R}_2-\boldsymbol{R}_1)\boldsymbol{R}_M^{-1} & (\boldsymbol{R}_3-\boldsymbol{R}_2)\boldsymbol{R}_M^{-1} & \dots & (\boldsymbol{R}_{M-1}-\boldsymbol{R}_{M-2})\boldsymbol{R}_M^{-1}\\ -\boldsymbol{I}_{Mn} & \boldsymbol{I}_{Mn} & \boldsymbol{0}_{Mn} & \dots & \boldsymbol{0}_{Mn}\\ \boldsymbol{0}_{Mn} & -\boldsymbol{I}_{Mn} & \boldsymbol{I}_{Mn} & \dots & \boldsymbol{0}_{Mn}\\ \dots & \dots & \dots & \dots & \dots\\ \boldsymbol{0}_{Mn} & \dots & \boldsymbol{0}_{Mn} & -\boldsymbol{I}_{Mn} & \boldsymbol{I}_{Mn}\end{bmatrix}, \tag{19}$$

is a block P-matrix [22] with respect to the partition $\{\boldsymbol{F}_1,\ldots,\boldsymbol{F}_{M-1}\}$ of $\{1,\ldots,(M-1)Mn\}$, with $\boldsymbol{F}_i=\{(M-1)Mn+1,\ldots,iMn\}$, $i=1,\ldots,M-1$.

Proof. The proof can be derived by a direct application of *Theorem 2* in [22]. Note also that $\boldsymbol{B}^M$ has a full row rank under *Assumption 1*.

Remark 1. Following [22], the matrix $\boldsymbol{V}$ is a block P-matrix with respect to any partition if its all principal minors are positive. This feature makes it possible to easily check the condition of Theorem 3.

Since the general framework for computing the fault estimate (14) is given, then its computational feasibility can be verified. Indeed, to obtain $\widehat{\boldsymbol{f}}_k$ it is necessary to have $\boldsymbol{x}_{f,k+1}$. Thus, the only choice to compensate $\boldsymbol{f}_k$ in (4) is to use $\widehat{\boldsymbol{f}}_{k-1}$. This determines the above-proposed control strategy

$$\boldsymbol{u}_{f,k}=-\widehat{\boldsymbol{f}}_{k-1}-\boldsymbol{K}(h_k)\boldsymbol{x}_{f,k}. \tag{20}$$

Note that this strategy is derived by setting $\boldsymbol{c}_k=\boldsymbol{0}$ in (10). Taking into account (15) and (7), it can be shown that

$$\widehat{\boldsymbol{f}}_k=\widehat{\boldsymbol{f}}_{k-1}+\boldsymbol{H}(h_k)\boldsymbol{W}(h_k)[\boldsymbol{w}_k-\boldsymbol{w}_{k-1}]+\overline{\boldsymbol{v}}_k, \tag{21}$$

and knowing that $\boldsymbol{w}_k,\overline{\boldsymbol{v}}_k\in l_2$ it is evident that there exists $\boldsymbol{v}_k$ such that:

$$\widehat{\boldsymbol{f}}_k=\widehat{\boldsymbol{f}}_{k-1}+\boldsymbol{v}_k,\qquad \boldsymbol{v}_k\in l_2. \tag{22}$$

Thus, $\boldsymbol{v}_k$ is related with both exogenous disturbances $\boldsymbol{w}_k$ and fault estimation uncertainty expressed by $\boldsymbol{\varepsilon}_{f,k}$. For the purpose of further analysis, it is additionally assumed that the above bounds have the following form

$$\boldsymbol{v}_k\in\mathcal{E}_v,\quad \mathcal{E}_v=\{\boldsymbol{v}\in\mathbb{R}^r|\,\boldsymbol{v}^T\boldsymbol{Q}\boldsymbol{v}\le 1\},\quad \boldsymbol{Q}\succ 0. \tag{23}$$

Thus, (20) can be written in an equivalent form, which will be used for further deliberations

$$\boldsymbol{u}_{f,k}=-\widehat{\boldsymbol{f}}_k+\boldsymbol{v}_k-\boldsymbol{K}(h_k)\boldsymbol{x}_{f,k}. \tag{24}$$

Substituting (24) into (4) gives

$$\begin{aligned}\boldsymbol{x}_{f,k+1}&=\boldsymbol{A}_1(h_k)\boldsymbol{x}_{f,k}\\&+[\boldsymbol{I}-\boldsymbol{B}(h_k)\boldsymbol{H}(h_k)]\,\boldsymbol{W}(h_k)\boldsymbol{w}_k+\boldsymbol{B}(h_k)\boldsymbol{v}_k,\end{aligned} \tag{25}$$

with $\boldsymbol{A}_1(h_k)=\sum_{i=1}^{M}\sum_{j=1}^{M}h_i(\boldsymbol{s}_k)h_j(\boldsymbol{s}_k)(\boldsymbol{A}^i-\boldsymbol{B}^i\boldsymbol{K}^j)$.

Further analysis of (25), and in particular

$$\left[\boldsymbol{I}-\boldsymbol{B}(h_k)\left[\boldsymbol{B}(h_k)^T\boldsymbol{B}(h_k)\right]^{-1}\boldsymbol{B}(h_k)^T\right]\boldsymbol{W}(h_k)\boldsymbol{w}_k= \tag{26}$$

$$\boldsymbol{W}(h_k)\boldsymbol{w}_k-\boldsymbol{B}(h_k)\left[\boldsymbol{B}(h_k)^T\boldsymbol{B}(h_k)\right]^{-1}\boldsymbol{B}(h_k)^T\boldsymbol{W}(h_k)\boldsymbol{w}_k, \tag{27}$$

along with the fact that (under *Assumption 1*) any vector $\boldsymbol{W}(h_k)\boldsymbol{w}_k\in\mathrm{col}(\boldsymbol{B}(h_k))$ can be written as $\boldsymbol{W}(h_k)\boldsymbol{w}_k=\boldsymbol{B}(h_k)\overline{\boldsymbol{w}}_k$ for some non-zero $\overline{\boldsymbol{w}}_k$, leads (27) to

$$\begin{aligned}&\boldsymbol{B}(h_k)\overline{\boldsymbol{w}}_k-\boldsymbol{B}(h_k)\left[\boldsymbol{B}(h_k)^T\boldsymbol{B}(h_k)\right]^{-1}\\&\cdot\boldsymbol{B}(h_k)^T\boldsymbol{B}(h_k)\overline{\boldsymbol{w}}_k=0.\end{aligned} \tag{28}$$

This significant simplification of (25) yields its new form:

$$\boldsymbol{x}_{f,k+1}=\boldsymbol{A}_1(h_k)\boldsymbol{x}_{f,k}+\boldsymbol{B}(h_k)\boldsymbol{v}_k. \tag{29}$$

Before providing the PDC control design procedure, let us remind the following lemma [23–25]:

Lemma 1. The following statements are equivalent

1. There exists $\boldsymbol{X}\succ 0$ such that
$$\boldsymbol{V}^T\boldsymbol{X}\boldsymbol{V}-\boldsymbol{W}\prec 0. \tag{30}$$
2. There exists $\boldsymbol{X}\succ 0$ such that
$$\begin{bmatrix}-\boldsymbol{W} & \boldsymbol{V}^T\boldsymbol{U}^T\\ \boldsymbol{U}\boldsymbol{V} & \boldsymbol{X}-\boldsymbol{U}-\boldsymbol{U}^T\end{bmatrix}\prec 0. \tag{31}$$

Remark 2. Note that the regularity of $\boldsymbol{U}$ is ensured by the last block diagonal element of (31), which implies $\boldsymbol{U}+\boldsymbol{U}^T\succ\boldsymbol{X}\succ\boldsymbol{0}$. This property will be exploited in further deliberations.

The following theorem constitutes the main result of this section.

Theorem 2. For a prescribed disturbance and fault estimation uncertainty attenuation level $\mu>0$ for the $\boldsymbol{x}_{f,k}$, the $\mathcal{H}_\infty$ controller design problem for the system (4) is solvable if there exist $\boldsymbol{U}$, $\boldsymbol{N}^i$ and $\boldsymbol{P}^i\succ\boldsymbol{0}$ $(i=1,\ldots,M)$ such that the following condition is satisfied:

$$\sum_{i=1}^{M}\sum_{j=1}^{M}\sum_{l=1}^{M}h_i(\boldsymbol{s}_k)h_j(\boldsymbol{s}_k)h_l(\boldsymbol{s}_{k+1})\Upsilon_{i,j}^l\prec\boldsymbol{0}, \tag{32}$$

where

$$\Upsilon_{i,j}^l=\begin{bmatrix}-\boldsymbol{P}^i & \boldsymbol{0} & \boldsymbol{U}^T\boldsymbol{A}^{(i)T}-\boldsymbol{N}^{(j)T}\boldsymbol{B}^{(i)T} & \boldsymbol{U}^T\\ \boldsymbol{0} & -\mu^2\boldsymbol{I}_r & \boldsymbol{B}^{(i)T} & \boldsymbol{0}\\ \boldsymbol{A}^i\boldsymbol{U}-\boldsymbol{B}^i\boldsymbol{N}^j & \boldsymbol{B}^i & \boldsymbol{P}^l-\boldsymbol{U}-\boldsymbol{U}^T & \boldsymbol{0}\\ \boldsymbol{U} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}\end{bmatrix} \tag{33}$$

with $\boldsymbol{N}^j=\boldsymbol{K}^j\boldsymbol{U}$.

Proof. The problem of $\mathcal{H}_\infty$ controller design (cf. [26,27]) is to determine the gain matrix $\boldsymbol{K}(h_k)$ such that

$$\lim_{k\to\infty}\boldsymbol{x}_{f,k}=\boldsymbol{0}\quad\text{for}\quad\boldsymbol{v}_k=\boldsymbol{0}, \tag{34}$$

$$\|\boldsymbol{x}_f\|_{l_2}\le\mu\|\boldsymbol{v}_k\|_{l_2}\quad\text{for}\quad\boldsymbol{v}_k\ne\boldsymbol{0},\ \ \boldsymbol{e}_0=\boldsymbol{0}. \tag{35}$$

In order to settle the above problem it is sufficient to find a Lyapunov function V_k such that:

$$\Delta V_k+\boldsymbol{x}_{f,k}^T\boldsymbol{x}_{f,k}-\mu^2\boldsymbol{v}_k^T\boldsymbol{v}_k<0,\qquad k=0,\ldots\infty, \tag{36}$$

where $\Delta V_k=V_{k+1}-V_k$. Indeed, if $\boldsymbol{v}_k=\boldsymbol{0}$ then (36) boils down to

$$\Delta V_k+\boldsymbol{x}_{f,k}^T\boldsymbol{x}_{f,k}<0,\qquad k=0,\ldots\infty, \tag{37}$$

and hence $\Delta V_k<0$, which leads to (34). If $\boldsymbol{v}_k\ne\boldsymbol{0}$ then (36) yields

$$J=\sum_{k=0}^{\infty}\left(\Delta V_k+\boldsymbol{x}_{f,k}^T\boldsymbol{x}_{f,k}-\mu^2\boldsymbol{v}_k^T\boldsymbol{v}_k\right)<0, \tag{38}$$

which can be written as

$$J = -V_0 + \sum_{k=0}^{\infty} \boldsymbol{x}_{f,k}^T \boldsymbol{x}_{f,k} - \sum_{k=0}^{\infty} \mu^2 \boldsymbol{v}_k^T \boldsymbol{v}_k < 0, \tag{39}$$

Knowing that $V_0 = 0$ for $\boldsymbol{x}_{f,0} = \boldsymbol{0}$, (39) leads to (35).

Selecting the Lyapunov function as (cf. Remark 2)

$$V_k = \boldsymbol{x}_{f,k}^T \boldsymbol{U}^{-T} \boldsymbol{P}(h_k) \boldsymbol{U}^{-1} \boldsymbol{x}_{f,k}, \tag{40}$$

where

$$\boldsymbol{P}(h_k) = \sum_{i=1}^{M} h_i(\boldsymbol{s}_k) \boldsymbol{P}^i, \tag{41}$$

the inequality (36) is

$$\Delta V + \boldsymbol{x}_{f,k}^T \boldsymbol{x}_{f,k} - \mu^2 \boldsymbol{v}_k^T \boldsymbol{v}_k < 0, \tag{42}$$

with

$$\begin{aligned} \Delta V &= V_{k+1} - V_k = \boldsymbol{x}_{f,k}^T \\ &\cdot \left[\boldsymbol{A}_1(h_k)^T \boldsymbol{U}^{-T} \boldsymbol{P}(h_{k+1}) \boldsymbol{U}^{-1} \boldsymbol{A}_1(h_k) - \boldsymbol{U}^{-T} \boldsymbol{P}(h_k) \boldsymbol{U}^{-1}\right] \boldsymbol{x}_{f,k} \\ &+ \boldsymbol{x}_{f,k}^T \left[\boldsymbol{A}_1(h_k)^T \boldsymbol{U}^{-T} \boldsymbol{P}(h_{k+1}) \boldsymbol{U}^{-1} \boldsymbol{B}(h_k)\right] \boldsymbol{v}_k \\ &+ \boldsymbol{v}_k^T \left[\boldsymbol{B}^T(h_k) \boldsymbol{U}^{-T} \boldsymbol{P}(h_{k+1}) \boldsymbol{U}^{-1} \boldsymbol{A}_1(h_k)\right] \boldsymbol{x}_{f,k} \\ &+ \boldsymbol{v}_k^T \left[\boldsymbol{B}(h_k)^T \boldsymbol{U}^{-T} \boldsymbol{P}(h_{k+1}) \boldsymbol{U}^{-1} \boldsymbol{B}(h_k)\right] \boldsymbol{v}_k. \end{aligned} \tag{43}$$

Note that by Rayleigh quotient and Remark 2:

$$\underline{\alpha} \le \lambda_i(\boldsymbol{U}^T \boldsymbol{U}) \le \overline{\alpha}, \quad \underline{\beta} \le \lambda_i(P) \le \overline{\beta} \quad i = 1, \ldots, n,$$

where $\lambda(\cdot)$ stands for an eigenvalue of its argument. This implies that

$$\underline{\alpha}\overline{\beta} \boldsymbol{x}_{f,k}^T \boldsymbol{x}_{f,k} \le V_k \le \overline{\alpha}\underline{\beta} \boldsymbol{x}_{f,k}^T \boldsymbol{x}_{f,k},$$

which clearly indicated that V_k is a proper Lyapunov candidate matrix. Thus, it can be shown that (42) is equivalent to

$$\begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} \prec \boldsymbol{0}. \tag{44}$$

where

$$\begin{aligned} a^* &= \boldsymbol{A}_1(h_k)^T \boldsymbol{U}^{-T} \boldsymbol{P}(h_{k+1}) \boldsymbol{U}^{-1} \boldsymbol{A}_1(h_k) + \boldsymbol{I}_n \\ &\quad - \boldsymbol{U}^{-T} \boldsymbol{P}(h_k) \boldsymbol{U}^{-1}, \end{aligned}$$

$$b^* = \boldsymbol{A}_1(h_k)^T \boldsymbol{U}^{-T} \boldsymbol{P}(h_{k+1}) \boldsymbol{U}^{-1} \boldsymbol{B}(h_k),$$

$$c^* = \boldsymbol{B}(h_k)^T \boldsymbol{U}^{-T} \boldsymbol{P}(h_{k+1}) \boldsymbol{U}^{-1} \boldsymbol{A}_1(h_k),$$

$$d^* = \boldsymbol{B}(h_k)^T \boldsymbol{U}^{-T} \boldsymbol{P}(h_{k+1}) \boldsymbol{U}^{-1} \boldsymbol{B}(h_k) - \mu^2 \boldsymbol{I}_r.$$

Multiplying it from left by $\mathrm{diag}(\boldsymbol{U}^T, \boldsymbol{I}_r)$ and from right by $\mathrm{diag}(\boldsymbol{U}, \boldsymbol{I}_r)$ gives

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{U}^T \boldsymbol{A}_1(h_k)^T \boldsymbol{U}^{-T} \\ \boldsymbol{B}(h_k)^T \boldsymbol{U}^{-T} \end{bmatrix} \boldsymbol{P}(h_{k+1}) \\ &\cdot \begin{bmatrix} \boldsymbol{U}^{-1} \boldsymbol{A}_1(h_k) \boldsymbol{U} & \boldsymbol{U}^{-1} \boldsymbol{B}(h_k) \end{bmatrix} \\ &+ \begin{bmatrix} \boldsymbol{U}^T \boldsymbol{U} - \boldsymbol{P}(h_k) & \boldsymbol{0} \\ \boldsymbol{0} & -\mu^2 \boldsymbol{I}_r \end{bmatrix} \prec \boldsymbol{0}. \end{aligned} \tag{45}$$

Applying Lemma 1 to (45) yields

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{U}^T \boldsymbol{U} - \boldsymbol{P}(h_k) & \boldsymbol{0} & \boldsymbol{U}^T \boldsymbol{A}_1(h_k)^T \\ \boldsymbol{0} & -\mu^2 \boldsymbol{I}_r & \boldsymbol{B}(h_k)^T \\ \boldsymbol{A}_1(h_k) \boldsymbol{U} & \boldsymbol{B}(h_k) & \boldsymbol{P}(h_{k+1}) - \boldsymbol{U} - \boldsymbol{U}^T \end{bmatrix} \prec \boldsymbol{0} \\ &\Leftrightarrow \begin{bmatrix} \boldsymbol{U}^T \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{I}_n \begin{bmatrix} \boldsymbol{U} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \\ &+ \begin{bmatrix} -\boldsymbol{P}(h_k) & \boldsymbol{0} & \boldsymbol{U}^T \boldsymbol{A}_1(h_k)^T \\ \boldsymbol{0} & -\mu^2 \boldsymbol{I}_r & \boldsymbol{B}(h_k)^T \\ \boldsymbol{A}_1(h_k) \boldsymbol{U} & \boldsymbol{B}(h_k) & \boldsymbol{P}(h_{k+1}) - \boldsymbol{U} - \boldsymbol{U}^T \end{bmatrix} \prec \boldsymbol{0}, \end{aligned} \tag{46}$$

which by Schur complements leads to

$$\begin{bmatrix} -\boldsymbol{P}(h_k) & \boldsymbol{0} & \boldsymbol{U}^T \boldsymbol{A}_1(h_k)^T & \boldsymbol{U}^T \\ \boldsymbol{0} & -\mu^2 \boldsymbol{I}_r & \boldsymbol{B}(h_k)^T & \boldsymbol{0} \\ \boldsymbol{A}_1(h_k) \boldsymbol{U} & \boldsymbol{B}(h_k) & \boldsymbol{P}(h_{k+1}) - \boldsymbol{U} - \boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{U} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I} \end{bmatrix} \prec \boldsymbol{0}, \tag{47}$$

Finally, substituting

$$\begin{aligned} \boldsymbol{A}_1(h_k) \boldsymbol{U} &= \sum_{i=1}^{M} \sum_{j=1}^{M} h_i(\boldsymbol{s}_k) h_j(\boldsymbol{s}_k) (\boldsymbol{A}^i \boldsymbol{U} - \boldsymbol{B}^i \boldsymbol{K}^j \boldsymbol{U}) \\ &= \sum_{i=1}^{M} \sum_{j=1}^{M} h_i(\boldsymbol{s}_k) h_j(\boldsymbol{s}_k) (\boldsymbol{A}^i \boldsymbol{U} - \boldsymbol{B}^i \boldsymbol{N}^j), \end{aligned}$$

gives (32), which completes the proof.

Note that (32) requires further relaxation procedure in order to be tractable within the effective LMI framework. A basic sufficient solution to this problem were described in [28] and further improved by many researchers (see, e.g., [25, 29] and the references therein). As indicated in [29], the conditions provided by [30] lead to a good compromise between complexity and conservatism, which in the case (32) leads to the following lemma:

Lemma 2. Condition (32) is fulfilled providing the following conditions hold:

$$\Upsilon_{i,i}^l \prec \boldsymbol{0}, \qquad i \in \{1, \ldots, M\}, \tag{48}$$

$$\begin{aligned} &\frac{2}{M-1} \Upsilon_{i,i}^l + \Upsilon_{i,j}^l + \Upsilon_{j,i}^l \prec \boldsymbol{0}, \\ &i, j, l \in \{1, \ldots, M\}, \qquad i \neq j. \end{aligned} \tag{49}$$

Finally, the design procedure boils down to solving (48), (49) with respect to $\boldsymbol{U}$, $\boldsymbol{N}^j$ and $\boldsymbol{P}^i$ ($i = 1, \ldots, M, j = 1, \ldots, M$), and then calculating

$$\boldsymbol{K}^j = \boldsymbol{N}^j \boldsymbol{U}^{-1}, \qquad j = 1, \ldots, M. \tag{50}$$

The objective of this section was to provide a fault estimation and compensation scheme without taking into the account the control limit. Thus, the objective of the subsequent section is to provide a useful description of the invariant set, taking into account the input constraints, while the Sec. 5 presents an on-line optimisation strategy that can be used for enlarging this set.

4. Derivation of a robust invariant set

As it was mentioned in the previous section, in order to maintain a desired system behaviour, the idea of a robust invariant set of state variables is to be employed [31–33]. To settle this problem the Quadratic Boundedness (QB) [34] will be recalled along with its further extension called Extended Non-Quadratic Boundedness (EQNB) [35].

Let us assume that $\boldsymbol{P}(h_k) = \boldsymbol{P}$ in , $\boldsymbol{P} \succ \boldsymbol{0}$, which makes it possible to formulate the following definitions (cf. [34]):

Definition 1. The system (29) is strictly quadratically bounded with $\boldsymbol{P} \succ \boldsymbol{0}$ for all allowable $\boldsymbol{v}_k \in \mathcal{E}_v$, $k \geq 0$, if $V_k > 1$ implies $V_{k+1} - V_k < 0$ for any $\boldsymbol{v}_k \in \mathcal{E}_v$.

Definition 2. A set $\mathcal{E}_{\boldsymbol{x}_f}$ is a robust invariant set for the system (29) for all allowable $\boldsymbol{v}_k \in \mathcal{E}_v$ if $\boldsymbol{x}_{f,k} \in \mathcal{E}_{\boldsymbol{x}_f}$ implies $\boldsymbol{x}_{f,k+1} \in \mathcal{E}_{\boldsymbol{x}_f}$, for any $\boldsymbol{v}_k \in \mathcal{E}_v$.

In this section the ellipsoidal bounding will be used for describing the robust invariant set, i.e.

$$\mathcal{E}_{\boldsymbol{x}_f} = \{\boldsymbol{x}_f \in \mathbb{R}^n | \, \boldsymbol{x}_f^T \boldsymbol{P} \boldsymbol{x}_f \leq 1\}. \tag{51}$$

The proposed ellipsoidal bounding strategy can be perceived as an inner approximation of the exact invariant set [36]. An obvious drawback to the proposed approach is that the obtained set is smaller than the exact one. However, the simplicity of the ellipsoidal description will make it possible to use it for on-line optimisation, which will be described in Sec. 5.

Using the above definitions and assumptions, it is possible to recall results provided in [34] that can be directly applied to (29):

Lemma 3. The following facts are equivalent

1. for all allowable $\boldsymbol{v}_k \in \mathcal{E}_v$, $k \geq 0$, the system (29) is strictly quadratically bounded with $\boldsymbol{P} \succ \boldsymbol{0}$,
2. for all allowable $\boldsymbol{v}_k \in \mathcal{E}_v$, $k \geq 0$, the ellipsoid (51) is a robust invariant set for the system (29).

In spite of the incontestable appeal of the above results, they inherit a drawback related to the fact that $\boldsymbol{P}(h_k) = \boldsymbol{P}$. To avoid such a limitation, the notion of EQNB was introduced [35]. In the light of this framework *Definition 1* and *Definition 2* can be suitably reformulated as:

Definition 3. The system (29) is strictly non-quadratically bounded for all allowable $\boldsymbol{v}_k \in \mathcal{E}_v$, $k \geq 0$, if $V_k = \boldsymbol{x}_{f,k}^T \boldsymbol{P}(h_k) \boldsymbol{x}_{f,k} > 1$ implies $\Delta V = V_{k+1} - V_k < 0$ for any $\boldsymbol{v}_k \in \mathcal{E}_v$.

Definition 4. A set $\mathcal{E}_{\boldsymbol{x}_{f,k}}$

$$\mathcal{E}_{\boldsymbol{x}_{f,k}} = \{\boldsymbol{x}_{f,k} \in \mathbb{R}^n | \, \boldsymbol{x}_{f,k}^T \boldsymbol{P}(h_k) \boldsymbol{x}_{f,k} \leq 1\} \tag{52}$$

is a robust invariant set for the system (29) for all allowable $\boldsymbol{v}_k \in \mathcal{E}_v$ if $\boldsymbol{x}_{f,k} \in \mathcal{E}_{\boldsymbol{x}_{f,k}}$ implies $\boldsymbol{x}_{f,k+1} \in \mathcal{E}_{\boldsymbol{x}_{f,k+1}}$, for any $\boldsymbol{v}_k \in \mathcal{E}_v$.
Following the same line of reasoning Lemma 4 can be reformulated in a similar fashion:

Lemma 4. The following facts are equivalent

1. for all allowable $\boldsymbol{v}_k \in \mathcal{E}_v$, $k \geq 0$, the system (29) is strictly non-quadratically bounded,
2. for all allowable $\boldsymbol{v}_k \in \mathcal{E}_v$, $k \geq 0$, the ellipsoid (52) is a robust invariant set for the system (29).

Using *Definition 3* and the fact that $\boldsymbol{v}_k^T \boldsymbol{Q} \boldsymbol{v}_k \leq 1$ (cf. (23)), it is possible to write:

$$\begin{aligned} &\boldsymbol{v}_k^T \boldsymbol{Q} \boldsymbol{v}_k < \boldsymbol{x}_{f,k}^T \boldsymbol{P}(h_k) \boldsymbol{x}_{f,k}, \\ &\Rightarrow \boldsymbol{x}_{f,k+1}^T \boldsymbol{P}(h_{k+1}) \boldsymbol{x}_{f,k+1} - \boldsymbol{x}_{f,k} \boldsymbol{P}(h_k) \boldsymbol{x}_{f,k} < 0. \end{aligned} \tag{53}$$

which by *Definition 3* gives

$$\boldsymbol{v}_k^T \boldsymbol{Q} \boldsymbol{v}_k < \boldsymbol{x}_{f,k}^T \boldsymbol{P}(h_k) \boldsymbol{x}_{f,k}, \Rightarrow \Delta V < 0, \tag{54}$$

which can be written as

$$\begin{bmatrix} \boldsymbol{x}_{f,k} \\ \boldsymbol{v}_k \end{bmatrix}^T \begin{bmatrix} -\boldsymbol{P}(h_k) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{f,k} \\ \boldsymbol{v}_k \end{bmatrix} \prec \boldsymbol{0}, \Rightarrow \tag{55}$$

$$\begin{bmatrix} \boldsymbol{x}_{f,k} \\ \boldsymbol{v}_k \end{bmatrix}^T \begin{bmatrix} e^* \boldsymbol{A}_1(h_k) - \boldsymbol{P}(h_k) & e^* \boldsymbol{B}(h_k) \\ f^* \boldsymbol{A}_1(h_k) & f^* \boldsymbol{B}(h_k) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{f,k} \\ \boldsymbol{v}_k \end{bmatrix} \prec \boldsymbol{0}, \tag{56}$$

where

$$e^* = \boldsymbol{A}_1(h_k)^T \boldsymbol{P}(h_{k+1}),$$

$$f^* = \boldsymbol{B}(h_k)^T \boldsymbol{P}(h_{k+1}).$$

By applying the S-Lemma [37], the relations (55), (56) can be written in the form:

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{A}_1(h_k)^T \\ \boldsymbol{B}(h_k)^T \end{bmatrix} \boldsymbol{P}(h_{k+1}) \begin{bmatrix} \boldsymbol{A}_1(h_k) \boldsymbol{B}(h_k) \end{bmatrix} \\ &+ \begin{bmatrix} -(1-\alpha)\boldsymbol{P}(h_k) & \boldsymbol{0} \\ \boldsymbol{0} & -\alpha \boldsymbol{Q} \end{bmatrix} \prec \boldsymbol{0}, \end{aligned} \tag{57}$$

where $\alpha > 0$ is some scalar. Thus, by applying Schur complement it can be written as

$$\begin{bmatrix} -(1-\alpha)\boldsymbol{P}(h_k) & \boldsymbol{0} & \boldsymbol{A}_1(h_k)^T \\ \boldsymbol{0} & -\alpha \boldsymbol{Q} & \boldsymbol{B}(h_k)^T \\ \boldsymbol{A}_1(h_k) & \boldsymbol{B}(h_k) & -\boldsymbol{P}(h_{k+1})^{-1} \end{bmatrix} \prec \boldsymbol{0}, \tag{58}$$

and subsequently, multiplying it from left nad right by $\mathrm{diag}(\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{P}(h_{k+1}))$

$$\begin{bmatrix} -(1-\alpha)\boldsymbol{P}(h_k) & \boldsymbol{0} & \boldsymbol{A}_1(h_k)^T \boldsymbol{P}(h_{k+1}) \\ \boldsymbol{0} & -\alpha \boldsymbol{Q} & \boldsymbol{B}^T \boldsymbol{P}(h_{k+1}) \\ \boldsymbol{P}(h_{k+1}) \boldsymbol{A}_1(h_k) & \boldsymbol{P}(h_{k+1}) \boldsymbol{B}(h_k) & -\boldsymbol{P}(h_{k+1}) \end{bmatrix} \prec \boldsymbol{0}. \tag{59}$$

Finally, substituting

$$\begin{aligned} \boldsymbol{A}_1(h_k) &= \sum_{i=1}^{M} \sum_{j=1}^{M} h_i(\boldsymbol{s}_k) h_j(\boldsymbol{s}_k) (\boldsymbol{A}^i - \boldsymbol{B}^i \boldsymbol{K}^j) \\ &= \sum_{i=1}^{M} \sum_{j=1}^{M} h_i(\boldsymbol{s}_k) h_j(\boldsymbol{s}_k) \boldsymbol{A}_1^{i,j} \end{aligned}$$

gives

$$\sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{l=1}^{M} h_i(\boldsymbol{s}_k) h_j(\boldsymbol{s}_k) h_l(\boldsymbol{s}_{k+1}) \Psi_{i,j}^l \prec \boldsymbol{0}, \tag{60}$$

where

$$\Psi_{i,j}^{l} = \begin{bmatrix} -(1-\alpha)\boldsymbol{P}^{i} & \boldsymbol{0} & \boldsymbol{A}_1^{(i,j)T}\boldsymbol{P}^{l} \\ \boldsymbol{0} & -\alpha\boldsymbol{Q} & \boldsymbol{B}^{(i)T}\boldsymbol{P}^{l} \\ \boldsymbol{P}^{l}\boldsymbol{A}_1^{i,j} & \boldsymbol{P}^{l}\boldsymbol{B}^{i} & -\boldsymbol{P}^{l} \end{bmatrix}. \tag{61}$$

Similarly as in the previous section, this allows writing the following lemma:

Lemma 5. Condition (61) is fulfilled providing the following conditions hold:

$$\Psi_{i,i}^{l} \prec \boldsymbol{0}, \qquad i \in \{1,\ldots,M\}, \tag{62}$$

$$\begin{aligned} &\frac{2}{M-1}\Psi_{i,i}^{l} + \Psi_{i,j}^{l} + \Psi_{j,i}^{l} \prec \boldsymbol{0}, \\ &i,j,l \in \{1,\ldots,M\}, \qquad i \neq j, \end{aligned} \tag{63}$$

$$0 < \alpha < 1. \tag{64}$$

Note that for a fixed α, the design procedure boils down to solving LMIs (62), (63) with respect to $\boldsymbol{P}^{i}$ $(i = 1,\ldots,M)$.

Remark 3. By *Proposition 2* from [34], for any $\boldsymbol{Q} \succ \boldsymbol{0}$, the system (29) is exponentially stable if and only if there exist $\alpha > 0$ and $\boldsymbol{P} \succ \boldsymbol{0}$ such that (58) is satisfied.

5. Efficient predictive FTC

The robust fault-tolerant control presented in Sec. 4 is based on the idea of estimating the fault, and then compensating it with a suitable increase or decrease of the control feeding the faulty actuator. In spite of the incontestable appeal of the proposed approach, its main drawback is that it does not take into account the fact that all actuators obey some saturation rules. Thus, the idea behind the approach presented in this section is as follows: when a saturation of a faulty actuator appears then perturb (or modify) the control strategy of the remaining actuators in such a way as to increase the robust invariant set and to make the overall control problem feasible. The subsequent part of this section is devoted to the implementation of such a strategy.

Thus, the objective of the subsequent part of this section is to develop a suitable control strategy that takes into account the actuator saturation. For this purpose, the efficient predictive control scheme introduced by [16] is utilised. In particular, the proposed scheme is suitably extended to cope with the external disturbances, and hence, achieving robustness.

Thus, predictions at time k are generated as follows [16]:

$$\boldsymbol{z}_{k+1} = \boldsymbol{Z}(h_k)\boldsymbol{z}_k + \widetilde{\boldsymbol{B}}(h_k)\boldsymbol{v}_k, \tag{65}$$

where

$$\boldsymbol{Z}(h_k) = \begin{bmatrix} \boldsymbol{A}(h_k) - \boldsymbol{B}(h_k)\boldsymbol{K}(h_k) & \boldsymbol{B}(h_k)\boldsymbol{T} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix},$$

$$\widetilde{\boldsymbol{B}}(h_k) = \begin{bmatrix} \boldsymbol{B}(h_k) \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} \boldsymbol{0}_{(n_c-1)r\times r} & \boldsymbol{I} \\ \boldsymbol{0}_{r\times r} & \boldsymbol{0}_{r\times(n_c-1)r} \end{bmatrix},$$

$$\boldsymbol{z}_k = \begin{bmatrix} \boldsymbol{x}_{f,k} \\ \boldsymbol{\omega}_k \end{bmatrix}, \quad \boldsymbol{\omega}_k = \begin{bmatrix} \boldsymbol{c}_k \\ \boldsymbol{c}_{k+1} \\ \ldots \\ \boldsymbol{c}_{k+n_c-1} \end{bmatrix},$$

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{I}_{r\times r} & \boldsymbol{0} & \ldots & \boldsymbol{0} \end{bmatrix}.$$

Following [16], it can be pointed out that if there exists robust invariant set $\mathcal{E}_{\boldsymbol{x}_f}$ (cf. (52)) for (65), then there must exist at least one robust invariant set $\mathcal{E}_{\boldsymbol{z}}$ for (65). Thus, (61) can be easily adapted for (65), which gives the robust invariant set for the proposed fault-tolerant predictive scheme:

$$\begin{bmatrix} -(1-\alpha)\boldsymbol{P}^{i} & \boldsymbol{0} & \boldsymbol{Z}^{(i,j)T}\boldsymbol{P}^{l} \\ \boldsymbol{0} & -\alpha\boldsymbol{Q} & \widetilde{\boldsymbol{B}}^{(i)T}\boldsymbol{P}^{l} \\ \boldsymbol{P}^{l}\boldsymbol{Z}^{i,j} & \boldsymbol{P}^{l}\widetilde{\boldsymbol{B}}^{i} & -\boldsymbol{P}^{l} \end{bmatrix} \quad 0 < \alpha < 1. \tag{66}$$

Note, for $\overline{\boldsymbol{w}}_k = \boldsymbol{0}$, the stability of (65) is guaranteed by the stability of $\boldsymbol{A}(h_k) - \boldsymbol{B}(h_k)\boldsymbol{K}(h_k)$. In case $\overline{\boldsymbol{w}}_k \neq \boldsymbol{0}$, the stability of (65) follows from *Remark 3*, which refers to the existence of a solution of (66) with respect to $\alpha > 0$ and $\boldsymbol{P}^{i} \succ \boldsymbol{0}$.

Since the robust invariant set for (65) is given then it is possible to introduce the input constraints (69). The easiest way to do this is to suitably scale $\boldsymbol{\omega}_k$ in (65) as follows, i.e. $\boldsymbol{\omega}_k$ is replaced by:

$$\overline{\boldsymbol{\omega}}_k = \begin{bmatrix} \boldsymbol{c}_k - \widehat{\boldsymbol{f}}_{k-1} \\ \boldsymbol{c}_{k+1} - \widehat{\boldsymbol{f}}_{k-1} \\ \ldots \\ \boldsymbol{c}_{k+n_c-1} - \widehat{\boldsymbol{f}}_{k-1} \end{bmatrix}. \tag{67}$$

Let us define

$$\boldsymbol{M}(h_k) = [-\boldsymbol{K}(h_k) \quad \boldsymbol{T}] \tag{68}$$

and hence

$$\boldsymbol{u}_{f,k} = \boldsymbol{M}(h_k)\boldsymbol{z}_k. \tag{69}$$

Let $\boldsymbol{e}_i$ denote i-th column of the r-order identity matrix, which makes it possible to rewrite the input constraints as follows

$$|\boldsymbol{e}_i^T\boldsymbol{u}_{f,k}| \leq \overline{u}_i \qquad i = 1,\ldots,r. \tag{70}$$

Subsequently, it can be observed that for $\boldsymbol{z}_k \in \mathcal{E}_z$ the above inequality implies

$$\begin{aligned} |\boldsymbol{e}_i^T\boldsymbol{u}_{f,k}|^2 &= |\boldsymbol{e}_i^T\boldsymbol{M}(h_k)\boldsymbol{z}_k|^2 \\ &= \boldsymbol{e}_i^T\boldsymbol{M}(h_k)\boldsymbol{P}(h_k)^{-1/2}\boldsymbol{P}(h_k)^{1/2}\boldsymbol{z}_k|^2 \\ &\leq ||\boldsymbol{e}_i^T\boldsymbol{M}(h_k)\boldsymbol{P}(h_k)^{-1/2}||^2||\boldsymbol{P}(h_k)^{1/2}\boldsymbol{z}_k||^2 \\ &\leq ||\boldsymbol{e}_i^T\boldsymbol{M}(h_k)\boldsymbol{P}(h_k)^{-1/2}||^2 \end{aligned} \tag{71}$$

and if there exists a symmetric matrix $\boldsymbol{Y}$ such that

$$\begin{aligned}\boldsymbol{M}(h_k)\boldsymbol{P}(h_k)^{-1}\boldsymbol{M}(h_k)^T - \boldsymbol{Y} \prec \boldsymbol{0},\\ \boldsymbol{Y}_{i,i} \leq \overline{u}_i^2, \qquad i = 1, \dots, r\end{aligned} \tag{72}$$

then $|u_{i,f,k}| \leq \overline{u}_i$, $(i = 1, \dots, r)$. Using the Schur complements, inequality (72) can be written as

$$\begin{bmatrix} -\boldsymbol{Y} & \boldsymbol{M}(h_k) \\ \boldsymbol{M}(h_k)^T & -\boldsymbol{P}(h_k) \end{bmatrix} \prec \boldsymbol{0}, \qquad \boldsymbol{Y}_{i,i} \leq \overline{u}_i^2. \tag{73}$$

Thus, in order to take into account the input constraints, (66) should be accompanied with

$$\begin{gathered}\begin{bmatrix} -\boldsymbol{Y} & (\boldsymbol{M}^j) \\ (\boldsymbol{M}^j)^T & -\boldsymbol{P}^i \end{bmatrix} \prec \boldsymbol{0}, \qquad \boldsymbol{Y}_{a,a} \leq \overline{u}_a^2,\\ a = 1, \dots, r, \qquad i = 1, \dots, M, \qquad j = 1, \dots, M.\end{gathered} \tag{74}$$

If the robust invariant set along with input constraints are described in a form of LMIs, then it is possible to solve them and simultaneously maximize the invariant set. For that purpose, various criteria can be selected, e.g.:

- minimization of the determinant of $\boldsymbol{P}(h_k)$, which corresponds to the maximization of volume of the invariant set,
- minimization of the trace of $\boldsymbol{P}(h_k)$, which corresponds to the maximization of the sum of the axes of the ellipsoid describing an invariant set.

Taking into account the structure of $\boldsymbol{P}(h_k)$, which is a weighted sum of matrices, to maximize the size of the $\mathcal{E}_{\boldsymbol{x}_f}$ the following sum of traces should be minimized:

$$\begin{aligned}&\min \operatorname{trace} \left(\sum_{i=1}^{M} \left(\boldsymbol{T}\boldsymbol{P}^i\boldsymbol{T}^T \right) \right)\\ &= \min \operatorname{trace} \left(\operatorname{diag} \left(\boldsymbol{T}\boldsymbol{P}^1\boldsymbol{T}^T, \dots, \boldsymbol{T}\boldsymbol{P}^M\boldsymbol{T}^T \right) \right)\end{aligned} \tag{75}$$

with

$$\boldsymbol{x}_{f,k} = \boldsymbol{T}\boldsymbol{z}_k, \tag{76}$$

under the constraints formed with (66) and (74). The algorithm for computing $\boldsymbol{c}_k$ in (65) is also inspired by [16] and boils down to perform, at each sampling time, the following minimisation

$$\boldsymbol{\omega}_k^* = \min_{\boldsymbol{\omega}_k} \boldsymbol{\omega}_k^T\boldsymbol{\omega}_k, \qquad s.t.\ \boldsymbol{z}_k^T\boldsymbol{P}(h_k)\boldsymbol{z}_k \leq 1, \tag{77}$$

which can be equivalently written as:

$$\begin{aligned}\boldsymbol{\omega}_k^* = \min_{\boldsymbol{\omega}_k} \boldsymbol{\omega}_k^T\boldsymbol{\omega}_k, \qquad s.t.\ &\boldsymbol{x}_{f,k}^T\boldsymbol{P}_{1,1}(h_k)\boldsymbol{x}_{f,k}\\ &+ 2\boldsymbol{x}_{f,k}^T\boldsymbol{P}_{1,2}(h_k)\overline{\boldsymbol{\omega}}_k\\ &+ \overline{\boldsymbol{\omega}}_k^T\boldsymbol{P}_{2,2}(h_k)\overline{\boldsymbol{\omega}}_k \leq 1,\end{aligned} \tag{78}$$

where $\boldsymbol{P}_{1,1}(h_k)$, $\boldsymbol{P}_{1,2}(h_k)$ and $\boldsymbol{P}_{2,2}(h_k)$ are block partitions of $\boldsymbol{P}(h_k)$ conformal to the partition of $\boldsymbol{z}_k = [\boldsymbol{x}_{f,k}^T\ \overline{\boldsymbol{\omega}}_k^T]^T$. Thus, if the $\mathcal{H}_\infty$ control is feasible then $\boldsymbol{\omega} = \boldsymbol{0}$, otherwise the solution lies on the boundary of $\mathcal{E}_{\boldsymbol{z}}$ described by (78). This means that when $\boldsymbol{\omega} = \boldsymbol{0}$ is contained in $\mathcal{E}_{\boldsymbol{z}}$ described by (78), then there is no need for optimisation and the optimal solution is $\boldsymbol{\omega} = \boldsymbol{0}$. Otherwise, as indicated in [16], the above optimisation problem has a unique solution and can be very efficiently solved with, e.g., the Newton-Raphson algorithm [38, 39]. Thus, the structure of whole robust predictive fault-tolerant control can be summarized as follows:

Off-line computation:

1. for a predefined disturbance attenuation level $\mu > 0$, design a robust controller $\boldsymbol{K}(h_k)$ by solving (33),
2. determine the robust invariant set by solving (75) under the constraints (66) and (74).

On-line computation: for each k,

1. compute the fault estimate $\widehat{\boldsymbol{f}}_{k-1}$ with (14),
2. solve the optimisation problem (78),
3. implement the first element of $\boldsymbol{\omega}_k$, i.e. $\boldsymbol{c}_k$.

An outline of the proposed scheme is depicted in Fig. 1.

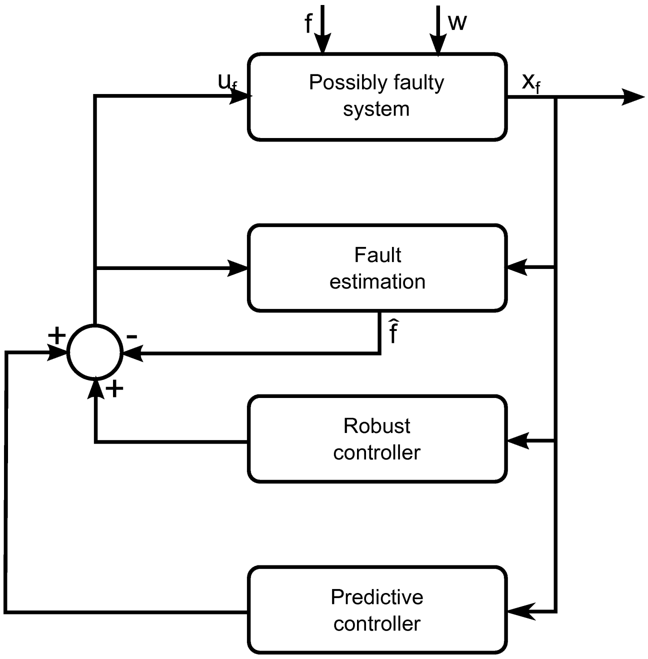

Fig. 1. Proposed robust predictive FTC scheme outline

6. Illustrative example

The selected nonlinear system is based on the Twin-Rotor MIMO System (TRMS), a laboratory set-up [40] developed by Feedback Instruments Limited. Extensive research regarding TRMS can be found in [41] and the references therein. The TRMS, as shown in Fig. 2, is driven by two DC motors. It has two perpendicular propellers joined by a beam pivoted on its base. Chassis have thus 2 degrees of freedom, allowing it to move inside imaginary static sphere. The joined beam can be moved by changing the input voltage of its motor, which controls the rotational speed of the propellers. The system is equipped with a pendulum counterweight fixed to the beam and it determines a stable equilibrium position. Additionally, the system is balanced in such a way that when the motors are switched off, the main rotor end of the beam is lowered.

The behaviour of the TRMS system resembles that of a helicopter [41]. For example, there is a strong cross-coupling between the main rotor (collective) and the tail rotor. However, the system is different from a helicopter in many ways, the main differences being the: location of the pivot point (midway between two rotors in TRMS vs. main rotor head in the helicopter), vertical control (speed control of main rotor vs. collective pitch control), yaw control (tail rotor speed vs. pitch angle of tail rotor blades) and lastly, cyclical control (none vs directional control).

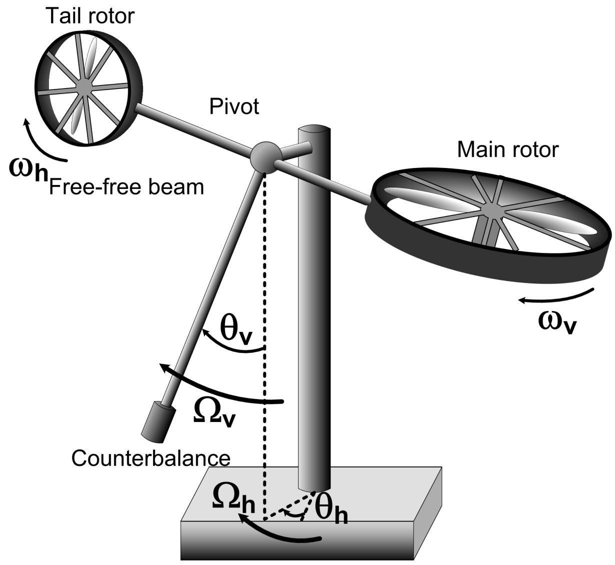

Fig. 2. Components of the twin-rotor MIMO system

The mathematical model of the TRMS can be described by a set of four nonlinear differential equations with two linear differential equations and four nonlinear functions [40]. Some of the parameters can be obtained from manual [40], whereas others should be collected as an experimental results, e.g., inertia, magnitudes of the physical propeller, coefficients of friction and impulse force. The inputs of the system are defined by the input vector $\boldsymbol{u} = [u_h, u_v]^T$, where u_h is the input voltage of the tail motor and u_v is the input voltage of the main motor. The input bounds are $\overline{u}_1 = 1$ and $\overline{u}_2 = 1$. The state vector is defined as $\boldsymbol{x} = [\Omega_h, \phi_h, \omega_h, \Omega_v, \phi_v, \omega_v]^T$, where Ω_h is the angular velocity around the vertical axis, ϕ_h is the azimuth angle of the beam, ω_h is the rotational velocity of the tail rotor, Ω_v is the angular velocity around the horizontal axis, ϕ_v is the pitch angle of the beam, ω_v is the rotational velocity of the main rotor. For the complete physical model of such a system refer to [40, 41].

A normalised TS model, which approximates the nonlinear TRMS system, is obtained by linearising a system around five operating points [2]. The system can be described in the following way:

$$\boldsymbol{x}_{f,k+1} = \sum_{i=1}^{5} h_i(\boldsymbol{s}_k)[\boldsymbol{A}^i\boldsymbol{x}_{f,k} + \boldsymbol{B}^i\boldsymbol{u}_{f,k} + \boldsymbol{B}^i\boldsymbol{f}_k + \boldsymbol{W}\boldsymbol{w}_k],. \tag{79}$$

The matrices $\boldsymbol{A}^i$, and $\boldsymbol{B}^i$, $(i = 1, \ldots, 5)$ are acquired by linearising the initial system around five points chosen in the operating range of the system considered, with the premise variable $\boldsymbol{s}_k = \phi_{h,k}$ and membership functions shown in Fig. 3. A detailed description of the model (79) can be found in [2]. Note also that, according to [2] the constant bias arising from the linearisation were removed due to their avoidably small values. Moreover, it was verified that the matrices $B^i, i = 1, \ldots, M$ satisfy the conditions of Theorem 3, which makes it possible to conduct the remaining design procedure. Five local models guarantee a relatively good approximation of the state of the real system by the TS model within the operating range.

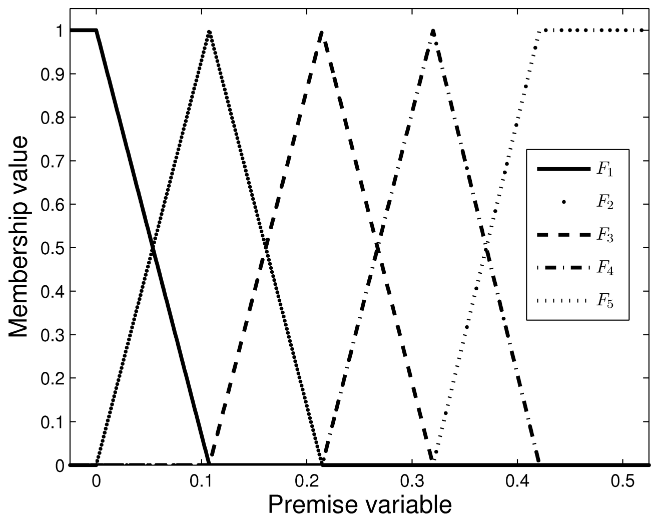

Fig. 3. Fuzzy sets used in the Takagi-Sugeno model

The robust $\mathcal{H}_\infty$ controller gain matrix $\boldsymbol{K}(h_k)$ has been obtained obtained by solving (33) with a predefined attenuation level $\mu = 0.2$ and for $\boldsymbol{W} = 0.01\boldsymbol{I}$. While, after numerous experiments, the prediction horizon was set to $n_c = 6$, which guarantees a good compromise between the complexity and quality of FTC. The actuator faults scenarios, i.e., a decrease in the performance of the main rotor ($\boldsymbol{f}_{1,k}$) and a rotor misalignment in tail electric motor ($\boldsymbol{f}_{2,k}$) are described as follows:

$$\boldsymbol{f}_{1,k} = \begin{cases} -0.2 & 50 \leq k \leq 90, \\ 0 & \text{otherwise} \end{cases}$$

$$\boldsymbol{f}_{2,k} = \begin{cases} 0.5\sin(2\pi + 1 + 0.1\pi * (k-80)) & 80 \leq k \leq 120, \\ 0 & \text{otherwise} \end{cases}$$

Figure 4 present the horizontal and vertical angular position of the beam, achieved for the proposed FTC strategy. As a result, Fig. 5 clearly shows that the faults can be estimated with a very high accuracy. The fault estimate exhibits some deviations from the nominal value due to exogenous disturbances and modelling errors being a consequence of the high nonlinearity of the system (its high cross-coupling between the main rotor and the tail rotor). Contrarily to the non-FTC scheme, the proposed strategy exhibits a small error only. Thus, the proposed control strategy deals with the faults in a satisfactory way. Figure 6 shows control signals for the chosen fault scenario. Nonlinearities in TRMS have also influence on the faulty behaviour of the system. Thus, Fig. 6 shows appropriate control that should be provided for both inputs in order to stabilize the system behaviour. This case proves efficiency of proposed control scheme for respectively large fault. Figure 4 presents stabilization of the beam position with FTC enabled while even small fault evolves into large deviation in the beam position for the non-FTC scheme. However, overshoots cannot be totally avoided, their existence have minor influence on the overall performance of the system, and hence, of the proposed scheme can be perceived as a reliable one.

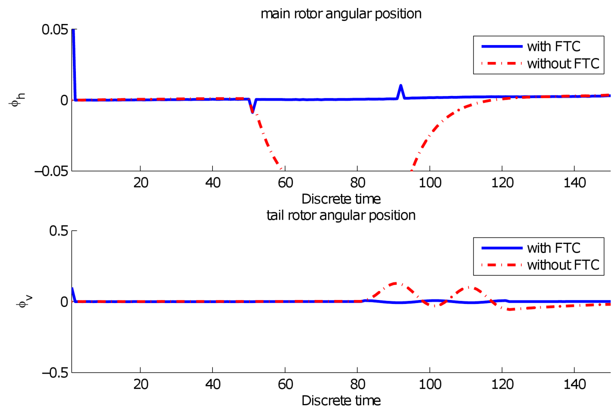

Fig. 4. Selected states of the system with with and without FTC

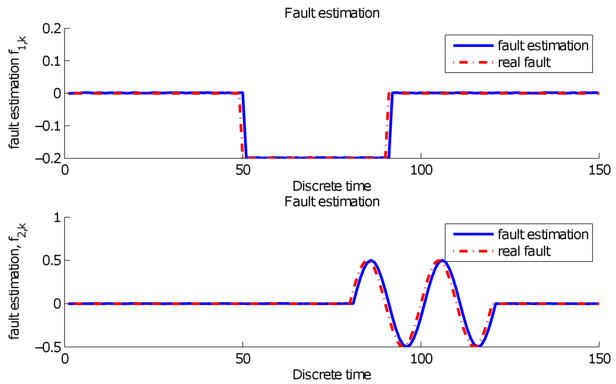

Fig. 5. Faults and their estimates

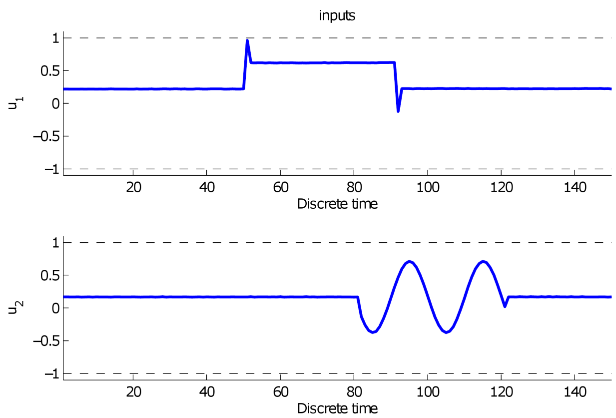

Fig. 6. Respective control inputs u_1, u_2

7. Conclusions

The main contribution of the paper is a fault-tolerant scheme for the nonlinear systems described in the Takagi-Sugeno framework. The proposed approach deals with the actuator faults and it is naturally assumed that the actuators have given performance limits. The scheme is composed of three main components: robust controller, fault estimator and the predictive controller. The robust controller is designed in such a way that a suitable disturbance attenuation level is achieved during the fault-free performance of the system. In a faulty situation the fault estimate is used for compensating the fault. However, such a compensation may lead to the faulty actuator saturation. This unappealing phenomenon is detected and maintained with the suitable predictive control action that employs the remaining actuators in order to bring the system back into the robust invariant set. Thus, this new triple-stage strategy faces a very challenging problem of fault-tolerant control for nonlinear input-constrained system. As indicates the state-of-the-art, there were no efficient solution to this problem so far. All the proposed approaches can be efficiently implemented, i.e., the off-line computations boils down to solving a number of linear matrix inequalities while the on-line computation reduces to the application of the Newton-Raphson method. The proposed approach was applied to the benchmark example of the twin-rotor system. The achieved results show the performance of the high performance of the proposed approach. In spite of the incontestable appeal of the proposed approach there are still some points, which may further improve its effectiveness. Indeed, in the proposed approach it is assumed the the state is available and, hence a natural approach is to relax this assumption by the introduction of a suitable state estimation strategy (cf. [15]). Another issue is associated with the inner-bounding of the real invariant set by an ellipsoid. It is evident that the drawback of such an approximation is the decreased size of the set while the advantage is related with the on-line computational burden. Finally, an interesting research direction is to develop an approach dedicated to the case when some of the premise variables are unmeasurable.

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