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VALUATION GRAPHS FOR PROPOSITIONAL LOGIC

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Abstract. In this paper we present the proof system, called *the valuation graphs system*, which is a new version of two proof procedures: Davis-Putnam and Stålmarck. The novelty is that in the rules we note which propositional variable occurring in some propositional formula does not determine the logical value of that formula. Due to Stålmarck, we define a notion of *proof width*, corresponding to the width of structure of valuation graph which is a number of applications of *dilemma rule*. The dilemma rule considers two cases, so the time of proof grows up exponentially.

1. Introduction

In recent years, there has been considerable renewed interest in the SATisfiability problem of propositional logic. The SAT is the question whether a propositional formula has a satisfying valuation. The SAT problem is known to be difficult to solve – it is the first known NP-complete problem, as it was proved by Stephen Cook in 1971. Because the SAT problem is fundamental to many practical problems in mathematics, computer science, and electrical engineering, efficient methods that can solve a large subset of SAT problems are eagerly sought. There are many competing algorithms for it and many implementations, most of them have been developed over the last two decades as highly optimized versions of the DPLL procedure of [3] and [4]. As a motivation, we refer to the Stålmarck patented method [7] for solving the propositional satisfiability problem in practical applications. For instance, it has been used successfully for industrial-scale problems [2]. This is the algorithm which is acceptably efficient in a large number of important cases (proof width).

Moreover, the algorithm itself is not yet widely known, and it is interesting to investigate how it performs. We have tried to do this, and the valuation graphs system has been created as a new version of Stålmarck procedure [1, 5, 6, 8].

The paper is organised as follows. Next section provides the preliminary notions. The definitions and rules of building the valuation graph are shown in section 3. In section 4 we give proofs of soundness and completness of our system. Section 5 presents the complexity of valuations graphs procedure. Section 6 completes the paper with some conclusions and future work.

2. Preliminary notions

We define the valuation graphs for propositional formulas which are built with logical connective $\rightarrow$, called *implication*, and the false symbol $\mathbf{F}$. The true symbol $\mathbf{T}$ can be defined as a formula: $\mathbf{F} \rightarrow \mathbf{F}$. Every propositional formula can be equivalently translated, in linear time, to implication form applying the Stålmarck procedure [6] and the following equivalences:

$$\begin{array}{rcl} p \vee q & \leftrightarrow & \neg\, p \rightarrow q \\ p \wedge q & \leftrightarrow & \neg\, (p \rightarrow \neg\, q) \\ \neg\, \neg\, p & \leftrightarrow & p \\ \neg\, p & \leftrightarrow & p \rightarrow \mathbf{F} \end{array}$$

Due to Stålmarck, every implication will be a triplet (c, β, γ), eventually with indices, where β and γ are subformulas and c is a new propositional variable which value is equivalent to the value of implication $\beta \rightarrow \gamma$; so $c \leftrightarrow (\beta \rightarrow \gamma)$. Each propositional formula α in implication form will be represented by a sequence $\overline{\mathbf{d}} = (\mathbf{d_1}, \ldots, \mathbf{d_n})$, where $\mathbf{d_i}$ is a triplet (c_i, β_i, γ_i) for $1 \leqslant i \leqslant n$, $\mathbf{d_n}$ is a main implication, n is the number of occurrences $\rightarrow$ in α. Then β_i and γ_i can be propositional variables $p, q, r, \ldots$, propositional constants $\mathbf{F}$ or $\mathbf{T}$ or a new triplet variable c_j. By $|\overline{\mathbf{d}}|$ we denote the number of triplets in the sequence $\overline{\mathbf{d}}$ ($|\overline{\mathbf{d}}| = n$). Our procedure inputs a propositional formula α in the following form:

$$\overline{\mathbf{d}} : \underbrace{c_1 \leftrightarrow (\beta_1 \rightarrow \gamma_1)}_{\mathbf{d_1}}, \underbrace{c_2 \leftrightarrow (\beta_2 \rightarrow \gamma_2)}_{\mathbf{d_2}}, \ldots, \underbrace{c_n \leftrightarrow (\beta_n \rightarrow \gamma_1)}_{\mathbf{d_n}}.$$

To check if the given formula is satisfiable, we construct its satisfying valuation. If constructing is failure, the propositional formula is unsatisfiable. Instead of the value of propositional variable p, we will say about substitution of propositional constant: p:=$\mathbf{F}$ or p:=$\mathbf{T}$. Similarly, by p:=q or p:=$-q$ we denote substitution of p. By "$-p$" we denote value of p contrary to its present value. For simplicity we will write "=" instead ":=". The substitution sets will be denoted by $\Sigma, \Delta, \ldots$, eventually with indices. By $\beta_{\surd}$ we denote a substitution of one of the constants $\{\mathbf{T},\ \mathbf{F}\}$ in place of β. This means that we do not need a value of β to determine a value of whole propositional formula. If in Σ we have contradiction of one of the form: $x{=}-x$; or $x{=}\mathbf{T}$ and $x{=}\mathbf{F}$; or $x{=}y$ and $x{=}-y$ for some propositional variables x and y, then we denote this contradiction by $\perp_x$ and we say that Σ is contradictory (we replace Σ by $\perp$).

3. Rules and definitions

First, in this section we present the rules of substitution of propositional values (constant) according to the truth table of $\rightarrow$: the reduction rules (RR) and the dilemma rule (RD). Next we define a valuation graph.

In general, the reduction rule has the following form:

$$\frac{\mathbf{d}\ [\ \Sigma\ ]}{[\ \Sigma'\ ]},$$

where **d** is a some triplet in a sequence of triplets representing a propositional formula α which tautology/satisfiability we check, and Σ' comes from Σ by adding the conclusions of that rule.

The special cases of **THE REDUCTION RULES**:

$$\frac{\mathbf{F} \leftrightarrow (\beta \rightarrow \gamma)}{\beta = \mathbf{T}, \gamma = \mathbf{F}}\ (RR1) \qquad \frac{c \leftrightarrow (\beta \rightarrow \mathbf{T})}{c = \mathbf{T}, \beta_{\surd}}\ (RR2)$$

$$\frac{c \leftrightarrow (\mathbf{F} \rightarrow \gamma)}{c = \mathbf{T}, \gamma_{\surd}}\ (RR3) \qquad \frac{c \leftrightarrow (\mathbf{T} \rightarrow \gamma)}{c = \gamma}\ (RR4)$$

$$\frac{c \leftrightarrow (\beta \rightarrow \mathbf{F})}{c = -\beta}\ (RR5) \qquad \frac{c \leftrightarrow (c \rightarrow \gamma)}{c = \mathbf{T}, \gamma = \mathbf{T}}\ (RR6)$$

$$\frac{c \leftrightarrow (\beta \rightarrow \beta)}{c = \mathbf{T}, \beta_{\surd}}\ (RR7)$$

In each reduction rule we have given some triplet representing subformula of propositional formula α, and a set of substitutions Σ. In particular, subformulas β and γ can be logical constants. Conclusions (substitutions) of each reduction rule are added to the set Σ, and the sequence of triplets representing α is reduced by removing the given triplet. Moreover, the RR2, RR3 and RR7 rules say that the value of implication does not depend on its subformulas. The logical value of implication, so the value of a triplet variable representing this implication too, can be sometimes deduced from the partial valuation with logical constants occurring in implication. Thus, in all of those rules the logical constant $\mathbf{T}$ substitutes a triplet variable.

As can be seen above, the reduction rules are not limited to deduce conclusions of the form: β *is* $\mathbf{T}$ and γ *is* $\mathbf{F}$ (see RR1), but includes also conclusions of the form: *c has the same value as* γ (see RR4) or *c and* β *have different values* (see RR5).

THE DILEMMA RULE:

$$\frac{\bar{\mathbf{d}}[\Sigma]}{\bar{\mathbf{d}}[\Sigma \cup \{\mathbf{x} = \mathbf{T}\}] \quad \Big| \quad \bar{\mathbf{d}}[\Sigma \cup \{\mathbf{x} = \mathbf{F}\}]} \quad (RD)$$

where x occurs in a sequence $\bar{\mathbf{d}}$ and for x there does not exist substitution in Σ.

When we cannot apply any reduction rule in a sequence, we have to use the dilemma rule. Then we obtain two sets of substitutions which arise from the set Σ: the first one by adding $x{=}\mathbf{T}$ and the second by adding $x{=}\mathbf{F}$. The dilemma rule is used to the variable from a sequence of a triplets if there does not exist a substitution in Σ for this variable. So, the sequence will not be reduced. Only one of substitutions of variable x is true, so we have the dilemma which set of substitutions is searched by us. Now, a merger of both sets of substitutions is necessary (see definition 1).

By $\bar{\mathbf{d}}[\Sigma]$ we denote the lable of a vertex of a valuation graph which is defined by induction on the length of propositional formula.

Definition 1. Let $\bar{\mathbf{d}}$ be a finite sequence of triplets $(d_1, d_2, \ldots, d_n)$ representing propositional formula α and $\Sigma = \emptyset$.

1. The single vertex labeled by $\bar{\mathbf{d}}[\Sigma \cup \{c_n {=} \mathbf{F}\}]$ is the *valuation graph* for α.

2. If $\mathcal{G}$ is a valuation graph for α, $\bar{\mathbf{d}}[\Sigma]$ is a label of a leaf, and $\mathcal{G}^*$ arises from $\mathcal{G}$ by adding a new vertex (and an edge from it to the leaf $\bar{\mathbf{d}}[\Sigma]$) labeled by $\bar{\mathbf{d}}'[\Sigma']$, which is deduced by applying one of the reduction rules, then $\mathcal{G}^*$ is the *valuation graph* for α.

3. If $\mathcal{G}$ is the valuation graph for α, $\bar{\mathbf{d}}[\Sigma]$ is a label of a leaf, and $\mathcal{G}^*$ arises from $\mathcal{G}$ by adding two new vertices (and two edges from them to the leaf $\bar{\mathbf{d}}[\Sigma]$, respectively) labeled by $\bar{\mathbf{d}}[\Sigma \cup \{\beta = \mathbf{T}\}]$ and $\bar{\mathbf{d}}[\Sigma \cup \{\beta = \mathbf{F}\}]$, which are deduced by applying dilemma rule, then $\mathcal{G}^*$ is *the valuation graph* for α.

4. If $\mathcal{G}$ is the valuation graph for α and $\bar{\mathbf{d}}_{\mathbf{1}}[\Delta_1]$ and $\bar{\mathbf{d}}_{\mathbf{2}}[\Delta_2]$ are labels of leaves in $\mathcal{G}$ obtained from vertices labeled by $\bar{\mathbf{d}}[\Sigma \cup \{\beta = \mathbf{T}\}]$ and $\bar{\mathbf{d}}[\Sigma \cup \{\beta = \mathbf{F}\}]$, respectively; $\bar{\mathbf{d}}_{\mathbf{1}}$ and $\bar{\mathbf{d}}_{\mathbf{2}}$ are empty sequences or in the set of substitutions there exists a contradictory, then $\mathcal{G}^*$ is the *valuation graph* for α obtained from $\mathcal{G}$ by adding a new vertex (and edge from it to those leaves) labeled by $\bar{\mathbf{d}}[\Delta]$, where

$$\Delta = \begin{cases} \bot & \text{when } \Delta_1 \text{ and } \Delta_2 \text{ are contradictory} \\ \Delta_1 & \text{when } \Delta_2 \text{ is contradictory and} \\ & \Delta_1 \text{ is not contradictory} \\ \Delta_2 & \text{when } \Delta_1 \text{ is contradictory and} \\ & \Delta_2 \text{ is not contradictory} \\ (\Delta_1 \cap \Delta_2) \cup \{\alpha = \gamma\} & \text{if } \Delta_1 \text{ and } \Delta_2 \text{ are not contradictory and} \\ & \{\alpha = \gamma\} \text{ occurs on one of the paths} \\ & \text{and } \alpha_{\surd} \text{ occurs on the second,} \\ & \text{simultaneously} \\ (\Delta_1 \cap \Delta_2) \cup \{\beta_{\surd}\} & \text{when } \Delta_1 \text{ and } \Delta_2 \text{ are not contradictory} \\ & \text{and } \Delta_1 \cap \Delta_2 = \Sigma \\ \Delta_1 \cap \Delta_2 & \text{otherwise, in particular,} \\ & \text{when } \Delta_1 \text{ and } \Delta_2 \text{ are not contradictory} \end{cases}$$

The set of substitutions Δ of path Θ of valuation graph $\mathcal{G}$ is a conjunction of substitutions, and the set of triplets $\bar{\mathbf{d}}$ is a conjunction of triplets.

Definition 2. The path Θ of valuation graph $\mathcal{G}$ is *closed* when the set of substitutions Δ of this path is contradictory.

Definition 3. The path Θ of valuation graph $\mathcal{G}$ is *maximal* when the set of triplets of this path is empty and the set of substitutions includes substitutions of all the propositional variables.

Definition 4. The valuation graph $\mathcal{G}$ is *closed* when all its paths are closed.

Definition 5. The valuation graph $\mathcal{G}$ is *maximal* when it is not closed.

In other words, if the maximal path exists in a valuation graph, then the valuation graph is maximal.

Definition 6. *Proof* of propositional formula α is the closed valuation graph for α.

To check if α is a tautology, we try to construct a model for $\neg\alpha$, i.e. we build a valuation v such that $v(\neg\alpha) = \mathbf{1}$. The valuation will be meant as a set of substitutions in which every propositional variable occurring in formula α has a substitution. Building the valuation, we start from substitution $\mathbf{F}$ into triplet variable representing the whole formula. Every next substitution is a conclusion of the reduction rules or the dilemma rule. If the set of substitutions includes contradictory, then α is a tautology, otherwise v is a model for $\neg\alpha$.

Example 1. The formula $(p \xrightarrow{b_1} p)$ in implication form is represented by the triplet (b_1, p, p). Building a valuation graph (Figure 1), we start from single vertex (I) labeled by a sequence of triplets, and the set of substitutions contains only the substitution $\mathbf{F}$ into b_1. The triplet and the substitution $b_1 = \mathbf{F}$ are premises of the reduction rule RR1. Thus, the conclusions of those rules are added to the set of substitutions, and the triplet (b_1, p, p) is removed from a sequence of triplets (II). Now we have empty sequence of triplets and contradictory $\bot_p$ in the set of substitution ($p = \mathbf{F}$ and $p = \mathbf{T}$). So, the set of substitutions is contradictory. Therefore, we have built the closed valuation graph for $p \to p$, so the formula $p \to p$ has the proof in the valuation graphs system.

$$\text{I} \quad ((b_1, p, p))[b_1 = \mathbf{F}]$$

$$\Big| \; (RR1_{b_1})$$

$$\text{II} \quad (\emptyset)[b_1 = \mathbf{F}, p = \mathbf{T}, p = \mathbf{F}]$$

$$\text{III} \quad (\emptyset)[b_1 = \mathbf{F}, \bot_p]$$

$$\text{IV} \quad (\emptyset)[\bot]$$

Figure 1. The closed valuation graph.

Example 2. For the propositional formula:

$$(p \xrightarrow{b_4} p) \xrightarrow{b_5} (((p \xrightarrow{b_1} q) \xrightarrow{b_2} p) \xrightarrow{b_3} q)$$

we get the following sequence of triplets:

$$\big((b_1, p, q), (b_2, b_1, p), (b_3, b_2, q), (b_4, p, p), (b_5, b_4, b_3)\big).$$

Now we are constructing the valuation graph:

$$\text{Let}\ \ \Sigma = \emptyset, \Delta_1 = \emptyset, \Delta_2 = \emptyset$$

$$((b_1, p, q), (b_2, b_1, p), (b_3, b_2, q), (b_4, p, p), (b_5, b_4, b_3))\ [\ \Sigma \cup \{b_5 = \mathbf{F}\}]$$

$$\mid\ (RR1_{b_5})$$

$$((b_1, p, q), (b_2, b_1, p), (b_3, b_2, q), (b_4, p, p))\ [\ \Sigma \cup \{b_4 = \mathbf{T}, b_3 = \mathbf{F}\}]$$

$$\mid\ (RR1_{b_3})$$

$$(b_1, p, q), (b_2, b_1, p), (b_4, p, p))\ [\ \Sigma \cup \{b_2 = \mathbf{T}, q = \mathbf{F}\}]$$

$$\mid\ (RR7_{b_4})$$

$$((b_1, p, q), (b_2, b_1, p))\ [\ \Sigma \cup \{b_4 = \mathbf{T}, p_{\surd}\}]$$

$$\mid\ (RR5_{b_1})$$

$$((b_2, b_1, p))[\ \Sigma \cup \{b_1 = -p\}]$$

$$(RD_p)$$

$$((b_2, b_1, p))\ [\ \Delta_1 \cup \Sigma \cup \{p = \mathbf{T}\}] \qquad ((b_2, b_1, p))\ [\ \Delta_2 \cup \Sigma \cup \{p = \mathbf{F}\}]$$

$$\mid (RR2_{b_2}) \qquad \mid (RR5_{b_2})$$

$$(\emptyset)[\ \Delta_1 \cup \{b_2 = \mathbf{T}, b_{1\surd}\}] \qquad (\emptyset)[\ \Delta_2 \cup \{b_2 = -b_1\}]$$

$$(\emptyset)[\ \Delta_2 \cup \perp_{b_2}]$$

$$(\emptyset)[\perp]$$

$$((b_2, b_1, p))[\ \Sigma \cup \Delta_1]$$

$$\mid (RR5_{b_2})$$

$$(\emptyset)\ [\ \Sigma \cup \{b_2 = \mathbf{T}, b_{1\surd}]$$

Figure 2. The maximal valuation graph.

4. Soundness and completness of valuation graphs system

Theorem 1 (Soundness). *If there exists a proof for α in the valuation graphs system, then α is a tautology.*

Proof: From the assumption that a proof for α exists in the valuation graphs system it follows that α has a closed valuation graph. Because the valuation graph representing all valuations is closed, every path in this graph is closed. Thus, contradictory occurs in the every path. So α is a tautology.

Theorem 2 (Completness). *If α is a tautology, then a proof of valuation graphs system exists for α.*

Proof: We assume that α is a tautology and there does not exist the proof for α in the valuation graphs system. This means that any valuation graph is not closed. Thus, in each valuation graph for α the contradictory does not occur at least in one path. Let $\mathcal{G}$ be one of valuation graphs for α. Because $\mathcal{G}$ has a path, which does not include a contradictory, hence the set of substitutions in label of leaf of this path contains the substitutions for all variables occurring in the triplet form of this formula. In particular, there are substitutions for propositional variables, subformulas and the whole formula α (because we have started building of valuation graph from $b_n = \mathbf{F}$ added to $\mathcal{G}$). Thus, a valuation v, constructed above, is a model for $\neg\alpha$. It is a contradictory to the assumption that α is a tautology.

The valuation graphs system with the reduction rules and the dilemma rule is sound and complete for propositional formulas built of propositional variable, implication and logical constant. Each propositional formula can be translated to implication form by applying the Stålmarck procedure (see [6]) in linear time. So the valuation graphs system is sound and complete for classical propositional logic.

Corollary 1. *A propositional formula α is satisfiable iff a valuation graphs system $\mathcal{G}$ for $\neg\alpha$ is maximal.*

Proof: A propositional formula α is satisfiable iff (by definition) there exists a Boolean valuation v such that α is true iff (by definition) a valuation graph for $\neg\alpha$ is maximal.

5. Complexity of procedure of valuation graphs system

The valuation graphs system allows searching for proofs and models for large class of formulas in linear or polynomial time with respect to the length of formula and width of its (maximal or closed) valuation graph. This estimation follows from analysis of branching valuation graphs of those formulas and depends exponentially on the width of valuation graph but not on the length of formula.

By *representation of structure* of valuation graph we mean its substructure consisted of vertices which labels are premises of the dilemma rule, its direct consequences and vertices of their merger (see item 4 of definition of valuation graph). Notice that for some valuation graph its representation of structure is unique. Consider all the possible substructures of representation of structure for some valuation graph. We associate each substructures with the number of leaves.

Definition 7. A *width* of valuation graph is the maximal number of leaves by all substructures of its representation of structure.

Definition 8. A formula α is *i-hard* when there exists a maximal or closed valuation graph for α with the width equal to $i+1$.

By definition 8, a formula is 1-hard when in its valuation graph we apply the dilemma rule once (by definition 7, the width of this valuation graph is equal to 2).

The time of building of valuation graph is growing up when a valuation graph branches. Due to Stålmarck, we present the recursively function of complexity:

$$\begin{aligned} g(0,n) &= 2 \cdot n, \\ g(k,n) &= \sum_{i=1}^{n} 2 \cdot i \cdot g(k-1,n), \end{aligned}$$

where: n is a length of formula,
k is a width of valuation graph.

The function $g(k,n)$ is at most n^{2k+1}, so the complexity of the presented procedure is $O(n^{2k+1})$.

6. Conclusions and future work

In the valuation graphs system the time of finding a satisfying valuation depends exponentially on the width of valuation graph but not on a length of formula. A width of valuation graph depends on the number of applications of dilemma rule. Hence, the best place for optimisation is a place where we must choose a propositional variable (a triplet variable representing a propositional formula) as a premise of the dilemma rule.

The valuation graphs system was implemented and the prototype version is tested. We are working out at experimental results which will be presented soon.

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