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(F, ρ)-invexity of higher order for multiobjective fractional variational problem*

by

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Abstract: Multiobjective fractional variational problem is considered and sufficient optimality conditions, characterizing efficiency of higher order, are obtained under the assumptions of $(F, \rho)-$invexity of higher order on the functionals involved. Parametric higher order dual of the above stated problem is proposed. Duality theorems are proved to relate efficient solutions of higher order for primal and its dual problem using generalized class of functionals.

Keywords: multiobjective, fractional, variational problem, efficiency of higher order, invexity, optimality, higher order duality

1. Introduction

The fractional variational programming problem is a problem of finding a piecewise smooth vector function in a way to optimize the ratio of two functionals, subject to differential inequality and boundary conditions (see Mishra and Mukherjee, 1994; Mititelu and Stancu-Minasian, 2009; Patel, 2005; Stancu -Minasian, 1997; Stancu-Minasian and Mititelu, 2008, 2011, 2012; Stancu-Minasian and Tigan, 2000). Due to their ratio structure, these problems have applications in various fields, like economics, information theory, engineering, heat exchange networking, and numerical analysis. In practical situations, fractional variational problems with more than a single objective function occur, so it is important to study the multiobjective fractional variational problem, which is stated below:

$$\text{(P) Minimize} \left(\frac{\int_a^b f^1(t, x(t), \dot{x}(t))dt}{\int_a^b k^1(t, x(t), \dot{x}(t))dt}, \frac{\int_a^b f^2(t, x(t), \dot{x}(t))dt}{\int_a^b k^2(t, x(t), \dot{x}(t))dt}, \ldots, \frac{\int_a^b f^r(t, x(t), \dot{x}(t))dt}{\int_a^b k^r(t, x(t), \dot{x}(t))dt} \right)$$

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subject to

$$g^i(t, x(t), \dot{x}(t)) \leq 0,\ t \in I,\ i = 1, 2, \ldots, p \tag{1}$$

$$x(a) = \alpha,\ x(b) = \beta,\ x \in X \tag{2}$$

where

(i) $I = [a, b], X$ is the space of piecewise smooth state functions $x : I \to \mathbb{R}^n$ with the derivative $\dot{x}$, equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differential operator D is given by $u = Dx \Leftrightarrow x(t) = x(a) + \int_a^t u(s)ds$. Therefore, $D = \frac{d}{dt}$ except at discontinuities;

(ii) $f^i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $k^i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, r$ and $g^i : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, p$ are continuously differentiable functions with respect to each of their arguments;

(iii) $\int_a^b f^i(t, x(t), \dot{x}(t))dt \geq 0$ and $\int_a^b k^i(t, x(t), \dot{x}(t))dt > 0$ for all $i \in \{1, 2, \ldots, r\}$ and for all $x \in X$.

The concept of optimal solution does not fit in multiobjective(vector) optimization problems as it becomes very difficult to find a single solution, which optimizes simultaneously each of the objectives. Considerable importance is given to the situation, where the solution for the vector optimization problem is described in terms of weak efficiency/efficiency/proper efficiency. Recently, much attention has been given to other types of solution concepts, one of them being higher order minimizer or minimizer of order m. This concept was introduced by Auslender (see Auslender, 1984) and Ward (see Ward, 1994). Jiménez (see Jiménez, 2002) extended the idea of Ward to define the notion of strict local efficient solution of order m for the vector optimization problem. Bhatia (see Bhatia, 2008) extended this idea further and defined the global strict minimizer of order m for the multiobjective optimization problem. Kumar and Sharma (2016, 2017) studied this solution concept for variational problem (fractional and non fractional case). These solutions are more stable than efficient solutions. The stablity is understood in the sense that under small perturbations of problem data these solutions preserve their type.

Using efficiency of higher order as the optimality criteria, sufficient optimality conditions and higher order duality results for a multiobjective fractional variational problem under the assumption of generalized invexity are established in this paper. The notion of efficiency of higher order used in this paper leads to stronger results, whereas generalized invexity broadens the class of functionals involved.

The paper is organized as follows: In Section 2 some basic definitions and preliminaries are given. Section 3 deals with the necessary optimality conditions for multiobjective fractional variational problem (P) using the concept of efficiency of higher order. Section 4 is devoted to sufficient optimality conditions for the problem undertaken. We propose higher order dual for (P), for which duality results are obtained under generalized invexity assumptions in Section 5.

2. Solution concepts

For any $x = (x^1, x^2, \ldots, x^n)^T$, $y = (y^1, y^2, \ldots, y^n)^T \in \mathbb{R}^n$.

(i) $x = y \Leftrightarrow x^i = y^i$ for all $i = 1, 2, \ldots, n$.

(ii) $x < y \Leftrightarrow x^i < y^i$ for all $i = 1, 2, \ldots, n$.

(iii) $x \leqq y \Leftrightarrow x^i \leqq y^i$ for all $i = 1, 2, \ldots, n$.

(iv) $x \leq y \Leftrightarrow x \leqq y$ and $x \neq y$.

$\mathbb{R}^n_+ = \{(x^1, x^2, \ldots, x^n)^T \in \mathbb{R}^n | \, x^i \geqq 0, \, i = 1, 2, \ldots, n\}$ and $\operatorname{int}\mathbb{R}^n_+$ denotes the interior of $\mathbb{R}^n_+$, that is, $\operatorname{int}\mathbb{R}^n_+ = \{(x^1, x^2, \ldots, x^n)^T \in \mathbb{R}^n | \, x^i > 0, \, i = 1, 2, \ldots, n\}$.

For notational convenience $x(t)$ will be written as x. Let X_0 be the set of all feasible solutions of (P), that is, $X_0 = \{x \in X | g^i(t, x, \dot{x}) \leq 0, \; t \in I, \; i = 1, \ldots, r, \; x(a) = \alpha, x(b) = \beta\}$.

Definition 1 *A point $\bar{x} \in X_0$ is said to be an efficient solution for (P) if there is no other $x \in X_0$ such that*

$$\frac{\int_a^b f^i(t, x, \dot{x})dt}{\int_a^b k^i(t, x, \dot{x})dt} \leq \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt},$$

for all $i \in \{1, 2, \ldots, r\}$ with strict inequality for at least one i.

Definition 2 *A point $\bar{x} \in X_0$ is said to be a weak efficient solution for (P) if there is no other $x \in X_0$ such that*

$$\frac{\int_a^b f^i(t, x, \dot{x})dt}{\int_a^b k^i(t, x, \dot{x})dt} < \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt}, \text{ for all } i \in \{1, 2, \ldots, r\}.$$

Let $m \geq 1$ be an integer and $\theta : I \times X \times X \to \mathbb{R}^n$ be a piecewise smooth function.

Definition 3 *A point $\bar{x} \in X_0$ is said to be an efficient solution of order m with respect to θ for (P) if there exist $c = (c^1, c^2, \ldots, c^r) \in int\mathbb{R}^r_+$ and $d = (d^1, d^2, \ldots, d^r) \in int\mathbb{R}^r_+$ such that for no other $x \in X_0$*

$$\frac{\int_a^b \{f^i(t, x, \dot{x}) - c^i \|\theta(t, x, \bar{x})\|^m\}dt}{\int_a^b \{k^i(t, x, \dot{x}) + d^i \|\theta(t, x, \bar{x})\|^m\}dt} \leq \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt},$$

for all $i \in \{1, 2, \ldots, r\}$ with strict inequality for at least one i.

Definition 4 *A point $\bar{x} \in X_0$ is said to be a weak efficient solution of order m with respect to θ for (P) if there exist $c = (c^1, c^2, \ldots, c^r) \in int\mathbb{R}^r_+$ and $d = (d^1, d^2, \ldots, d^r) \in int\mathbb{R}^r_+$ such that for no other $x \in X_0$*

$$\frac{\int_a^b \{f^i(t, x, \dot{x}) - c^i \|\theta(t, x, \bar{x})\|^m\}dt}{\int_a^b \{k^i(t, x, \dot{x}) + d^i \|\theta(t, x, \bar{x})\|^m\}dt} < \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt}, \text{ for all } i \in \{1, 2, \ldots, r\}.$$

Remark 1 *Efficient solution of higher order and weak efficient solution of higher order are more stable than efficient and weak efficient solutions. By stability, we mean that they still remain the same type of solutions under small perturbations of the problem data. For details a reader is referred to Ginchev et al. (2005).*

The main objective of this paper is to study these solution concepts through optimality conditions and duality results. Let us denote the partial derivative of f^i, $i = 1, 2, \ldots, r$ with respect to t, x and $\dot{x}$ by f^i_t, f^i_x, $f^i_{\dot{x}}$, respectively. Analogously, we write the partial derivative of k^i, $i = 1, 2, \ldots, r$ and g^i, $i = 1, 2, \ldots, p$.

3. Necessary optimality conditions

Necessary optimality conditions are important, because these conditions lay down the foundation for many computational techniques in optimization problems, as they indicate when a feasible point is not an efficient solution of higher order. At the same time, these conditions are useful in the development of numerical algorithms for solving certain optimization problems. Further, these conditions are also the basis for the development of duality theory, on which there exists an extensive literature and a substantial use of which has been made in theoretical as well as computational applications in many diverse fields.

Consider the following parametric multiobjective variational problem (P_v)

$$\text{Minimize} \Big(\int_a^b \{f^1(t,x,\dot{x}) - v^1 k^1(t,x,\dot{x})\}dt, \ldots, \int_a^b \{f^r(t,x,\dot{x}) - v^r k^r(t,x,\dot{x})\}dt \Big)$$

subject to

$$g(t,x,\dot{x}) = (g^1(t,x,\dot{x}), g^2(t,x,\dot{x}), \ldots, g^p(t,x,\dot{x})) \leqq 0, \;\; t \in I$$

$$x(a) = \alpha, \;\; x(b) = \beta$$

$$v = (v^1, v^2, \ldots, v^r) \in \mathbb{R}^r_+.$$

Lemma 1 *If $\bar{x}$ is an efficient solution of order m with respect to θ for (P) then there exists $\bar{v} = (\bar{v}^1, \bar{v}^2, \ldots, \bar{v}^r) \in \mathbb{R}^r_+$ such that $\bar{x}$ is an efficient solution of order m with respect to θ for $(P_{\bar{v}})$.*

proof Let $\bar{x}$ be an efficient solution of order m with respect to θ for (P).
Take $\bar{v}^i = \frac{\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt}{\int_a^b k^i(t,\bar{x},\dot{\bar{x}})dt}$, $i = 1, 2, \ldots, r$.
If possible, suppose that $\bar{x}$ is not an efficient solution of order m with respect to θ for $(P_{\bar{v}})$. Then, for any $\rho = (\rho^1, \rho^2, \ldots, \rho^r) \in \operatorname{int} \mathbb{R}^r_+$, there exists $\hat{x} \in X_0$ such that

$$\int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) - \bar{v}^i k^i(t,\hat{x},\dot{\hat{x}})\}dt \leq \int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) - \bar{v}^i k^i(t,\bar{x},\dot{\bar{x}}) + \rho^i \|\theta(t,\hat{x},\bar{x})\|^m\}dt,$$

for all $i \in \{1, 2, ..., r\}$ with strict inequality for at least one i.

That is,

$$\int_a^b \{f^i(t, \hat{x}, \dot{\hat{x}}) - \bar{v}^i k^i(t, \hat{x}, \dot{\hat{x}})\}dt \leq \rho^i \int_a^b \|\theta(t, \hat{x}, \bar{x})\|^m dt,$$

for all $i \in \{1, 2, ..., r\}$ with strict inequality for at least one i.

This means that

$$\frac{\int_a^b f^i(t, \hat{x}, \dot{\hat{x}})dt}{\int_a^b k^i(t, \hat{x}, \dot{\hat{x}})dt} \leq \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt} + \frac{\rho^i \int_a^b \|\theta(t, \hat{x}, \bar{x})\|^m dt}{\int_a^b k^i(t, \hat{x}, \dot{\hat{x}})dt}, \tag{3}$$

for all $i \in \{1, 2, ..., r\}$, with strict inequality for at least one i.

Case (i)

If $\|\theta(t, \hat{x}, \bar{x})\|^m = 0$, then (3) becomes

$$\frac{\int_a^b f^i(t, \hat{x}, \dot{\hat{x}})dt}{\int_a^b k^i(t, \hat{x}, \dot{\hat{x}})dt} \leq \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt},$$

for all $i \in \{1, 2, ..., r\}$ with strict inequality for at least one i.

Then, for any $c = (c^1, c^2, \dots, c^r) \in int\mathbb{R}^r_+$ and $d = (d^1, d^2, \dots, d^r) \in int\mathbb{R}^r_+$, there exists $\hat{x} \in X_0$ such that

$$\frac{\int_a^b \{f^i(t, \hat{x}, \dot{\hat{x}}) - c^i\|\theta(t, \hat{x}, \bar{x})\|^m\}dt}{\int_a^b \{k^i(t, \hat{x}, \dot{\hat{x}}) + d^i\|\theta(t, \hat{x}, \bar{x})\|^m\}dt} \leq \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt},$$

for all $i \in \{1, 2, \dots, r\}$ with strict inequality for at least one i, which contradicts the fact that $\bar{x}$ is an efficient solution of order m with respect to θ for (P).

Case (ii)

If $\|\theta(t, \hat{x}, \bar{x})\|^m \neq 0$, then we apply the following reasoning.
For any $c = (c^1, c^2, \dots, c^r) \in \operatorname{int}\mathbb{R}^r_+$ and $d = (d^1, d^2, \dots, d^r) \in \operatorname{int}\mathbb{R}^r_+$, define

$$\rho^i = \frac{c^i \int_a^b k^i(t, \hat{x}, \dot{\hat{x}})dt + d^i \int_a^b f^i(t, \hat{x}, \dot{\hat{x}})dt}{\int_a^b k^i(t, \hat{x}, \dot{\hat{x}})dt + d^i \int_a^b \|\theta(t, \hat{x}, \bar{x})\|^m dt}, i = 1, 2, \dots, r.$$

Substituting this in (3) yields

$$\frac{\int_a^b \{f^i(t, \hat{x}, \dot{\hat{x}}) - c^i\|\theta(t, \hat{x}, \bar{x})\|^m\}dt}{\int_a^b \{k^i(t, \hat{x}, \dot{\hat{x}}) + d^i\|\theta(t, \hat{x}, \bar{x})\|^m\}dt} \leq \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt},$$

for all $i \in \{1, 2, \dots, r\}$ with strict inequality for at least one i.

We arrive at a contradiction to the fact that $\bar{x}$ is an efficient solution of order m with respect to θ for (P). Hence, the result follows. □

Theorem 1 *(Necessary optimality conditions) Let $\bar{x}$ be an efficient solution of order m with respect to θ for (P). Then, there exist $\bar{\lambda} = (\bar{\lambda}^1, \dots, \bar{\lambda}^r) \in \mathbb{R}^r, \bar{v} = (\bar{v}^1, \bar{v}^2, \dots, \bar{v}^r) \in \mathbb{R}^r_+$ and a piecewise smooth function $\bar{y} : I \to \mathbb{R}^p$ such that*

$$\sum_{i=1}^{r} \bar{\lambda}^i (f^i_{\bar{x}}(t) - \bar{v}^i k^i_{\bar{x}}(t)) + \bar{y}(t)^T g_{\bar{x}}(t) = \frac{d}{dt}\left[\sum_{i=1}^{r} \bar{\lambda}^i (f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)) + \bar{y}(t)^T g_{\dot{\bar{x}}}(t)\right], \quad t \in I, \tag{4}$$

$$\int_a^b \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}}) dt = 0, \tag{5}$$

$$\bar{\lambda} \geqq 0,\ \bar{y}(t) \geqq 0,\ t \in I,\ \bar{v}^i = \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}}) dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}}) dt},\ i = 1, 2, \dots, r. \tag{6}$$

Proof Let $\bar{x}$ be an efficient solution of order m with respect to θ for (P). By Lemma 3.1., there exists $\bar{v} = (\bar{v}^1, \bar{v}^2, \dots, \bar{v}^r)$ such that $\bar{x}$ is an efficient solution of order m with respect to θ for $(P_{\bar{v}})$. Since every efficient solution of order m with respect to θ for $(P_{\bar{v}})$ is an efficient solution for $(P_{\bar{v}})$, proof follows directly by writing necessary optimality conditions for $(P_{\bar{v}})$ (see Mititelu and Stancu-Minasian, 2009).

The above conditions are only necessary, but not sufficient, with the following example validating this fact:

Example 1

$$\textit{(P1) Minimize} \left(\frac{\int_0^1 \{x(t)(x(t) - t(t-1))\}^2 dt}{\int_0^1 \{x^2(t) + 1\} dt}, \frac{\int_0^1 \{\dot{x}(t)(\dot{x}(t) - 2t + 1)\}^2 dt}{\int_0^1 \{\dot{x}^2(t) + 1\} dt}\right)$$

subject to

$$x(t) \leq 0,\ t \in I = [0, 1],$$
$$x(0) = 0,\ x(1) = 0.$$

$\bar{x}(t) = 0,\ t \in I$ is a feasible solution, which satisfies (4), (5) *and* (6) *for $\bar{\lambda} = (1, 0)^T, \bar{v} = (0, 0)^T$ and $\bar{y}(t) = 0,\ t \in I$. But $\bar{x}$ is not an efficient solution of order m for (P1) with respect to θ, where*

$$\theta(t, x, \bar{x}) = \begin{cases} \frac{x(t) - \bar{x}(t)}{t(t-1)} & 0 < t < 1 \\ 1 & t = 0, 1. \end{cases}$$

Since for each $c = (c^1, c^2) \in int\mathbb{R}^2_+$ and $d = (d^1, d^2) \in int\mathbb{R}^2_+$, we can take $\hat{x}(t) = (t^2 - t) \in X_0,\ t \in I$, then

$$\frac{\int_a^b \{f^1(t, x, \dot{x}) - c^1 \|\theta(t, x, \bar{x})\|^m\} dt}{\int_a^b \{k^1(t, x, \dot{x}) + d^1 \|\theta(t, x, \bar{x})\|^m\} dt} = \frac{-105c^1}{113 + 105d^1} < 0 = \frac{\int_a^b f^1(t, \bar{x}, \dot{\bar{x}}) dt}{\int_a^b k^1(t, \bar{x}, \dot{\bar{x}}) dt},$$

$$\frac{\int_a^b \{f^2(t,\, x,\, \dot{x}) - c^2\|\theta(t,\, x,\, \bar{x})\|^m\}dt}{\int_a^b \{k^2(t,\, x,\, \dot{x}) + d^2\|\theta(t,\, x,\, \bar{x})\|^m\}dt} = \frac{-3c^2}{4+3d^2} < 0 = \frac{\int_a^b f^2(t,\, \bar{x},\, \dot{\bar{x}})dt}{\int_a^b k^2(t,\, \bar{x},\, \dot{\bar{x}})dt}.$$

DEFINITION 5 $\bar{x} \in X_0$ *is said to be a normal efficient solution of order* m *with respect to* θ *for (P) if it is an efficient solution of order* m *with respect to* θ *for (P) and* $\bar{\lambda} \neq 0$.

4. Sufficient optimality conditions

In order to proceed towards sufficient optimality conditions, we have to impose certain conditions on the functionals involved. The notion of $(F,\rho)-$invexity of higher order, introduced in this paper, serves this purpose.

Let $\Phi : X \to \mathbb{R}$, defined by $\Phi(x) = \int_a^b \phi(t,x,\dot{x})dt$, be Fréchet differentiable, where $\phi(t,x,\dot{x})$ is a scalar function with continuous derivatives up to and including second order with respect to each of its arguments.

Let there exist a real number ρ and a functional $F : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ such that $F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\cdot)$ is sublinear on $\mathbb{R}^n$, i.e. for any $x(t),\bar{x}(t) \in \mathbb{R}^n$

$$F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\xi_1+\xi_2) \leq F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\xi_1) + F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\xi_2),\ \text{for any } \xi_1,\xi_2 \in \mathbb{R}^n, \tag{7}$$

$$F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\gamma\xi) = \gamma F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\xi),\ \text{for any } \gamma \in \mathbb{R}_+ \text{ and } \xi \in \mathbb{R}^n. \tag{8}$$

It is quite evident from (8) that

$$F(t,x,\dot{x},\bar{x},\dot{\bar{x}},0) = 0. \tag{9}$$

For the sake of convenience, $\phi_x(t)$ represents $\phi_x(t,x(t),\dot{x}(t))$ and $\phi_{\dot{x}}(t)$ represents $\phi_{\dot{x}}(t,x(t),\dot{x}(t))$.

DEFINITION 6 *A functional* $\Phi(x)$ *is said to be* $(F,\rho)-$*invex of order* m *at* $\bar{x} \in X$ *with respect to* θ *if*

$$\Phi(x) - \Phi(\bar{x}) \geq \int_a^b \{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\phi_{\bar{x}}(t) - \frac{d}{dt}[\phi_{\dot{\bar{x}}}(t)]) + \rho\|\theta(t,x,\bar{x})\|^m\}dt,$$

for all $x \in X$.

DEFINITION 7 *A functional* $\Phi(x)$ *is said to be* (F,ρ)*-pseudoinvex type 2 of order* m *at* $\bar{x} \in X$ *with respect to* θ *if*

$$\int_a^b \{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\phi_{\bar{x}}(t) - \frac{d}{dt}[\phi_{\dot{\bar{x}}}(t)])\}dt \geq 0 \Rightarrow \Phi(x) \geq \Phi(\bar{x}) + \rho\int_a^b \{\|\theta(t,x,\bar{x})\|^m\}dt,\ \textit{for all } x \in X.$$

Definition 8 *A functional $\Phi(x)$ is said to be (F,ρ)-quasiinvex type 1 of order m at $\bar{x}\in X$ with respect to θ if*

$$\Phi(x)\le\Phi(\bar{x})\Rightarrow\int_a^b\{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\phi_{\bar{x}}(t)-\frac{d}{dt}[\phi_{\dot{\bar{x}}}(t)])+\rho\|\theta(t,x,\bar{x})\|^m\}dt\le 0,$$

for all $x\in X$.

Definition 9 *A functional $\Phi(x)$ is said to be (F,ρ)-quasiinvex type 2 of order m at $\bar{x}\in X$ with respect to θ if*

$$\Phi(x)\le\Phi(\bar{x})+\rho\int_a^b\{\|\theta(t,x,\bar{x})\|^m\}dt\Rightarrow$$
$$\int_a^b\{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\phi_{\bar{x}}(t)-\frac{d}{dt}[\phi_{\dot{\bar{x}}}(t)])\}dt\le 0,\ \textit{for all } x\in X.$$

Definition 10 *A functional $\Phi(x)$ is said to be strictly (F,ρ)-pseudoinvex type 2 of order m at $\bar{x}\in X$ with respect to θ if*

$$\int_a^b\{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\phi_{\bar{x}}(t)-\frac{d}{dt}[\phi_{\dot{\bar{x}}}(t)])\}dt\ge 0\Rightarrow$$
$$\Phi(x)>\Phi(\bar{x})+\rho\int_a^b\{\|\theta(t,x,\bar{x})\|^m\}dt,\ \textit{for all } x\in X\setminus\{\bar{x}\}.$$

Definition 11 *A functional $\Phi(x)$ is said to be strictly (F,ρ)-quasiinvex type 1 of order m at $\bar{x}\in X$ with respect to θ if*

$$\Phi(x)\le\Phi(\bar{x})\Rightarrow\int_a^b\{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\phi_{\bar{x}}(t)-\frac{d}{dt}[\phi_{\dot{\bar{x}}}(t)])+\rho\|\theta(t,x,\bar{x})\|^m\}dt<0,$$

for all $x\in X\setminus\{\bar{x}\}$.

Theorem 2 *(Sufficient optimality conditions) Let $\bar{x}\in X_0$ and suppose that there exist $\bar{\lambda}=(\bar{\lambda}^1,\ldots,\bar{\lambda}^r)\in\mathbb{R}^r,\bar{v}=(\bar{v}^1,\bar{v}^2,\ldots,\bar{v}^r)\in\mathbb{R}^r_+$ and a piecewise smooth function $\bar{y}:I\to\mathbb{R}^p$ such that conditions (4), (5) and (6) are satisfied. Let us write $\theta^i(x)=\int_a^b\{f^i(t,x,\dot{x})-\bar{v}^ik^i(t,x,\dot{x})\}dt, i=1,2,\ldots,r$ and $G(x)=\int_a^b\{\bar{y}(t)^Tg(t,x,\dot{x})dt$. Further, if any one of the following conditions holds:*

(a) $\theta^i(x)$, for $i=1,2,\ldots,r$ are strictly (F,ρ^i)-pseudoinvex type 2 functionals of order m at $\bar{x}$ with respect to η and θ, and $G(x)$ is (F,ρ')-quasiinvex type 1 functional of order m at $\bar{x}$ with respect to η and θ, where ρ', $\rho^i\in int\,\mathbb{R}_+$, for $i=1,2,\ldots,r$.

(b) $\theta^i(x)$, for $i=1,2,\ldots,r$ are (F,ρ^i)-quasiinvex type 2 functionals of order m at $\bar{x}$ with respect to η and θ, and $G(x)$ is strictly (F,ρ')-quasiinvex type 1 functional of order m at $\bar{x}$ with respect to η and θ, where ρ', $\rho^i\in int\mathbb{R}_+$, for $i=1,2,\ldots,r$.

Then $\bar{x}$ is an efficient solution of order m with respect to θ for (P).

Proof (a) Suppose $\bar{x}$ is not an efficient solution of order m with respect to θ for (P). Then, for any $c = (c^1, c^2, \ldots, c^r) \in \operatorname{int} \mathbb{R}^r_+$ and $d = (d^1, d^2, \ldots, d^r) \in \operatorname{int} \mathbb{R}^r_+$, there exists $\hat{x} \in X_0$ such that

$$\frac{\int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) - c^i\|\theta(t,\hat{x},\bar{x})\|^m\}dt}{\int_a^b \{k^i(t,\hat{x},\dot{\hat{x}})dt + d^i\|\theta(t,\hat{x},\bar{x})\|^m\}dt} \leq \frac{\int_a^b f^i(t,\bar{x},\dot{\bar{x}})dt}{\int_a^b k^i(t,\bar{x},\dot{\bar{x}})dt} = \bar{v}^i,$$

for all $i \in \{1, 2, \ldots, r\}$ with strict inequality for at least one i.

That is

$$\int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) - \bar{v}^i k^i(t,\hat{x},\dot{\hat{x}})\}dt \leq (c^i + d^i\bar{v}^i)\int_a^b \|\theta(t,\hat{x},\bar{x})\|^m dt,$$

for all $i \in \{1, 2, \ldots, r\}$ with strict inequality for at least one i.

Choose $\rho^i = c^i + d^i\bar{v}^i$, for $i = 1, 2, ..., r$. The above inequalities, along with (6), imply

$$\int_a^b \{f^i(t,\hat{x},\dot{\hat{x}}) - \bar{v}^i k^i(t,\hat{x},\dot{\hat{x}})\}dt \leq \int_a^b \{f^i(t,\bar{x},\dot{\bar{x}}) - \bar{v}^i k^i(t,\bar{x},\dot{\bar{x}}) + \rho^i\|\theta(t,\hat{x},\bar{x})\|^m\}dt,$$

for all $i \in \{1, 2, \ldots, r\}$ with strict inequality for at least one i.

Since $\theta^i(x)$, for $i = 1, 2, \ldots, r$, are strictly (F, ρ^i)-pseudoinvex type 2 functionals of order m at $\bar{x}$ with respect to η and θ, we obtain

$$\int_a^b \{F(t,\hat{x},\dot{\hat{x}},\bar{x},\dot{\bar{x}}, f^i_{\bar{x}}(t) - \bar{v}^i k^i_{\bar{x}}(t) - \frac{d}{dt}[f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)])\}dt < 0.$$

By multiplying the above inequalities by $\bar{\lambda}^i$, $i = 1, 2, \ldots, r$, and then adding, we get

$$\int_a^b \{\sum_{i=1}^r \bar{\lambda}^i F(t,\hat{x},\dot{\hat{x}},\bar{x},\dot{\bar{x}}, f^i_{\bar{x}}(t) - \bar{v}^i k^i_{\bar{x}}(t) - \frac{d}{dt}[f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)])\}dt < 0.$$

Upon using (7) and (8), we get

$$\int_a^b \{F(t,\hat{x},\dot{\hat{x}},\bar{x},\dot{\bar{x}}, \sum_{i=1}^r \bar{\lambda}^i(f^i_{\bar{x}}(t) - \bar{v}^i k^i_{\bar{x}}(t) - \frac{d}{dt}[f^i_{\dot{\bar{x}}}(t) - \bar{v}^i k^i_{\dot{\bar{x}}}(t)]))\}dt < 0. \tag{10}$$

Using $\bar{y}(t) \geqq 0,\ t \in I,\ \hat{x} \in X_0$, and (5), we obtain

$$G(\hat{x}) \leq 0 = G(\bar{x}).$$

Since $G(x)$ is (F, ρ')-quasiinvex type 1 functional of order m at $\bar{x}$ with respect to θ,

$$\int_a^b \{F(t,\hat{x},\dot{\hat{x}},\bar{x},\dot{\bar{x}}, \bar{y}(t)^T g_{\bar{x}}(t) - \frac{d}{dt}[\bar{y}(t)^T g_{\dot{\bar{x}}}(t)]) + \rho'\|\theta(t,\hat{x},\bar{x})\|^m\}dt \leq 0. \tag{11}$$

Using (4), (9) and (7) after adding both sides of the inequalities (10) and (11), we get

$$\rho' \int_a^b \{\|\theta(t,\hat{x},\bar{x})\|^m\} dt < 0.$$

This is a contradiction, as $\rho' > 0, \|\theta(t,\hat{x},\bar{x})\|^m \geq 0$, for all positive integer m. This completes the proof. □

(b) Proof runs along the same lines as the proof of part (a) and is hence omitted.

Theorem 3 *Let $\bar{x} \in X_0$ and suppose that there exist $\bar{\lambda} = (\bar{\lambda}^1, \ldots, \bar{\lambda}^r) \in \mathbb{R}^r, \bar{v} = (\bar{v}^1, \bar{v}^2, \ldots, \bar{v}^r) \in \mathbb{R}^r_+$ and a piecewise smooth function $\bar{y} : I \to \mathbb{R}^p$ such that conditions (4), (5) and (6) are satisfied.*
Let us write $\theta^i(x) = \int_a^b \{f^i(t,x,\dot{x}) - \bar{v}^i k^i(t,x,\dot{x})\} dt, i = 1,2,\ldots,r$ and $G(x) = \int_a^b \{\bar{y}(t)^T g(t,x,\dot{x}) dt$. Further, if any one of the following conditions holds:

(a) $\theta^i(x)$, for $i = 1,2,\ldots,r$ are (F,ρ^i)-pseudoinvex type 2 functionals of order m at $\bar{x}$ with respect to η and θ, and $G(x)$ is (F,ρ')-quasiinvex type 1 functional of order m at $\bar{x}$ with respect to η and θ, where ρ', $\rho^i \in int\, \mathbb{R}_+$, for $i = 1,2,\ldots,r$.

(b) $\theta^i(x)$, for $i = 1,2,\ldots,r$ are (F,ρ^i)-quasiinvex type 2 functionals of order m at $\bar{x}$ with respect to η and θ, and $G(x)$ is strictly (F,ρ')-quasiinvex type 1 functional of order m at $\bar{x}$ with respect to η and θ, where ρ', $\rho^i \in int\mathbb{R}_+$, for $i = 1,2,\ldots,r$.

Then $\bar{x}$ is a weak efficient solution of order m with respect to θ for (P).

Proof Proof runs along the similar lines as the proof of Theorem 3 and is hence omitted.

Remark 2 *If $F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\xi) = (\eta(t,x,\bar{x}))^T \xi$, where $\eta : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $\eta(t,x,\bar{x}) = 0$ at t if $x(t) = \bar{x}(t)$, then Definition 6 reduces to $\rho - (\eta,\theta)-$invexity of order m. So, the above results hold good for this class also.*

5. Higher order duality results

Duality results are important, because they lay foundation for many computational techniques in optimization problems. Recently, several researchers (see Antczak, 2014, 2015; Arana and Ortegon Gallego, 2013, Kumar and Sharma, 2017, Sharma et al., 2017) contributed to the development of duality theory for multiobjective variational problems. Higher order duality has even greater significance than the first order duality, since it provides tighter bounds for the value of the objective function when approximations are used, because it involves more parameters. One more advantage of higher order duality, when applicable, is that if a feasible point in the primal problem is given and first order duality does not apply, then we can use higher order duality to provide a lower bound on the value of the primal. Following the parametric approach of Bector (see Bector et al., 1993), the higher order dual (D) to multiobjective fractional variational problem is defined as follows:

$$(D) \text{ Maximize } \bar{v} = (\bar{v}^1, \bar{v}^2, \ldots, \bar{v}^r)$$

subject to

$$\sum_{i=1}^{r} \bar{\lambda}^i [\nabla_p h^i(t, \bar{x}, \dot{\bar{x}}, p) - \bar{v}^i \nabla_p l^i(t, \bar{x}, \dot{\bar{x}}, p)] +$$
$$\sum_{j=1}^{p} \bar{y}^j(t) \nabla_p \xi^j(t, \bar{x}, \dot{\bar{x}}, p) = 0 \quad t \in I, \tag{12}$$

$$\int_a^b \{\sum_{j=1}^{p} \bar{y}^j(t)(g^j(t, \bar{x}, \dot{\bar{x}}) + \xi^j(t, \bar{x}, \dot{\bar{x}}, p)) -$$
$$p^T(\sum_{j=1}^{p} \bar{y}^j(t) \nabla_p \xi^j(t, \bar{x}, \dot{\bar{x}}, p))\} dt \geqq 0. \tag{13}$$

$$\bar{y}^j(t) \geq 0, t \in I, j \in \{1, 2, \ldots, p\}, \bar{x} \in X. \tag{14}$$

$$\int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) + h^i(t, \bar{x}, \dot{\bar{x}}, p) - p^T \nabla_p h^i(t, \bar{x}, \dot{\bar{x}}, p)$$
$$-\bar{v}^i(k^i(t, \bar{x}, \dot{\bar{x}}) + l^i(t, \bar{x}, \dot{\bar{x}}, p) - p^T \nabla_p l^i(t, \bar{x}, \dot{\bar{x}}, p))\} dt \geq 0, i = 1, 2, \ldots, r \tag{15}$$

$$\bar{x} \in X,\, \bar{\lambda} \geq 0,\, \sum_{i=1}^{r} \bar{\lambda}^i = 1,\, \bar{y}(t) \geqq 0, t \in I,\, \bar{v}^i \geq 0, i = 1, 2, \ldots, r. \tag{16}$$

$$\bar{x}(a) = \alpha,\, \bar{x}(b) = \beta, \tag{17}$$

where $h^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $l^i : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, i \in \{1, 2, \ldots, r\}$, $\xi^j : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are differentiable functions.

Let Y_0 be the set of all feasible solutions of (D). In order to facilitate the derivation of higher order duality for this solution concept, we introduce the following class of functionals.

Let there exist a real number ρ and a functional $F : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $F(t, x, \dot{x}, \bar{x}, \dot{\bar{x}}, \cdot)$ is sub-linear on $\mathbb{R}^n$. Let $p \in \mathbb{R}^n$ and $h : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a differentiable function.

Definition 12 *A functional $\Phi(x)$ is said to be higher order $(F,\rho,\theta,m,h)-$invex at $\bar{x}\in X$ if*

$$\Phi(x)-\Phi(\bar{x})-\int_a^b\{h(t,\bar{x},\dot{\bar{x}},p)-p^T\bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p)\}dt$$
$$\geq\int_a^b\{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\bigtriangledown_p h(t,\bar{x},\dot{\bar{x}},p))+\rho\|\theta(t,x,\bar{x})\|^m\}dt,\ \textit{for all } x\in X.$$

Remark 3 *(a) If $h(t,\bar{x},\dot{\bar{x}},p)=p^T[\phi_{\bar{x}}(t)-\frac{d}{dt}(\phi_{\dot{\bar{x}}}(t))]$ then Definition 12 reduces to Definition 6.*
(b) If $h(t,\bar{x},\dot{\bar{x}},p)=p^T[\phi_{\bar{x}}(t)-\frac{d}{dt}(\phi_{\dot{\bar{x}}}(t))]+\frac{1}{2}p^T[\phi_{\bar{x}\bar{x}}(t)-2\frac{d}{dt}\phi_{\bar{x}\dot{\bar{x}}}(t)+\frac{d^2}{dt^2}\phi_{\dot{\bar{x}}\dot{\bar{x}}}(t)]p$, then Definition 6 reduces to the definition of second order $(F,\alpha,\rho,\theta)-$ convexity with $\alpha(x,\bar{x})=1,\theta:I\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R},\beta(t)=p,t\in I$ (see Jayswal et al., 2015).

Various duality results, connecting efficient solutions of primal and its dual problem are established in the sequel.

Theorem 4 *(Weak duality) Let $x\in X_0$, $(\bar{x},\bar{\lambda},\bar{v},\bar{y},p)\in Y_0$, and let us write $F^i(x)=\int_a^b\{f^i(t,x,\dot{x})-\bar{v}^ik^i(t,x,\dot{x})\}dt$, $i=1,2,\ldots,r$ and $G(x)=\int_a^b\{\sum\limits_{i=1}^{r}\bar{y}^i(t)g^i(t,x,\dot{x})\}dt$.*
Assume
(i) $F^i(x),i=1,2,\ldots,r$ to be higher order $(F,\rho^i,\theta,m,\eta^i)$-strictly invex at $\bar{x}$, where $\eta^i(t,x,\dot{x},p)=h^i(t,x,\dot{x},p)-\bar{v}^il^i(t,x,\dot{x},p),i=1,2,\ldots,r$.
(ii) $G(x)$ to be higher order $(F,\rho',\theta,m,h')-$invex at $\bar{x}$, where $h'(t,x,\dot{x},p)=\sum\limits_{j=1}^{m}\bar{y}^j(t)(\xi^j(t,x,\dot{x},p))$.
(iii) $(\rho^1,\rho^2,\ldots,\rho^r)\in int\mathbb{R}^r_+$ and $\rho'>0$.

Then there exist $c=(c^1,c^2,\ldots,c^r)\in int\mathbb{R}^r_+$ and $d=(d^1,d^2,\ldots,d^r)\in int\mathbb{R}^r_+$ such that the following cannot hold:

$$\frac{\int_a^b\{f^i(t,x,\dot{x})-c^i\|\theta(t,x,\bar{x})\|^m\}dt}{\int_a^b\{k^i(t,x,\dot{x})+d^i\|\theta(t,x,\bar{x})\|^m\}dt}\leq\bar{v}^i,$$

for all $i\in\{1,2,\ldots,r\}$ with strict inequality for at least one i.

Proof Using (i) and (15), we get

$$F^i(x)>\int_a^b\{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\bigtriangledown_p\eta^i(t,\bar{x},\dot{\bar{x}},p))+\rho^i\|\theta(t,x,\bar{x})\|^m\}dt,$$

By multiplying the above inequalities by $\bar{\lambda}^i,i=1,2,\ldots,r$ and then adding, we obtain

$$\sum_{i=1}^{r}\bar{\lambda}^iF^i(x)>\int_a^b\{\sum_{i=1}^{r}\bar{\lambda}^iF(t,x,\dot{x},\bar{x},\dot{\bar{x}},\bigtriangledown_p\eta^i(t,\bar{x},\dot{\bar{x}},p))+\sum_{i=1}^{r}\bar{\lambda}^i\rho^i\|\theta(t,x,\bar{x})\|^m\}dt,$$

On using (7) and (8), we get

$$\sum_{i=1}^{r} \bar{\lambda}^i F^i(x) > \int_a^b \{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\sum_{i=1}^{r} \bar{\lambda}^i \nabla_p \eta^i(t,\bar{x},\dot{\bar{x}},p)) + \sum_{i=1}^{r} \bar{\lambda}^i \rho^i \|\theta(t,x,\bar{x})\|^m\} dt.$$

Using feasibility of x and $(\bar{x},\bar{\lambda},\bar{v},\bar{y},p)$ along with condition (ii), we obtain

$$0 \geq \int_a^b \{F(t,x,\dot{x},\bar{x},\dot{\bar{x}},\nabla_p h'(t,\bar{x},\dot{\bar{x}},p)) + \rho' \|\theta(t,x,\bar{x})\|^m\} dt.$$

By adding the above two inequalities and using (12), (7) and (8), we obtain

$$\sum_{i=1}^{r} \bar{\lambda}^i F^i(x) > \int_a^b \{\sum_{i=1}^{r} \bar{\lambda}^i \rho^i + \rho'\} \|\theta(t,x,\bar{x})\|^m dt.$$

Contrary to the result, assume that for any $c = (c^1, c^2, \ldots, c^r) \in \operatorname{int} \mathbb{R}^r_+$ and $d = (d^1, d^2, \ldots, d^r) \in \operatorname{int} \mathbb{R}^r_+$

$$\frac{\int_a^b \{f^i(t,x,\dot{x}) - c^i \|\theta(t,x,\bar{x})\|^m\} dt}{\int_a^b \{k^i(t,x,\dot{x}) + d^i \|\theta(t,x,\bar{x})\|^m\} dt} \leq \bar{v}^i,$$

for all $i \in \{1, 2, \ldots, r\}$ with strict inequality for at least one i.
That is,

$$\sum_{i=1}^{r} \bar{\lambda}^i F^i(x) \leq \sum_{i=1}^{r} \bar{\lambda}^i (c^i + d^i \bar{v}^i) \int_a^b \|\theta(t,x,\bar{x})\|^m dt,$$

for all $i \in \{1, 2, \ldots, r\}$ with strict inequality for at least one i.
Choose c and d such that $\sum\limits_{i=1}^{r} \bar{\lambda}^i (c^i + d^i \bar{v}^i) = \sum\limits_{i=1}^{r} \bar{\lambda}^i \rho^i + \rho'$, we arrive at a contradiction. Hence the result follows. □

Strong duality guarantees the existence of feasible solution of the higher order dual problem if the primal problem has an efficient solution, which is evident from the previous literature, but the solution concept considered earlier was an efficient solution, whereas in this paper efficiency of higher order is taken as a new solution concept. The strong duality theorem, proved here, provides more stable solutions for the dual problem.

Theorem 5 *(Strong duality) Let $\bar{x}$ be an efficient solution of order m with respect to θ for (P), which is normal. Then, there exist $\bar{\lambda}, p \in \mathbb{R}^r, \bar{v} \in \mathbb{R}^r_+$ and a piecewise smooth function $\bar{y} : I \to \mathbb{R}^p$ such that $(\bar{x},\bar{\lambda},\bar{v},\bar{y},p) \in Y_0$. Further, if weak duality theorem holds, then $(\bar{x},\bar{\lambda},\bar{v},\bar{y},p)$ is an efficient solution of order m with respect to θ for (D).*

Proof Since $\bar{x}$ is an efficient solution of order m with respect to θ for (P) which is normal, hence, by Theorem 1, there exist $\bar{\lambda} \in \mathbb{R}^r, \bar{v} \in \mathbb{R}^r_+$, a piecewise

smooth function $\bar{y}: I \to \mathbb{R}^p$ and $p = 0$ such that $(\bar{x}, \bar{\lambda}, \bar{v}, \bar{y}, p) \in Y_0$.
Let, if possible, $(\bar{x}, \bar{\lambda}, \bar{v}, \bar{y}, p)$ be not an efficient solution of order m with respect to θ for (P), then for any $\rho = (\rho^1, \rho^2, ..., \rho^r) \in \text{int}\mathbb{R}^r_+$, there exist $(\hat{x}, \hat{\lambda}, \hat{v}, \hat{y}, \hat{p}) \in Y_0$ such that

$$\hat{v}^i + \rho^i \int_a^b \|\theta(t, \hat{x}, \bar{x})\|^m dt \geq \bar{v}^i = \frac{\int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{\int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt} \tag{18}$$

for all $i \in \{1, 2, \ldots, r\}$ with strict inequality for at least one i.

Case (i)
If $\|\theta(t, \hat{x}, \bar{x})\|^m = 0$, then for any $c = (c^1, c^2, \ldots, c^r) \in \text{int}\,\mathbb{R}^r_+$ and $d = (d^1, d^2, \ldots, d^r) \in \text{int}\,\mathbb{R}^r_+$, we have

$$\frac{\int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) - c^i\|\theta(t, \hat{x}, \bar{x})\|^m\}dt}{\int_a^b \{k^i(t, \bar{x}, \dot{\bar{x}}) + d^i\|\theta(t, \hat{x}, \bar{x})\|^m\}dt} \leq \hat{v}^i$$

for all $i \in \{1, 2, ..., r\}$ with strict inequality for at least one i, which is a contradiction to the weak duality theorem.

Case (ii)
Assume $\|\theta(t, \hat{x}, \bar{x})\|^m \neq 0$.
For any $c = (c^1, c^2, \ldots, c^r) \in \text{int}\,\mathbb{R}^r_+$ and $d = (d^1, d^2, \ldots, d^r) \in \text{int}\,\mathbb{R}^r_+$, define

$$\rho^i = \frac{c^i \int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt + d^i \int_a^b f^i(t, \bar{x}, \dot{\bar{x}})dt}{(\int_a^b \{k^i(t, \bar{x}, \dot{\bar{x}}) + d^i\|\theta(t, \hat{x}, \bar{x})\|^m\}dt) \int_a^b k^i(t, \bar{x}, \dot{\bar{x}})dt}, i = 1, 2, \ldots, r.$$

Substituting this in (18) yields

$$\frac{\int_a^b \{f^i(t, \bar{x}, \dot{\bar{x}}) - c^i\|\theta(t, \hat{x}, \bar{x})\|^m\}dt}{\int_a^b \{k^i(t, \bar{x}, \dot{\bar{x}}) + d^i\|\theta(t, \hat{x}, \bar{x})\|^m\}dt} \leq \hat{v}^i,$$

for all $i \in \{1, 2, \ldots, r\}$ with strict inequality for at least one i, which contradicts the weak duality theorem. Thus, $(\bar{x}, \bar{\lambda}, \bar{v}, \bar{y}, p)$ is an efficient solution of order m with respect to θ for (D). □

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