Random models of coupled inductors

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This paper describes the method of determining the first two moments of the state equation solution. The presented method is applied to the analysis of the coupled coils system. Parameters of coupled coils are random variables. There are supplied by the ideal voltage sources. It is assumed that the forces are stochastic processes. The results are illustrated by the example.

KEYWORDS: random dynamical systems, coupled coils, moments equations

1. Linear systems with random variables coefficients

Let be given the state equation [11]:

$$\begin{cases} \dfrac{d\mathbf{X}(t)}{dt} = \mathbf{AX}(t) + \mathbf{BF}(t) \\ \mathbf{X}(0) = \mathbf{X}_0 \end{cases}, \tag{1}$$

where **A** is the matrix of random variables describing the model; **B** is the matrix of random variables; $\mathbf{X}_0$ is a vector containing the random variables of the initial condition; $\mathbf{F}(t)$ is a vector containing the stochastic processes - forces; $\mathbf{X}(t)$ is a vector-valued stochastic process - a solution of the state equation.

Analytical solution of the equation (1) in the mean square sense is known [12] and determined by the formula:

$$\begin{cases} \mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{X}_0 + \displaystyle\int_0^t \mathbf{\Phi}(t-\tau)\mathbf{BF}(t)\, d\tau \\ \mathbf{\Phi}(t) = \exp(\mathbf{A}t) \end{cases}. \tag{2}$$

Applying the expected value operator to the equation (2) [13], one can determine the basic moments of the solution of the state equation (1). Moments mentioned above are [13]:

– the expected value of the response:

$$\mu_X(t) = \mathrm{E}[\mathbf{\Phi}(t)\mathbf{X}_0] + \int_0^t \mathrm{E}\left[\mathbf{\Phi}(t-\tau)\mathbf{BF}(t)\right]\, d\tau, \tag{3}$$

– cross-correlation of the force and the response:

$$\mathrm{R}_{FX}(t_1,t_2)=\mathrm{E}\left[\mathbf{F}(t_1)\mathbf{X}^{\mathrm{T}}(t_2)\right], \tag{4}$$

– correlation of the response:

$$\mathrm{R}_{X}(t_1,t_2)=\mathrm{E}\left[\mathbf{X}(t_1)\mathbf{X}^{\mathrm{T}}(t_2)\right]. \tag{5}$$

By the equations showed below, one can also determine the moments associated with force $\mathbf{F}(t)$:

– the expected value of the force:

$$\mu_F(t)=\mathrm{E}[\mathbf{F}(t)], \tag{6}$$

– correlation of the force:

$$\mathrm{R}_{F}(t_1,t_2)=\mathrm{E}\left[\mathbf{F}(t_1)\mathbf{F}^{\mathrm{T}}(t_2)\right]. \tag{7}$$

2. Coupled coils

Circuit of the analysed system is shown on Figure 1:

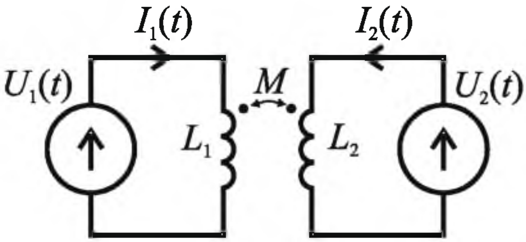

Fig. 1. Coupled coils system

It is assumed that L_1, L_2, M are independent random variables. Forces $U_1(t)$ and $U_2(t)$ are independent stochastic processes of the second order.

The currents and voltages in the analysed system are related with the following equations:

$$\begin{cases} U_1(t)=L_1\dfrac{dI_1(t)}{dt}+M\dfrac{dI_2(t)}{dt} \\ U_2(t)=M\dfrac{dI_1(t)}{dt}+L_2\dfrac{dI_2(t)}{dt} \end{cases}. \tag{8}$$

System of equations (8) in the matrix form follows:

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}U_1(t)\\ U_2(t)\end{bmatrix}=\begin{bmatrix}L_1 & M\\ M & L_2\end{bmatrix}\begin{bmatrix}\frac{dI_1(t)}{dt}\\ \frac{dI_2(t)}{dt}\end{bmatrix}. \tag{9}$$

Let:

$$\mathbf{B}^{-1} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}, \tag{10}$$

then:

$$\mathbf{B} = \frac{1}{L_1 L_2 - M^2} \begin{bmatrix} L_2 & -M \\ -M & L_1 \end{bmatrix}. \tag{11}$$

Finally, the system of equations (8) is:

$$\begin{bmatrix} \frac{dI_1(t)}{dt} \\ \frac{dI_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} \frac{L_2}{L_1 L_2 - M^2} & -\frac{M}{L_1 L_2 - M^2} \\ -\frac{M}{L_1 L_2 - M^2} & \frac{L_1}{L_1 L_2 - M^2} \end{bmatrix} \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix}. \tag{12}$$

Comparing equation (12) with the state equation (1), it can be obtained:

$$\mathbf{X}(t) = \mathbf{I}(t), \quad \mathbf{F}(t) = \mathbf{U}(t), \quad \mathbf{X}_0 = \mathbf{I}_0,$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{L_2}{L_1 L_2 - M^2} & -\frac{M}{L_1 L_2 - M^2} \\ -\frac{M}{L_1 L_2 - M^2} & \frac{L_1}{L_1 L_2 - M^2} \end{bmatrix}. \tag{13}$$

Using the formulas (13) and (3), (4), (5) presented in the chapter 1, it can be determined:

– the expected value of the current of the coils:

$$\mu_I(t) = \mu_{I_0} + \mathrm{E}[\mathbf{B}] \int_0^t \mu_U(\tau)\, d\tau, \tag{14}$$

– cross-correlation of the force and the current of the coils:

$$\mathrm{R}_{UI}(t_1, t_2) = \mu_U(t_1)\mu_{I_0}^{\mathrm{T}} + \int_0^{t_2} \mathrm{R}_U(t_1, \tau)\, d\tau\, \mathrm{E}\left[\mathbf{B}^{\mathrm{T}}\right], \tag{15}$$

– correlation of the current of the coils:

$$\begin{aligned} \mathrm{R}_I(t_1, t_2) = {} & \mathrm{R}_{I_0} + \mu_{I_0} \int_0^{t_2} \mu_U^{\mathrm{T}}(\tau)\, d\tau\, \mathrm{E}\left[\mathbf{B}^{\mathrm{T}}\right] \\ & + \mathrm{E}[\mathbf{B}] \int_0^{t_1} \mu_U(\tau)\, d\tau\, \mu_{I_0}^{\mathrm{T}} \\ & + \mathrm{E}[\mathbf{B}] \int_0^{t_1}\int_0^{t_2} \mathrm{R}_U(\tau_1, \tau_2)\, d\tau_1\, d\tau_2\, \mathrm{E}\left[\mathbf{B}^{\mathrm{T}}\right] \end{aligned} \tag{16}$$

where:

$$\mu_{I_0} = \mathrm{E}[\mathbf{I}_0], \quad \mathrm{R}_{I_0} = \mathrm{E}[\mathbf{I}_0\mathbf{I}_0^{\mathrm{T}}].$$

Computational problem occurs when the matrix E[**B**] is determined, due to the elements of the matrix are functions of random variables L_1, L_2, M. To simplify the calculations, one can define the new random variables:

$$L_1' = \frac{L_1}{L_1L_2 - M^2}, \quad L_2' = \frac{L_2}{L_1L_2 - M^2}, \quad M' = \frac{M}{L_1L_2 - M^2}. \tag{17}$$

The expected value operator E[ ] is defined as:

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx, \tag{18}$$

where $g(x)$ is any deterministic function, while $f_X(x)$ is the probability density function of the random variable X.

Using the formulas (17) and (18) the matrix E[**B**] can be determined:

$$\mathrm{E}\left[\mathbf{B}\right] = \begin{bmatrix} \int_{-\infty}^{\infty} l_2' f_{L_2'}(l_2')\, dl_2' & -\int_{-\infty}^{\infty} m' f_{M'}(m')\, dm' \\ -\int_{-\infty}^{\infty} m' f_{M'}(m')\, dm' & \int_{-\infty}^{\infty} l_1' f_{L_1'}(l_1')\, dl_1' \end{bmatrix}. \tag{19}$$

Each integral of the matrix (19) can be determined by the below formulas:

$$\int_{-\infty}^{\infty} l_1' f_{L_1'}(l_1')\, dl_1' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{l_1}{l_1l_2 - m^2} f_{L_1}(l_1) f_{L_2}(l_2) f_M(m)\, dl_1\, dl_2\, dm, \tag{20}$$

$$\int_{-\infty}^{\infty} l_2' f_{L_2'}(l_2')\, dl_2' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{l_2}{l_1l_2 - m^2} f_{L_1}(l_1) f_{L_2}(l_2) f_M(m)\, dl_1\, dl_2\, dm, \tag{21}$$

$$\int_{-\infty}^{\infty} m' f_{M'}(m')\, dm' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{m}{l_1l_2 - m^2} f_{L_1}(l_1) f_{L_2}(l_2) f_M(m)\, dl_1\, dl_2\, dm. \tag{22}$$

In the case that any of the variable L_1, L_2, M is a real number instead a random variable, triple integrals in formulas (21), (22), (23) reduce to double or even single integrals (when two variables are real numbers).

3. Example

To simplify the calculation, some of the parameters are assumed to be real numbers. Let:

$$\begin{aligned} L_1 &= 5 \text{ mH} \\ L_2 &= 7 \text{ mH} \\ M &= \mathcal{U}(a = 3, b = 4) \text{ mH} \\ u_1(t) &= \sin(\omega t) \text{ V} \\ u_2(t) &= \sin(\omega t + \varphi) \text{ V} \\ \omega &= 314 \text{ rad/s} \\ \varphi &= \mathcal{N}(\mu_\varphi = 0, \sigma_\varphi^2 = \pi) \text{ rad} \\ I_1(0) &= I_2(0) = 0 \end{aligned} \tag{23}$$

where U means the uniform distribution, while N means the normal distribution.

Using the formulas (6) and (7) the expected value and correlation of the force are determined:

$$\mu_U(t) = \begin{bmatrix} \sin(314t) \\ \sin(314t)\exp(-\pi/2) \end{bmatrix}, \tag{24}$$

$$\mathrm{R}_U(t_1,t_2) = \begin{bmatrix} \sin(314t_1)\sin(314t_2) & \sin(314t_1)\sin(314t_2)\exp(-\pi/2) \\ \sin(314t_1)\sin(314t_2)\exp(-\pi/2) & r_{U22}(t_1,t_2) \end{bmatrix}, \tag{25}$$

where:

$$\begin{aligned} r_{U22}(t_1,t_2) &= \frac{1}{2}\left(\cos(314t_1)\cos(314t_2) + \sin(314t_1)\sin(314t_2)\right) \\ &- \frac{1}{4}\left(\exp(314jt_1 + 314jt_2 - 2\pi) + \exp(-314jt_1 - 314jt_2 - 2\pi)\right). \end{aligned} \tag{26}$$

Using the formulas (19), (20), (21), (22) the matrix E[**B**] is be determined:

$$b_{11} = \frac{L2}{\sqrt{L_1L_2}}\left(\operatorname{arctanh}\left(\frac{b}{\sqrt{L_1L_2}}\right) - \operatorname{arctanh}\left(\frac{a}{\sqrt{L_1L_2}}\right)\right) = 0.311, \tag{27}$$

$$b_{12} = b_{21} = \frac{1}{2}\ln(L_1L_2 - b^2) - \frac{1}{2}\ln(L_1L_2 - a^2) = -0.157, \tag{28}$$

$$b_{22} = \frac{L1}{\sqrt{L_1 L_2}} \left(\text{arctanh} \left(\frac{b}{\sqrt{L_1 L_2}} \right) - \text{arctanh} \left(\frac{a}{\sqrt{L_1 L_2}} \right) \right) = 0.222. \tag{29}$$

To determine the expected value of the response (currents of coils), the equation (14) can be used:

$$\mu_I(t) = \begin{bmatrix} 0.311 & -0.157 \\ -0.157 & 0.222 \end{bmatrix} \begin{bmatrix} \frac{1}{314}(1 - cos(314t)) \\ \frac{1}{314}(1 - cos(314t)) \exp(-\pi/2) \end{bmatrix}. \tag{30}$$

Cross-correlation of the force and response is determined by the formula (15):

$$\mathrm{R}_{UI}(t_1, t_2) = \begin{bmatrix} r_{x11}(t_1, t_2) & r_{x12}(t_1, t_2) \\ r_{x21}(t_1, t_2) & r_{x22}(t_1, t_2) \end{bmatrix} \begin{bmatrix} 0.311 & -0.157 \\ -0.157 & 0.222 \end{bmatrix}, \tag{31}$$

where:

$$\begin{aligned} r_{x11}(t_1, t_2) =& \frac{\sin(314t_1)}{314}(1 - \cos(314t_2)) \\ r_{x12}(t_1, t_2) =& r_{x21}(t_1, t_2) = \frac{\sin(314t_1)\exp(-\pi/2)}{314}(1 - \cos(314t_2)) \\ r_{x22}(t_1, t_2) =& -\frac{1}{628}((\sin(314t_1 + 314t_2) - \sin(314t_1))\exp(-2\pi) \\ & - \cos(314t_1)\sin(314t_2) + \sin(314t_1)\cos(t_2)). \end{aligned} \tag{32}$$

Following in the same way it can be determined (by the formula (16)) correlation of the response. However, the analytical formula is too long to put it into this paper.

4. Summary

The presented method allows to obtain efficient solutions of the moments of the state equation solution of the random dynamical systems. In the first chapter, the problem is formalized and the solution is presented in the moments domain. The second chapter concerns of the coupled coils. It is assumed that the parameters describing the model are random variables and forces are stochastic processes. There are presented an analytical formulas for the moments of the system response. In the third chapter a concrete example of a calculation is presented.

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