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VARIATIONAL EQUATIONS ON THE MÖBIUS STRIP

43.1 INTRODUCTION

The aim of this paper is to analyse restrictions of ordinary differential equations, defined on a Euclidean space, with respect a topologically *non*-trivial example of an embedded submanifold the 2-dimensional *Möbius strip*. The general problem is what can be said about the influence of the topology on variationality of differential equations, given on submanifolds, and on the existence and structure of the corresponding local and global variational principles (see Krupka, Urban, and Volná [4]).

For this purpose, we introduce an adapted smooth atlas on a Euclidean space with respect to the Möbius strip, and study its properties. We apply results of the variational sequence theory on sheaves of differential forms, although we do *not* introduce this concept here (cf. Krupka [2], Volná and Urban [8]; see also Takens [7]). In particular, we refer to our recent work Krupka, Urban, and Volná [4], where a general theory of variational submanifolds has been developed.

For standard notions of smooth manifolds (chart, atlas, submanifold, adapted chart) we refer to Lee [5]. Basic notions of the jet theory as well as an introduction to variational principles on fibered manifolds can be found in Krupka and Saunders [3]. The classical textbook on sheaf theory, needed for understanding of the local and global variationality, is Warner [9].

The (global) Cartesian coordinates on the Euclidean space $\mathbb{R}^2$, resp. $\mathbb{R}^3$, are denoted by (x, y), resp. (x, y, z). Throughout we apply the Einstein summation convention. All figures are created in GeoGebra.

43.2 POLAR COORDINATES AND THE ARCTANGENT WITH TWO ARGUMENTS

First we recall a standard modification of the arctangent function, needed for further considerations. The function atan2 on $\mathbb{R}^2$ *without* the origin, defined by

$$\operatorname{atan2}(y,x) = \begin{cases} \operatorname{atan}(y/x), & x > 0 \text{ (quadrants I and IV, positive } x\text{-axis)} \\ \operatorname{atan}(y/x) + \pi, & x < 0, y \geq 0 \text{ (quadrant II, negative } x\text{-axis)} \\ \operatorname{atan}(y/x) - \pi, & x < 0, y < 0 \text{ (quadrant III)} \\ \pi/2, & x = 0, y > 0 \text{ (positive } y\text{-axis)} \\ -\pi/2, & x = 0, y < 0 \text{ (negative } y\text{-axis)} \\ \text{undefined}, & x = 0, y = 0 \text{ (the origin in } (0,0) \in \mathbb{R}^2) \end{cases} \quad (43.1)$$

is sometimes also called the *arctangent with two arguments*. Note that this function has values in the interval $(-\pi, \pi]$, and the graphs of atan2 and $\operatorname{atan}(y/x)$ coincide in the quadrants I and IV, as demonstrated by the following figure Fig. 43.1.

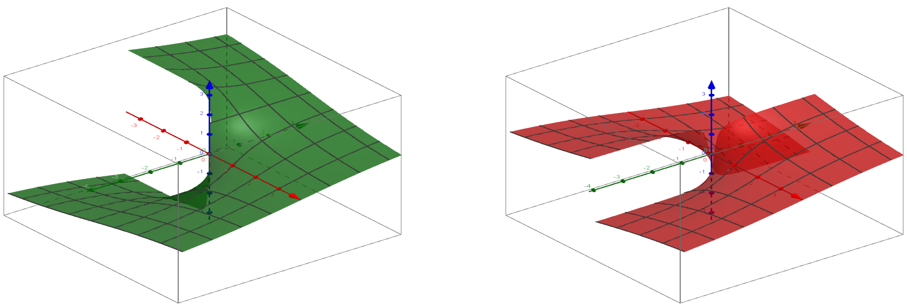

Fig. 43.1 Graph of function $(x,y) \to \operatorname{atan2}(y,x)$ **(green) and function** $(x,y) \to \operatorname{atan}(y/x)$ **(red)**
Source: own elaboration

We remark that the function atan2 was first introduced in programming languages, and later became common in other fields of science and engineering. In particular, one can use this function for instance in Matlab, Java, C++, GeoGebra, Mathematica, or in most spreadsheets software.

In this paper, we shall employ the function atan2 for constructing adapted coordinates to the Möbius strip in $\mathbb{R}^3$. This is motivated by the following well-known fact. On the open set $U = \mathbb{R}^2 \setminus ((-\infty, 0] \times \{0\})$ in $\mathbb{R}^2$, the *polar coordinates* $\Phi = (r, \varphi)$ are defined, where r is the distance of a point of U from the origin $(0,0) \in \mathbb{R}^2$, and φ is the angle with values in $(-\pi, \pi)$ between the vector with endpoints the origin and a point of U, and the positive x-axis. The transformation from polar to Cartesian coordinates $\Phi^{-1} : \Phi(U) \to U$, where $\Phi(U) = (0, \infty) \times (-\pi, \pi)$, and the inverse transformation from Cartesian to polar coordinates $\Phi : U \to \Phi(U)$ are given by the equations

$$\Phi^{-1} : x = r\cos\varphi,\ y = r\sin\varphi, \qquad \Phi : r = \sqrt{x^2 + y^2},\ \varphi = \operatorname{atan2}(y, x),$$

where atan2 is given by (43.1).

43.3 ADAPTED ATLAS TO THE MÖBIUS STRIP $M_{R,a}$

In this section, we consider the *Möbius strip* $M_{R,a}$ (without boundary) of *radius* R and *wideness* $2a$, where $0 < a < R$, as a two-dimensional embedded submanifold of the Euclidean space $\mathbb{R}^3$. $M_{R,a}$ is also called the *open* Möbius strip. We observe that $M_{R,a}$ can be parametrized by the equations

$$x = \left(R + \tau\cos\frac{\varphi}{2}\right)\cos\varphi, \quad y = \left(R + \tau\cos\frac{\varphi}{2}\right)\sin\varphi, \quad z = \tau\sin\frac{\varphi}{2}, \tag{43.2}$$

where $0 \le \varphi < 2\pi$, $-a < \tau < a$, and then it corresponds to the figure Fig. 43.2.

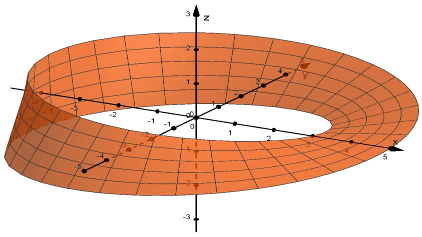

Fig. 43.2 Möbius strip

Source: own elaboration

Equations (43.2) can be used for construction of a smooth atlas on the open subset X of $\mathbb{R}^3$,

$$X = \mathbb{R}^3 \setminus \{(0,0,z)\,|\,z \in \mathbb{R}\}, \tag{43.3}$$

which consists of two *adapted charts* to $M_{R,a}$. We put

$$V = \mathbb{R}^3 \setminus ((-\infty,0] \times \{0\} \times \mathbb{R}) \tag{43.4}$$

an open subset of $\mathbb{R}^3$, and $\Psi = (\varphi,\tau,\kappa)$, where

$$\begin{aligned}
\varphi &= \operatorname{atan2}(y,x),\\
\tau &= \left(\sqrt{x^2+y^2} - R\right)\frac{\sqrt{2}}{2}\sqrt{1 + \frac{x}{\sqrt{x^2+y^2}}} + \operatorname{sgn}(y) z\frac{\sqrt{2}}{2}\sqrt{1 - \frac{x}{\sqrt{x^2+y^2}}},\\
\kappa &= -\left(\sqrt{x^2+y^2} - R\right)\operatorname{sgn}(y)\frac{\sqrt{2}}{2}\sqrt{1 - \frac{x}{\sqrt{x^2+y^2}}} + z\frac{\sqrt{2}}{2}\sqrt{1 + \frac{x}{\sqrt{x^2+y^2}}}.
\end{aligned} \tag{43.5}$$

It is straightforward to check that (V,Ψ) is a chart on X. Indeed, Ψ has the inverse mapping $\Psi^{-1} : \Psi(V) \to V$, defined by the equations

$$\begin{aligned}
&x = \left(R + \tau\cos\frac{\varphi}{2} - \kappa\sin\frac{\varphi}{2}\right)\cos\varphi, \quad y = \left(R + \tau\cos\frac{\varphi}{2} - \kappa\sin\frac{\varphi}{2}\right)\sin\varphi,\\
&z = \tau\sin\frac{\varphi}{2} + \kappa\cos\frac{\varphi}{2},
\end{aligned} \tag{43.6}$$

where the domain $\Psi(V)$ is an open subset of $\mathbb{R}^3$, expressed by

$$\Psi(V) = \left\{(\varphi,\tau,\kappa) \in \mathbb{R}^3 \mid -\pi < \varphi < \pi,\ \tau > \frac{-R + \kappa\sin(\varphi/2)}{\cos(\varphi/2)},\ \kappa \in \mathbb{R}\right\}.$$

The Jacobian $\det J$ of Ψ^{-1} does *not* vanish. Indeed, we have

$$\det J = -R - \tau \cos\frac{\varphi}{2} + \kappa \sin\frac{\varphi}{2} < 0.$$

Note that the open subset $\Psi(V)$ of $\mathbb{R}^3$ is *bounded* by the surface of equation

$$\tau = \frac{-R + \kappa \sin(\varphi/2)}{\cos(\varphi/2)},$$

and it contains the origin of $\mathbb{R}^3$; for constant φ, $\Psi(V)$ is bounded by lines (see Fig. 43.3, where a line of boundary is highlighted for $\varphi = 2\pi/3$). The corresponding half-plane of τ and κ-axis is given by Fig. 43.3.

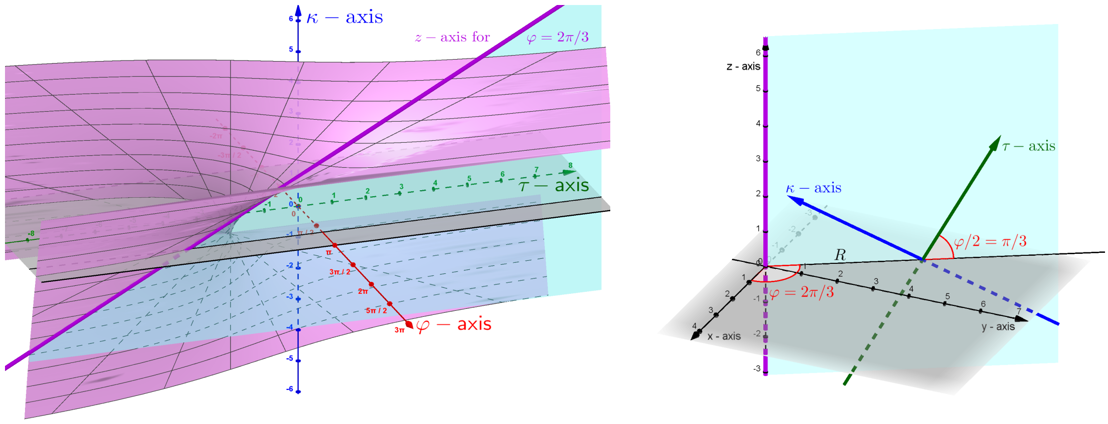

Fig. 43.3 Border surface and one of underlying halfplane for second coordinate system
Source: own elaboration

Using the parametrization (43.2) of $M_{R,a}$, it is easily seen from the chart transformations (43.5), (43.6), that $M_{R,a}$ is characterized by the equation $\kappa = 0$, where $\tau \in (-a, a)$. We call the pair (V, Ψ), $\Psi = (\varphi, \tau, \kappa)$, given by (43.4) and (43.5), the *first adapted chart* to the Möbius strip $M_{R,a}$.

In order to obtain an atlas on X (43.3), we complete (V, Ψ) by an additional chart on X as follows. Consider an open subset $\bar{V}$ of $\mathbb{R}^3$,

$$\bar{V} = \mathbb{R}^3 \setminus \left([0, \infty) \times \{0\} \times \mathbb{R}\right), \tag{43.7}$$

and $\bar{\Psi} = (\bar{\varphi}, \bar{\tau}, \bar{\kappa})$, where

$$\begin{aligned}
\bar{\varphi} &= \begin{cases} \operatorname{atan2}(y, x), & y \geq 0, \\ \operatorname{atan2}(y, x) + 2\pi, & y < 0, \end{cases} \\
\bar{\tau} &= -\left(\sqrt{x^2+y^2} - R\right)\frac{\sqrt{2}}{2}\sqrt{1 + \frac{x}{\sqrt{x^2+y^2}}} - \operatorname{sgn}(y) z \frac{\sqrt{2}}{2}\sqrt{1 - \frac{x}{\sqrt{x^2+y^2}}}, \\
\bar{\kappa} &= \left(\sqrt{x^2+y^2} - R\right)\operatorname{sgn}(y)\frac{\sqrt{2}}{2}\sqrt{1 - \frac{x}{\sqrt{x^2+y^2}}} - z\frac{\sqrt{2}}{2}\sqrt{1 + \frac{x}{\sqrt{x^2+y^2}}}.
\end{aligned} \tag{43.8}$$

The inverse mapping $\bar{\Psi}^{-1} : \bar{\Psi}(\bar{V}) \to \bar{V}$ has the equations

$$x = \left(R + \bar{\tau}\cos\frac{\bar{\varphi}}{2} - \bar{\kappa}\sin\frac{\bar{\varphi}}{2}\right)\cos\bar{\varphi}, \quad y = \left(R + \bar{\tau}\cos\frac{\bar{\varphi}}{2} - \bar{\kappa}\sin\frac{\bar{\varphi}}{2}\right)\sin\bar{\varphi},$$

$$z = \bar{\tau}\sin\frac{\bar{\varphi}}{2} + \bar{\kappa}\cos\frac{\bar{\varphi}}{2},$$

where the domain $\bar{\Psi}(\bar{V})$ is an open subset of $\mathbb{R}^3$, expressed by

$$\bar{\Psi}(\bar{V}) = \left\{(\bar{\varphi}, \bar{\tau}, \bar{\kappa}) \in \mathbb{R}^3 \mid 0 < \bar{\varphi} < 2\pi,\ \bar{\tau} \in \mathbb{R},\ \bar{\kappa} < \frac{R + \bar{\tau}\cos(\bar{\varphi}/2)}{\sin(\bar{\varphi}/2)}\right\}.$$

Again, it is easy to observe that $\bar{\Psi}(\bar{V})$ is bounded by the surface (see Fig. 43.4) of the equation

$$\bar{\kappa} = \frac{R + \bar{\tau}\cos(\bar{\varphi}/2)}{\sin(\bar{\varphi}/2)},$$

and it lies below this surface.

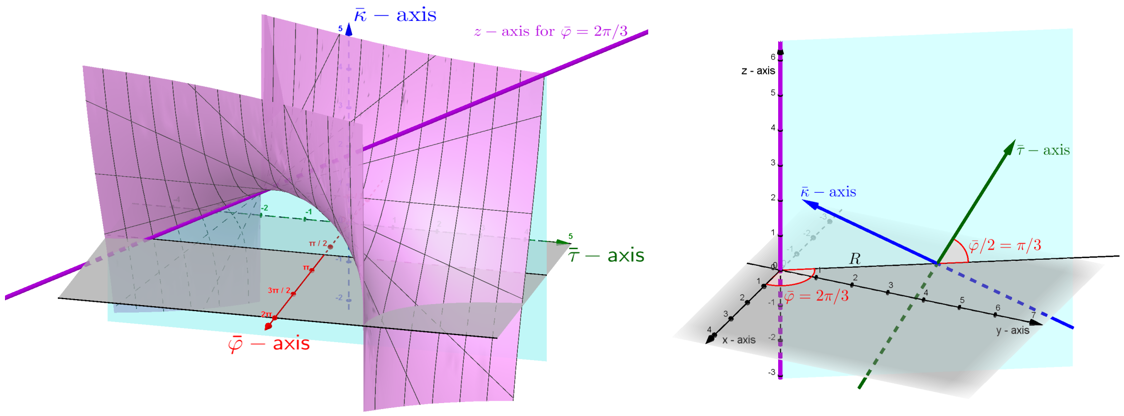

Fig. 43.4 Border surface and one of underlying halfplane for second coordinate system

Source: own elaboration

In the chart $(\bar{V}, \bar{\Psi})$, $M_{R,a}$ is characterized by the equation $\bar{\kappa} = 0$, where $\bar{\tau} \in (-a, a)$. We call $(\bar{V}, \bar{\Psi})$ $\bar{\Psi} = (\bar{\varphi}, \bar{\tau}, \bar{\kappa})$, given by (43.7) and (43.8), the *second adapted chart* to the Möbius strip $M_{R,a}$. Clearly $X = V \cup \bar{V}$. It remains to show that the coordinate transformations between (V, Ψ) and $(\bar{V}, \bar{\Psi})$ are differentiable. Indeed, restricting both charts onto $V \cap \bar{V} = \mathbb{R}^3 \setminus (\mathbb{R} \times \{0\} \times \mathbb{R})$,we have the coordinate transformations $\Psi \circ \bar{\Psi}^{-1} : \bar{\Psi}(\bar{V}) \setminus \{\bar{\varphi} = \pi\} \to \Psi(V) \setminus \{\varphi = 0\}$,

$$\Psi \circ \bar{\Psi}^{-1}(\bar{\varphi}, \bar{\tau}, \bar{\kappa}) = \begin{cases} (\bar{\varphi}, \bar{\tau}, \bar{\kappa}), & \bar{\varphi} \in (0, \pi), \\ (\bar{\varphi} - 2\pi, -\bar{\tau}, -\bar{\kappa}), & \bar{\varphi} \in (\pi, 2\pi), \end{cases}$$

and $\bar{\Psi} \circ \Psi^{-1} : \Psi(V) \setminus \{\varphi = 0\} \to \bar{\Psi}(\bar{V}) \setminus \{\bar{\varphi} = \pi\}$,

$$\bar{\Psi} \circ \Psi^{-1}(\varphi, \tau, \kappa) = \begin{cases} (\varphi, \tau, \kappa), & \varphi \in (0, \pi), \\ (\varphi + 2\pi, -\tau, -\kappa), & \varphi \in (-\pi, 0). \end{cases}$$

Thus we obtained a smooth atlas on X (43.3), consisting of the charts (V, Ψ) and $(\bar{V}, \bar{\Psi})$, which is called *adapted* to the Möbius strip $M_{R,a}$.

43.4 VARIATIONAL EQUATIONS ON THE MÖBIUS STRIP $M_{R,a}$

Consider now the canonical embedding

$$\iota : \mathbb{R} \times M_{R,a} \to \mathbb{R} \times \mathbb{R}^3, \tag{43.9}$$

as a morphism of fibered manifolds over the identity $\mathrm{id}_{\mathbb{R}}$ of the real line $\mathbb{R}$. The jet prolongation

$$J^2\iota : J^2(\mathbb{R} \times M_{R,a}) \to J^2(\mathbb{R} \times \mathbb{R}^3) \tag{43.10}$$

of ι is defined for any section $\gamma : U \to \mathbb{R} \times M_{R,a}$, $U \subset \mathbb{R}$ open, by $J^2\iota(J^2_t\gamma) = J^2_t(\iota \circ \gamma)$. Note that the mapping $J^2\iota$ (43.10) acts on differential forms, defined on $J^2(\mathbb{R} \times \mathbb{R}^3)$, by means of the pull-back operation. We find a chart expression of $J^2\iota$ by a straightforward calculation. Let $(V \cap M_{R,a}, \Psi|_{V\cap M_{R,a}})$, $\Psi|_{V\cap M_{R,a}} = (\varphi, \tau)$, be the chart on $M_{R,a}$, associated with the first adapted chart (V, Ψ), $\Psi = (\varphi, \tau, \kappa)$, (43.4), (43.5). Suppose ι (43.9) is expressed by equations

$$t \circ \iota = t, \quad x \circ \iota = f^1(\varphi, \tau), \quad y \circ \iota = f^2(\varphi, \tau), \quad z \circ \iota = f^3(\varphi, \tau),$$

where

$$f^1(\varphi, \tau) = \left(R + \tau \cos\frac{\varphi}{2}\right)\cos\varphi, \;\; f^2(\varphi, \tau) = \left(R + \tau \cos\frac{\varphi}{2}\right)\sin\varphi, \;\; f^3(\varphi, \tau) = \tau \sin\frac{\varphi}{2},$$

with respect to the Cartesian coordinates (t, x, y, z) on $\mathbb{R} \times \mathbb{R}^3$ and the associated coordinates (t, φ, τ) on $\mathbb{R} \times (V \cap M_{R,a})$. Then $J^2\iota$ is expressed in the associated coordinates $(t, \varphi, \tau, \dot\varphi, \dot\tau, \ddot\varphi, \ddot\tau)$ by

$$\begin{aligned}
&t \circ J^2\iota = t, \quad x \circ J^2\iota = f^1(\varphi, \tau), \quad y \circ J^2\iota = f^2(\varphi, \tau), \quad z \circ J^2\iota = f^3(\varphi, \tau),\\
&\dot x \circ J^2\iota = \frac{d}{dt}f^1(\varphi, \tau), \quad \dot y \circ J^2\iota = \frac{d}{dt}f^2(\varphi, \tau), \quad \dot z \circ J^2\iota = \frac{d}{dt}f^3(\varphi, \tau),\\
&\ddot x \circ J^2\iota = \frac{d^2}{dt^2}f^1(\varphi, \tau), \quad \ddot y \circ J^2\iota = \frac{d^2}{dt^2}f^2(\varphi, \tau), \quad \ddot z \circ J^2\iota = \frac{d^2}{dt^2}f^3(\varphi, \tau),
\end{aligned}$$

where $df^i/dt = (\partial f^i/\partial\varphi)\dot\varphi + (\partial f^i/\partial\tau)\dot\tau$, $i = 1, 2, 3$, is the formal derivative operator acting on functions.

Consider a system of second-order differential equations on $J^2(\mathbb{R} \times \mathbb{R}^3)$,

$$\varepsilon_i(x, y, z, \dot x, \dot y, \dot z, \ddot x, \ddot y, \ddot z) = 0, \quad i = 1, 2, 3, \tag{43.11}$$

for unknowns the curves $t \to (x(t), y(t), z(t))$. We assign to the left-hand sides of (43.11) a differential source form

$$\varepsilon = \varepsilon_i \omega^i \wedge dt, \tag{43.12}$$

where $\omega^1 = dx - \dot x dt$, $\omega^2 = dy - \dot y dt$, and $\omega^3 = dz - \dot z dt$, are contact 1-forms satisfying the property $(J^1\gamma)^*\omega^i = 0$ for every section γ of $\mathbb{R}\times\mathbb{R}^3$. In accordance with the general theory of variational equations we say that the system (43.11) (or the source form (43.12)) is *locally variational*, if there exists a real-valued function $\mathscr{L} = \mathscr{L}(x, y, z, \dot{x}, \dot{y}, \dot{z})$ for which (43.11) is the system of the *Euler–Lagrange equations*, i.e.

$$\varepsilon_1 = \frac{\partial \mathscr{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{x}}, \quad \varepsilon_2 = \frac{\partial \mathscr{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{y}}, \quad \varepsilon_3 = \frac{\partial \mathscr{L}}{\partial z} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{z}}. \tag{43.13}$$

If $\mathscr{L}$ exists, it is called the Lagrange function for the system (43.11), and (43.11) represents the system of equations for extremals of a variational functional, associated with $\mathscr{L}$.

The necessary and sufficient conditions for (local) variationality of systems of differential equations are the well-known *Helmholtz conditions*. For second-order system (43.11) we have the following result. Denote $x^1 = x$, $x^2 = y$, $x^3 = z$, and analogously the dot coordinates.

Theorem 43.1 *Let ε be a source form given by* (43.12). *The following conditions are equivalent:*

(a) *The equations* (43.13) *has a solution.*

(b) *The functions ε_i satisfy the conditions*

$$\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} - \frac{\partial \varepsilon_j}{\partial \ddot{x}^i} = 0, \quad \frac{\partial \varepsilon_i}{\partial \dot{x}^j} + \frac{\partial \varepsilon_j}{\partial \dot{x}^i} - \frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \ddot{x}^j} + \frac{\partial \varepsilon_j}{\partial \ddot{x}^i}\right) = 0,$$
$$\frac{\partial \varepsilon_i}{\partial x^j} - \frac{\partial \varepsilon_j}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \varepsilon_i}{\partial \dot{x}^j} - \frac{\partial \varepsilon_j}{\partial \dot{x}^i}\right) = 0.$$

Proof For the proof see e.g. Sarlet [6], or Krupka [1].

The *induced source form* $(J^2\iota)^*\varepsilon$ on $J^2(\mathbb{R} \times M_{R,a})$ has the expression

$$(J^2\iota)^*\varepsilon = \varepsilon_\varphi \omega^\varphi \wedge dt + \varepsilon_\tau \omega^\tau \wedge dt, \tag{43.14}$$

where $\omega^\varphi = d\varphi - \dot{\varphi}dt$, $\omega^\tau = d\tau - \dot{\tau}dt$, and

$$\begin{aligned}
\varepsilon_\varphi = {} & (\varepsilon_1 \circ J^2\iota)\big(-R\sin\varphi + (\tau/2)\sin(\varphi/2) - 3\tau\cos^2(\varphi/2)\sin(\varphi/2)\big) \\
& + (\varepsilon_2 \circ J^2\iota)\big(R\cos\varphi + \tau\cos(\varphi/2) - 3\tau\cos(\varphi/2)\sin^2(\varphi/2)\big) \\
& + (\varepsilon_3 \circ J^2\iota)(\tau/2)\cos(\varphi/2), \\
\varepsilon_\tau = {} & (\varepsilon_1 \circ J^2\iota)\big(\cos(\varphi/2) - 2\cos(\varphi/2)\sin^2(\varphi/2)\big) \\
& + 2(\varepsilon_2 \circ J^2\iota)\sin(\varphi/2)\cos^2(\varphi/2) + (\varepsilon_3 \circ J^2\iota)\sin(\varphi/2).
\end{aligned}$$

By Theorem 43.1, the induced source form $(J^2\iota)^*\varepsilon$ is locally variational if and only if its components ε_φ and ε_τ satisfy

$$\begin{gathered}
\frac{\partial \varepsilon_\varphi}{\partial \ddot{\tau}} - \frac{\partial \varepsilon_\tau}{\partial \ddot{\varphi}} = 0, \quad \frac{\partial \varepsilon_\varphi}{\partial \dot{\varphi}} - \frac{d}{dt}\frac{\partial \varepsilon_\varphi}{\partial \ddot{\varphi}} = 0, \quad \frac{\partial \varepsilon_\tau}{\partial \dot{\tau}} - \frac{d}{dt}\frac{\partial \varepsilon_\tau}{\partial \ddot{\tau}} = 0, \\
\frac{\partial \varepsilon_\varphi}{\partial \dot{\tau}} + \frac{\partial \varepsilon_\tau}{\partial \dot{\varphi}} - \frac{d}{dt}\left(\frac{\partial \varepsilon_\varphi}{\partial \ddot{\tau}} + \frac{\partial \varepsilon_\tau}{\partial \ddot{\varphi}}\right) = 0, \quad \frac{\partial \varepsilon_\varphi}{\partial \tau} - \frac{\partial \varepsilon_\tau}{\partial \varphi} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_\varphi}{\partial \dot{\tau}} - \frac{\partial \varepsilon_\tau}{\partial \dot{\varphi}}\right) = 0.
\end{gathered} \tag{43.15}$$

If these conditions are satisfied, then the variational sequence theory implies that $(J^2\iota)^*\varepsilon$, or the system of equations $\varepsilon_\varphi = 0$ and $\varepsilon_\tau = 0$, is also *globally variational*, i.e. for $(J^2\iota)^*\varepsilon$ there exists a Lagrange function $\widetilde{\mathscr{L}}$ defined on the jet prolongation $J^2(\mathbb{R} \times M_{R,a})$ of the Möbius strip $M_{R,a}$.

CONCLUSION

We present the application of the variational theory on smooth manifolds. A smooth atlas in $\mathbb{R}^3$, adapted to the Möbius strip is described and used for analysis of restrictions of ordinary differential equations from a Euclidean space to the Möbius strip.

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VARIATIONAL EQUATIONS ON THE MÖBIUS STRIP

Abstract: *In this paper, systems of second-order ordinary differential equations (or dynamical forms in Lagrangian mechanics), induced by the canonical embedding of the two-dimensional Möbius strip into the Euclidean space, are considered in the class of variational equations. For a given non-variational system, the conditions assuring variationality (Helmholtz conditions) for the induced system on the Möbius strip are formulated. The theory contributes to variational foundations of geometric mechanics.*

Keywords: *Lagrangian; Euler-Lagrange equations; Helmholtz conditions; fibered manifold; Möbius strip.*

VARIAČNÍ ROVNICE NA MÖBIOVĚ PÁSCE

Abstrakt: *V tomto článku je studována variačnost systémů obyčejných diferenciálních rovnic (dynamických forem v geometrické mechanice) druhého řádu, kterou indukuje kanonické vložení dvojrozměrné Möbiovy pásky do Euklidova prostoru. Pro daný nevariační systém rovnic jsou formulovány nutné a postačující podmínky variačnosti (Helmholtzovy podmínky). Práce je příspěvkem k variačním základům geometrické mechaniky na Möbiově pásce.*

Klíčová slova: *Lagrangian; Euler-Lagrangeovy rovnice; Helmholtzovy podmínky; fibrovaná varieta; Möbiova páska.*

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