Guo Renkuan
University of Cape Town, Cape Town, South Africa

An univariate DEMR modelling on repair effects

Keywords

grey theory, DEMR, repair effect, credibility measure theory, random fuzzy variable

Abstract

Repairable system analysis is in nature an evaluation of repair effects. Recent tendency in reliability engineering literature was estimating system repair effects or linking repair to certain covariate to extract repair impacts by imposing repair regimes during system reliability analysis. In this paper, we develop a differential equation motivated regression (abbreviated as DEMR) model with a random fuzzy error term based on the axiomatic framework of self-dual fuzzy credibility measure theory proposed by Liu [5] and grey differential equation models. The fuzzy variable indexes the random fuzzy error term will be used to facilitate the evaluation of repair effects. We further propose a parameter estimation approach for the fuzzy variable (repair effect) under the maximum entropy principle.

1. Introduction

Repairable system analysis is in nature an evaluation of repair effects. Recent tendency in reliability engineering literature was estimating system repair effects or linking repair to certain covariate to extract repair impacts by imposing repair regimes to the system. Guo [3], [4] proposed an approach to isolate repair effects in terms of grey differential equation modelling, particularly, the one-variable first order differential equation model, abbreviated as GM (1,1) model, initiated by Deng [2]. The efforts of modelling of system repair effects in terms of grey differential equation models has attracted attention from because it is easy to calculated, for example, in Microsoft Excel. However, there were two fundamental problems necessary to be addressed. The first issue is the nature of the GM(1,1) model. In The second fundamental problem is GM(1,1) model is a deterministic approach and is just a delicate approximation approach and in nature ignores the regression error structure, which may be very reasonable if the sample size is too small, however, in general, Deng's approach results in information loss, particularly he used the adjective word "grey", implying grey uncertainty involved, but there was not uncertainty structure build up to describe "grey uncertainty". In other words, the existing GM(1,1) model has a good idea without a convincingly rigorous mathematical foundation yet.

In this paper, we will review the coupling principle materialization in GM(1,1) model in section 2. In section 3, will propose a families of first order differential equation motivated regression models under unequal-gaped data, which is suitable for the usages in system functioning time analysis. In section 4, we argue that the differential equation motivated regression model is a coupling regression model with random fuzzy error terms in nature. In section 5, review Liu's [5] fuzzy credibility measure theory and then discuss the random fuzzy variable theory in order to establish the differential equation motivated regression models as a coupling regression with random fuzzy error terms. In section 6, we will discuss the parameter estimation for the fuzzy variable repair effect indexing the random fuzzy error terms of the differential equation motivated regression modelling on system functioning time sequence under maximum entropy principle. Section 7 concludes the paper.

2. An univariate DEMR model

The success of GM(1,1) model lies on the following two aspects: data accumulative generation operator (abbreviated as AGO), which is the partial sum operation in algebra, and a simple regression model coupled with a first-order linear constant coefficient differential equation model, which Deng [2] called is as whitening differential equation or the shadow differential equation. Let $X^{(0)}=(x^{(0)}(1), x^{(0)}(2),\ldots, x^{(0)}(n))$ be a data sequence, and the partial sum with respect to $X^{(0)}$

$$x^{(1)}(k)=\sum_{i=1}^{k}x^{(0)}(i),\ k=2,3,4,\ldots,n, \tag{1}$$

and the mean of two consecutive partial sums, which is used as an approximation to the primitive function of $x^{(1)}(t)$

$$z^{(1)}(k)=\frac{1}{2}\left[x^{(1)}(k)+x^{(1)}(k-1)\right]. \tag{2}$$

Definition 1. Given a (strictly positive) discrete real-valued data sequence $X^{(0)}=(x^{(0)}(1), x^{(0)}(2),\ldots, x^{(0)}(n))$, the equation

$$x^{(0)}(k)=\alpha+\beta\left(-z^{(1)}(k)\right)+\varepsilon_k, \tag{3}$$
$$k=2,3,4,\ldots,n,$$

"coupled" with the first-order constant coefficient linear ordinary differential equation.

$$\begin{cases}\dfrac{dx^{(1)}(t)}{dt}+\beta x^{(1)}(t)=\alpha \\ x^{(0)}(k)=\alpha+\beta\left(-z^{(1)}(k)\right)+\varepsilon_k,\ k=2,3,4,\ldots,n\end{cases} \tag{4}$$

is called a univariate DEMR model with respect to the data sequence $X^{(0)}=(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))$. Parameter β is called the developing coefficient, parameter α is the grey input, term $x^{(0)}$ is called a *grey derivative* and term $x^{(1)}(k)$ is called the k^{th} 1-AGO of $X^{(0)}$ value (partial sum in fact). Furthermore, the differential equation $dx^{(1)}/dt+\beta x^{(1)}=\alpha$ in Eq. (4) is called the whitening differential equation or the *shadow* equation of the grey differential equation Eq. (3) by Deng [2]. The unknown parameter values (α,β) can be estimated in terms of a standard regression. Note that Eq. (3) can be re-written as in a simple regression form,

$$y_k=\alpha+\beta x_k+\varepsilon_k,\ k=2,3,4,\ldots,n, \tag{5}$$

where

$$y_k=x^{(0)}(k),\ x_k=-z^{(1)}(k),\ k=2,3,4,\ldots,n. \tag{6}$$

The estimate for regression parameter pair (α,β), denoted as (a,b), can be calculated by,

$$(a,b)^T=\left(X^TX\right)^{-1}X^TY \tag{7}$$

where

$$X=\begin{bmatrix}1 & -z^{(1)}(2)\\ 1 & -z^{(1)}(3)\\ \vdots & \vdots\\ 1 & -z^{(1)}(n)\end{bmatrix},\ Y=\begin{bmatrix}x^{(0)}(2)\\ x^{(0)}(3)\\ \vdots\\ x^{(0)}(n)\end{bmatrix}. \tag{8}$$

The grey filtering-prediction equation is thus

$$\hat{x}^{(0)}(k)=\hat{x}^{(1)}(k)-\hat{x}^{(1)}(k-1), \tag{9}$$

where

$$\hat{x}^{(1)}(k+1)=\left[x^{(0)}(1)-\frac{a}{b}\right]e^{-bk}+\frac{a}{b}. \tag{10}$$

Note that Eq. (10) is the discrete version of the solution to the differential equation (Eq. (4))

$$x^{(1)}=\left[x^{(1)}(0)-\frac{\alpha}{\beta}\right]e^{-\beta t}+\frac{\alpha}{\beta}. \tag{11}$$

The typical goodness-of-fit measure of GM(1,1) model is the (absolute) relative error described by Deng [2], i.e.

$$e(k)=\frac{\left|x^{(0)}(k)-\hat{x}^{(0)}(k)\right|}{x^{(0)}(k)},\ k=2,3,4,\ldots,n, \tag{12}$$

and the model efficiency is defined as

$$E=\frac{1}{n-1}\sum_{i=2}^{n}e(i). \tag{13}$$

The nature of the univariate DEMR model can be identified as that the model couples a differential equation model and a simple regression model together organically. The form of the motivated differential equation (i.e., Deng's whitening differential equation) in Eq. (4) determines the form of the coupling regression (i.e., CREG) in Eq. (3). The data assimilated parameter pair (a,b) in CREG determines the system parameter pair (α,β). The coupling translation rule is listed in *Table 1*.

Table 1. Coupling Rule in Univariate DEMT Model

Term	Motivated DE	Coupling REG
Translation between MDE and CREG		
Intrinsic feature	Continuous	Discrete
Independent Variable	t	k
1st-order Derivative	$dx^{(1)}(t)/dt$	$x^{(0)}(k)$
2st-order Derivative	$d^{(2)}x^{(1)}(t)/dt^2$	$x^{(-1)}(k)$
Primitive function	$x^{(1)}(t)$	$z^{(1)}(k)$
Model Formation	$\frac{dx^{(1)}(t)}{dt}+\beta x^{(1)}(t)=\alpha$	$x^{(0)}(k)+bz^{(1)}(k)=a$
Parameter Coupling		
Parameter	(α,β)	(a,b)
Dynamics (Solution)	$x^{(1)}(t)=$ $\left[x^{(1)}(0)-\frac{\alpha}{\beta}\right]e^{-\beta t}+\frac{\alpha}{\beta}$	$\hat{x}^{(1)}(k+1)=$ $\left[x^{(0)}(1)-\frac{a}{b}\right]e^{-bk}+\frac{a}{b}$
Filtering (Prediction)	$x^{(0)}(t)=$ $\left[\alpha-\beta x^{(0)}(1)\right]e^{-\beta t}$	$\hat{x}^{(0)}(t)=$ $\hat{x}^{(1)}(k+1)-\hat{x}^{(1)}(k)$

In DEMR modelling, the motivated differential equation and the coupling regression model are not separable but are organic integration. The DEMR models are differential equation motivated but defined by system data. A DEMR model starts with a motivated differential equation, then the coupling regression model is specified in the form "translated" from the form of the motivated differential equation, in return, in terms of coupling regression model, the parameters specifying the motivated differential equation are estimated under L_2-optimality, and finally, the solution to the motivated differential equation (or the discredited solution) equipped with data-assimilated parameters is used for system analysis or prediction. In nature a DEMR model is a coupling of a motivated differential equation and a regression formed by the discredited version of the motivated differential equation. We call the "translation" rule in grey differential equation modelling as a coupling principle.

3. Unequal-gapped differential equation motivated regression model with term of product of exponential and sine function

The basic form of the first order linear differential equation with constant term in right side is

$$\frac{dx}{dt}+\beta x=\alpha e^{\delta t}\sin(\omega t+\varpi) \tag{14}$$

Note here, the proposal of the motivated differential equation in Eq. (14) is featured by the term $\alpha e^{\delta t}\sin\omega t$ to replacing the constant term α in Eq. (4) with an intention that the fluctuating pattern of $e^{\delta t}\sin(\omega t+\varpi)$ will help the model goodness-of-fit.

Then the solution to Eq. (14) is

$$x=x_h+x_p \tag{15}$$

where

$$x_h=c_0 e^{-\beta t}, \tag{16}$$

is the solution to the homogeneous equation

$$\frac{dx}{dt}+\alpha x=0 \tag{17}$$

while a particular solution to the motivated differential equation Eq. (14) takes a form

$$x_p=e^{\delta t}\left(A_0\sin(\omega t+\varpi)+B_0\cos(\omega t+\varpi)\right). \tag{18}$$

Note that x_p satisfies Eq. (14), thus substitute the particular solution into Eq. (14), we obtain

$$\begin{aligned}\frac{dx_p}{dt}+\beta x_p &= A_0\delta e^{\delta t}\sin(\omega t+\varpi)+B_0\delta e^{\delta t}\cos(\omega t+\varpi)\\ &\quad+A_0\omega e^{\delta t}\cos(\omega t+\varpi)+B_0\omega e^{\delta t}\sin(\omega t+\varpi)\\ &\quad+A_0\beta e^{\delta t}\sin(\omega t+\varpi)+B_0\beta e^{\delta t}\cos(\omega t+\varpi)\\ &=\alpha e^{\delta t}\sin(\omega t+\varpi),\end{aligned} \tag{19}$$

which leads to an equation system by comparing the coefficients of term $e^{\delta t}\sin(\omega t+\varpi)$ and term $e^{\delta t}(\omega t+\varpi)$ respectively,

$$\begin{cases}A_0(\delta+\beta)-B_0\omega=\alpha\\ A_0\omega+B_0(\delta+\beta)=0\end{cases} \tag{20}$$

Solving the linear equation Eq. (19), we obtain the coefficients A_0 and B_0 respectively as follows

$$\begin{cases} A_0 = \dfrac{(\delta+\beta)\alpha}{\omega^2+(\beta+\delta)^2} \\ \\ B_0 = -\dfrac{\alpha\omega}{\omega^2+(\beta+\delta)^2} \end{cases} \tag{21}$$

In theory, the expressions of A_0 and B_0 will determine the particular solution x_p

$$x_p = A_0 e^{\delta t}\sin(\omega t+\varpi) + B_0 e^{\delta t}\cos(\omega t+\varpi) \tag{22}$$

which will result in the general solution to Eq. (14) as

$$x = c_1 e^{-\beta t} + A_0 e^{\delta t}\sin(\omega t+\varpi) + B_0 e^{\delta t}\cos(\omega t+\varpi) \tag{23}$$

Note that for the unequal-gapped data sequence, $X^{(0)} = \left(x^{(0)}(t_1), x^{(0)}(t_2), \cdots, x^{(0)}(t_n)\right)$, the coupling (or translation) rule is slightly different from the equal-gapped data sequence.

Table 2. Coupling Principle in unequal gapped GM(1,1) Model.

Term	Motivated DE	Coupling REG
Model Formation		
Intrinsic feature	Continuous	Discrete
Independent Variable	t	t_k
Response	$x^{(0)}(t)$	$x^{(0)}(t_k)$
1st-order Derivative	$dx^{(1)}(t)/dt$	$x^{(0)}(t_k)$
2nd-order Derivative	$d^2x^{(1)}(t)/dt^2$	$\dfrac{x^{(0)}(t_k) - x^{(0)}(t_{k-1})}{t_k - t_{k-1}}$
Primitive function	$x^{(1)}(t)$	$z^{(1)}(t_k)$
Data Assimilation in Model		
Parameter	(α,β)	(a,b)
Dynamic law	$\frac{dx}{dt}+\beta x = \alpha e^{\delta t}\sin(\omega t+\varpi)$	$x^{(0)}(t_k) = \alpha e^{\delta t_k}\sin(\omega t_k+\varpi)+\beta\left(-z^{(1)}(t_k)\right)$
Dynamics (Solution)	$x^{(1)}(t) = c_1 e^{-\beta t}$ $+A_0 e^{\delta t}\sin(\omega t+\varpi)$ $+B_0 e^{\delta t}\cos(\omega t+\varpi)$	$x^{(1)}(t_k) = c_1 e^{-\beta t_k}$ $+A_0 e^{\delta t_k}\sin(\omega t_k+\varpi)$ $+B_0 e^{\delta t_k}\cos(\omega t_k+\varpi)$
Filtering (Prediction)	$x^{(0)}(t) = -\beta c_1 e^{-\beta t}$ $+(A_0\omega+B_0\delta)e^{\delta t}\cos(\omega t+\varpi)$ $+(A_0\delta-B_0\omega)e^{\delta t}\sin(\omega t+\varpi)$	$x^{(0)}(t_k) = -\beta c_1 e^{-\beta t_k}$ $+(A_0\omega+B_0\delta)e^{\delta t}\cos(\omega t_k+\varpi)$ $+(A_0\delta-B_0\omega)e^{\delta t}\sin(\omega t_k+\varpi)$

The coupling regression is

$$x^{(0)}(t_k) = \alpha e^{\delta t_k}\sin(\omega t_k+\varpi) + \beta\left(-z^{(1)}(t_k)\right) + \varepsilon_k,\ k = 2,3,4,\ldots,n, \tag{24}$$

where

$$z^{(1)}(t_1) = z^{(0)}(t_1)t_1$$

$$z^{(1)}(t_k) = z^{(1)}(t_{k-1}) + z^{(0)}(t_k)(t_k - t_{k-1})$$

$$k = 2,3,4,\ldots,n. \tag{25}$$

The parameter pair (α,β) is obtained by least-square estimation $(a,b)^T = \left(X^TX\right)^{-1}X^TY$, where

$$X = \begin{bmatrix} e^{\delta t_2}\sin(\omega t_1+\varpi) & -z^{(1)}(t_1) \\ e^{\delta t_3}\sin(\omega t_2+\varpi) & -z^{(1)}(t_2) \\ \vdots & \vdots \\ e^{\delta t_n}\sin(\omega t_n+\varpi) & -z^{(1)}(t_n) \end{bmatrix},$$

$$Y = \begin{bmatrix} z^{(0)}(t_1) \\ z^{(0)}(t_2) \\ \vdots \\ z^{(0)}(t_n) \end{bmatrix} \tag{26}$$

since δ and ω are given (in a manner by trials and errors).
Formally, we have a DEMR model as

$$\begin{cases} \dfrac{dx}{dt} + \beta x = \alpha e^{\delta t}\sin(\omega t+\varpi) \\ \\ x^{(0)}(t_k) = \alpha e^{\delta t}\sin(\omega t+\varpi) + \beta\left(-z^{(1)}(t_k)\right) + \varepsilon_k. \end{cases} \tag{27}$$

4. Fuzzy repair effect structure

In standard regression modelling exercises, it is often to assume that the error terms ε_i, $i = 1,2,\cdots,n$ are random with zero mean and constant variance, i.e., $\mathrm{E}[\varepsilon_i] = 0$ and $\mathrm{VAR}[\varepsilon_i] = \sigma^2$, $i = 1,2,\cdots,n$. It is typically assuming a normal distribution with zero mean and constant variance, i.e., $N\left(0,\sigma^2\right)$.

Furthermore, as we pointed out that a grey differential equation model is a motivated differential equation motivated regression, which takes the form translated from the motivated differential equation, as shown in *Table 1* for GM(1,1) case. However, we should be fully aware that translation back and forward between the motivated differential equation and the coupling regression will bring in new error which is different from the random sampling error $N\left(0,\sigma^2\right)$. The errors brought in come from the steps of the usage of difference $x^{(0)}(k)=x^{(1)}(k)-x^{(1)}(k-1)$ to replace the derivative $\left(dx/dt\right)_{t=k}$ and the usage of the average accumulated partial sum $z^{(1)}(t_k)$ to replace the primitive function $x^{(1)}(t_k)$ during the translation between the motivated differential equation and the coupling regression.

Our simulation studies have shown that the coupling-introduced error is dependent upon the grids size Δ, or equivalent to the total number of approximation N. The simulation evidences have shown that the larger the number of approximating grid, or equivalently, the smaller the approximating grid, the coupling translation error is smaller. However, the coupling translation error and the approximating grid do not hold a deterministic functional relation. What we can see is the functional relation has a certain degree of belongingness. In other words, the coupling translation process induces a fuzzy error term, denoted as ς with a membership function.

We perform a simulation study of the error occurrence frequencies of approximating $\cos\left(\pi/2\right)$ by $\left(\sin\left(\pi/2\right)-\sin\left(\pi/2+\Delta x\right)\right)/\Delta x$.

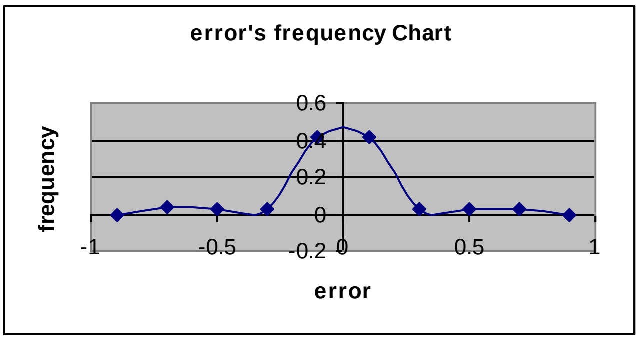

Figure 1. Error occurrence frequency

Therefore, in general the error terms of a differential equation motivated regression model (i.e., grey differential equation in current grey theory literature) is fuzzy because the vague nature of the error occurrences.

As a standard exercise, the fuzzy error component e_i may be assumed as triangular fuzzy variable with a membership function

$$\mu_e(s)=\begin{cases}\dfrac{s+o}{o} & \text{if } -o\le s<0\\[2mm] \dfrac{o-s}{o} & \text{if } 0\le s\le o\\[2mm] 0 & \text{otherwise}\end{cases} \qquad (28)$$

which has a fuzzy mean zero.

However, in the modelling of system functioning times, we further note that the repair will reset the system dynamic rule so that the repair impact may be understood as a fuzzy variable having a triangular membership

$$\mu_r(z)=\begin{cases}\dfrac{z-a}{b-a} & \text{if } a\le z<b\\[2mm] \dfrac{c-z}{c-b} & \text{if } b\le z<c\\[2mm] 0 & \text{otherwise}\end{cases} \qquad (29)$$

The fuzzy mean of the fuzzy repair effect is thus

$$E_\mu(r)=\frac{1}{4}(a+2b+c), \qquad (30)$$

which provides a repair effect structure. Therefore, the "composite" fuzzy "error" term appearing in the differential equation motivated regression for modelling a system function time will be

$$\zeta_i=e_i+r_i,\ \ i=2,3,\ldots,n, \qquad (31)$$

with a triangular membership function, i.e.,

$$\mu_\zeta(w)=\begin{cases}\dfrac{w-a+\varpi}{b-a+\varpi} & \text{if } a-\varpi\le w<b\\[2mm] \dfrac{c+\varpi-w}{c+\varpi-b} & \text{if } b\le w<c+\varpi\\[2mm] 0 & \text{otherwise}\end{cases} \qquad (32)$$

because the sum of two triangular fuzzy variables is still a triangular fuzzy variable. The total error

$$\xi_i=\zeta_i+\varepsilon_i=(r_i+e_i)+\varepsilon_i,\ \ i=2,3,\ldots,n, \qquad (33)$$

which is a sequence of random fuzzy variables because the summation nature of a random fuzzy variable and a fuzzy variable according to Liu [5]. Now, we reach a point that the random fuzzy variable concept is involved and therefore it is necessary to have a quick review on the relevant theoretical foundation.

5. A random fuzzy variable foundation

First we need to review the fuzzy credibility measure theory foundation proposed by Liu [5], then we will establish the normal random fuzzy variable theory for a facilitation of error analysis in the differential equation motivated regression models.

Let Θ be a nonempty set, and 2^{Θ} the power set on Θ. Each element, let us say, $A\subset\Theta$, $A\in 2^{\Theta}$ is called an event. A number denoted as $\mathrm{Cr}\{A\}$, $0\le \mathrm{Cr}\{A\}\le 1$, is assigned to event $A\in 2^{\Theta}$, which indicates the credibility grade with which event $A\in 2^{\Theta}$ occurs. $\mathrm{Cr}\{A\}$ satisfies following axioms given by Liu [5]:

Axiom 1. $\mathrm{Cr}\{\Theta\}=1$.

Axiom 2. $\mathrm{Cr}\{\cdot\}$ is non-decreasing, i.e., whenever $A\subset B$, $\mathrm{Cr}\{A\}\le \mathrm{Cr}\{B\}$.

Axiom 3. $\mathrm{Cr}\{\cdot\}$ is self-dual, i.e., for any $A\in 2^{\Theta}$, $\mathrm{Cr}\{A\}+\mathrm{Cr}\{A^c\}=1$.

Axiom 4. $\mathrm{Cr}\{\cup_i A_i\}\wedge 0.5=\sup_i\left[\mathrm{Cr}\{A_i\}\right]$ for any $\{A_i\}$ with $\mathrm{Cr}\{A_i\}\le 0.5$.

Axiom 5. Let set functions $Cr_k\{\cdot\}:2^{\Theta_k}\to[0,1]$ satisfy *Axioms 1-4*, and $\Theta=\Theta_1\times\Theta_2\times\cdots\times\Theta_p$, then:

$$Cr\{\theta_1,\theta_2,\ldots,\theta_p\}=Cr\{\theta_1\}\wedge Cr\{\theta_2\}\wedge\cdots\wedge Cr\{\theta_p\} \quad (34)$$

for each $\{\theta_1,\theta_2,\ldots,\theta_p\}\in 2^{\Theta}$.

Definition 5.1. Liu [5] Any set function $\mathrm{Cr}:2^{\Theta}\to 0,1$ satisfies *Axioms 1-4* is called a $(\vee,\wedge)$-credibility measure (or classical credibility measure). The triple $(\Theta,2^{\Theta},Cr)$ is called the $(\vee,\wedge)$-credibility measure space.

Definition 5.2. Liu [5] A fuzzy variable ξ is a mapping from credibility space $(\Theta,2^{\Theta},Cr)$ to the set of real numbers, i.e., $\xi:(\Theta,2^{\Theta},\mathrm{Cr})\to\mathbb{R}$.

Definition 5.3. Liu [5] The (induced) membership function of a fuzzy variable ξ on $(\Theta,2^{\Theta},Cr)$ is:

$$\mu(x)=\left(2Cr\{\xi=x\}\right)\wedge 1,\ \ x\in\mathbf{R} \quad (35)$$

Conversely, for given membership function the credibility measure is determined by the credibility inversion theorem.

Theorem 5.4. Liu [5] Let ξ be a fuzzy variable with membership function μ. Then for $\forall B\subset\mathbf{R}$,

$$Cr\{\xi\in B\}=\frac{1}{2}\left(\sup_{x\in B}\mu(x)+1-\sup_{x\in B^C}\mu(x)\right) \quad (36)$$

As an example, if the set B is degenerated into a point x, then

$$Cr\{\xi=x\}=\frac{1}{2}\left(\mu(x)+1-\sup_{y\ne x}\mu(y)\right),\ \forall x\in\mathbf{R} \quad (37)$$

Definition 5.5. Liu [5] The credibility distribution $\Phi:\mathbb{R}\to[0,1]$ of a fuzzy variable ξ on $(\Theta,2^{\Theta},Cr)$ is

$$\Phi(x)=Cr\{\theta\in\Theta|\xi(\theta)\le x\}. \quad (38)$$

The credibility distribution $\Phi(x)$ is the accumulated *credibility grade* that the fuzzy variable ξ takes a value less than or equal to a real number $x\in\mathbb{R}$. Generally speaking, the credibility distribution Φ is neither left-continuous nor right-continuous.

Theorem 5.6. Liu [5] Let ξ be a fuzzy variable on $(\Theta,2^{\Theta},Cr)$ with membership function μ. Then its credibility distribution,

$$\Phi(x)=\frac{1}{2}\left(\sup_{y\le x}\mu(y)+1-\sup_{y>x}\mu(y)\right),\ \forall x\in\mathbf{R} \quad (39)$$

Definition 5.7. Liu [5] Let Φ be the credibility distribution of the fuzzy variable ξ. Then function $\phi:\mathbb{R}\to[0,+\infty)$ of a fuzzy variable ξ is called a credibility density function such that,

$$\Phi(x)=\int_{-\infty}^{x}\phi(y)dy,\ \ \forall x\in\mathbf{R}. \quad (40)$$

Now we are ready to state the normal random fuzzy variable theory for the error analysis in the repairable system modelling.

Liu [5] defines a random fuzzy variable as a mapping from the credibility space $(\Theta,2^{\Theta},Cr)$ to a set of random variables. We would like to present a definition similar to that of stochastic process in probability theory and expect readers who are familiar with the basic concept of stochastic processes can understand the comparative definition.

Definition 5.8. A random fuzzy variable, denoted as $\xi = \left\{ X_{\beta(\theta)}, \theta \in \Theta \right\}$, is a collection of random variables X_β defined on the common probability space $(\Omega, \mathfrak{A}, \Pr)$ and indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(\Theta, 2^\Theta, Cr)$.
Similar to the interpretation of a stochastic process, $X = \left\{ X_t, t \in \mathbb{R}^+ \right\}$, a random fuzzy variable is a bivariate mapping from $(\Omega \times \Theta, \mathfrak{A} \times 2^\Theta)$ to the space $(\mathbb{R}, \mathfrak{B})$. As to the index, in stochastic process theory, index used is referred to as time typically, which is a positive (scalar variable), while in the random fuzzy variable theory, the "index" is a fuzzy variable, say, β. Using uncertain parameter as index is not starting in random fuzzy variable definition. In stochastic process theory we already know that the stochastic process $X = \left\{ X_{\tau(\omega)}, \omega \in \Omega \right\}$ uses stopping time $\tau\ \omega$, $\omega \in \Omega$, which is an (uncertain) random variable as its index.
In random fuzzy variable theory, we may say that that average chance measure, denoted as ch, plays a similar role similar to a probability measure, denoted as Pr, in probability theory.

Definition 5.9. Liu and Liu [6] Let ξ be a random fuzzy variable, then the average chance measure denoted by ch $\cdot$, of a random fuzzy event $\{\xi \le x\}$, is

$$ch\{\xi \le x\} = \int_0^1 Cr\{\theta \in \Theta | \Pr\{\xi(\theta) \le x\} \ge \alpha\} d\alpha. \quad (41)$$

Then function $\Psi\ \cdot$ is called as average chance distribution if and only if

$$\Psi(x) = ch\{\xi \le x\}. \quad (42)$$

Liu [5] stated that if a random variable η has zero mean and a fuzzy variable ζ, then the sum of the two, $\eta + \zeta$, results in a random fuzzy variable ξ. Now, it is time to find the average chance distribution for a normal random fuzzy variable $\xi \overset{d}{\sim} N\ \zeta, \sigma^2$, where ζ is a triangular fuzzy variable and σ^2 is a given positive real number. Note that fuzzy event

$$\{\theta \in \Theta : \Pr\{\xi(\theta) \le x\} \ge \alpha\}$$

$$\Leftrightarrow \left\{ \theta \in \Theta : \Phi\left(\frac{x - \zeta(\theta)}{\sigma} \right) \ge \alpha \right\}$$

(43)

$$\Leftrightarrow \left\{ \theta \in \Theta : \mathrm{x} \ge \zeta(\theta) + \sigma\Phi^{-1}(\alpha) \right\}$$

$$\Leftrightarrow \left\{ \theta \in \Theta : \zeta(\theta) \le x - \sigma\Phi^{-1}(\alpha) \right\} \quad (43)$$

The fuzzy mean is assumed to have a triangular membership function

$$\mu_\zeta(w) = \begin{cases} \dfrac{w - a_\zeta}{b_\zeta - a_\zeta} & a_\zeta \le w \le b_\zeta \\ \dfrac{c_v - w}{c_v - b_v} & b_\zeta \le w \le a_\zeta \\ 0 & \text{otherwise} \end{cases} \quad (44)$$

and

$$\Phi(w) = Cr\{\zeta \le w\} = \begin{cases} 0 & w < a_v \\ \dfrac{w - a_\zeta}{2(b_\zeta - a_\zeta)} & a_\zeta \le w \le b_\zeta \\ \dfrac{w + c_\zeta - 2b_\zeta}{2(c_\zeta - b_\zeta)} & b_\zeta \le w < c_\zeta \\ 1 & w \ge c_\zeta \end{cases} \quad (45)$$

which gives the credibility distribution for the fuzzy mean, ζ.
Then the critical step is to derive the expression of $Cr\{\zeta(\theta) \in \Theta | \Pr\{\xi(\omega, \theta) \le x\} \ge \alpha\}$. For normal random fuzzy variable with a triangular fuzzy mean,

$$\{\zeta(\theta) : \Pr\{\xi(\omega, \theta) \le x\} \ge \alpha\}$$

$$\Leftrightarrow \left\{ \theta \in \Theta : \zeta(\theta) \le x - \sigma\Phi^{-1}(\alpha) \right\}. \quad (46)$$

Then the range for the integration of the integrand $Cr\left\{ \theta \in \Theta : \zeta(\theta) \le x - \sigma\Phi^{-1}(\alpha) \right\}$ with respect to α is listed in *Table 3*.

$g(\alpha)$	Range for α	$Cr\ \theta\in\Theta:\zeta\ \theta\ \le x-\sigma\Phi^{-1}\ \alpha$
$-\infty<g(\alpha)<a_\zeta$	$\Phi\left(\frac{x-a_\zeta}{\sigma}\right)<\alpha<1$	0
$a_\zeta\le g(\alpha)<b_\zeta$	$\Phi\left(\frac{x-b_\zeta}{\sigma}\right)<\alpha<\Phi\left(\frac{x-a_\zeta}{\sigma}\right)$	$\frac{x-\sigma\Phi^{-1}(\alpha)-a_\zeta}{2(b_\zeta-a_\zeta)}$
$b_\zeta\le g(\alpha)<c_\zeta$	$\Phi\left(\frac{x-c_\zeta}{\sigma}\right)<\alpha<\Phi\left(\frac{x-b_\zeta}{\sigma}\right)$	$\frac{x-\sigma\Phi^{-1}(\alpha)+c_\zeta-2b_\zeta}{2(c_\zeta-b_\zeta)}$
$g(\alpha)\ge c_\zeta$	$0<\alpha<\Phi\left(\frac{x-c_\zeta}{\sigma}\right)$	1

Table 3. Integration range with respect to α

where $\zeta=g(\alpha)=x-\sigma\Phi^{-1}(\alpha)$.
Then we obtain the average chance measure for the event $\{\xi(\omega,\theta)\le x\}$

$$ch\{\xi(\omega,\theta)\le x\}=\int_{\Phi\left(\frac{x-b_\zeta}{\sigma}\right)}^{\Phi\left(\frac{x-a_\zeta}{\sigma}\right)}\frac{x-\sigma\Phi^{-1}(\alpha)-a_\zeta}{2(b_\zeta-a_\zeta)}d\alpha$$

$$+\int_{\Phi\left(\frac{x-c_\zeta}{\sigma}\right)}^{\Phi\left(\frac{x-b_\zeta}{\sigma}\right)}\frac{x-\sigma\Phi^{-1}(\alpha)+c_\zeta-2b_\zeta}{2(c_\zeta-b_\zeta)}d\alpha+\int_0^{\Phi\left(\frac{x-c_\zeta}{\sigma}\right)}1x \qquad (47)$$

which leads to the average chance distribution

$$\Psi(x)=\frac{x-a_\zeta}{2(b_\zeta-a_\zeta)}\left(\Phi\left(\frac{x-a_\zeta}{\sigma}\right)-\Phi\left(\frac{x-b_\zeta}{\sigma}\right)\right)$$

$$+\frac{x+c_\zeta-2b_\zeta}{2(c_\zeta-b_\zeta)}\left(\Phi\left(\frac{x-b_\zeta}{\sigma}\right)-\Phi\left(\frac{x-c_\zeta}{\sigma}\right)\right)$$

$$+\Phi\left(\frac{x-c_\zeta}{\sigma}\right)-\frac{\sigma}{2(b_\zeta-a_\zeta)}\int_{\frac{x-b_\zeta}{\sigma}}^{\frac{x-a_\zeta}{\sigma}}u\phi(u)du$$

$$-\frac{\sigma}{2(c_\zeta-b_\zeta)}\int_{\frac{x-c_\zeta}{\sigma}}^{\frac{x-b_\zeta}{\sigma}}u\phi(u)du \qquad (48)$$

6. Fuzzy repair effect estimation under fuzzy maximum entropy principle

Entropy is a measure of uncertainty. The entropy of De Luca and Termini [1] characterizes uncertainty resulting primarily from the linguistic vagueness rather than resulting from information deficiency, and vanishes when the fuzzy variable takes all the values with membership degree 1. However, we hope that the degree of uncertainty is 0 when the fuzzy variable degenerates to a crisp number, and is maximum when the fuzzy variable is an equi-possible one, i.e., all values have the same possibility. In order to address such a requirement, Li and Liu [6] provided a new definition based on credibility measure.

Definition 6.1.(Fuzzy Entropy) Let ξ be a continuous fuzzy variable defined on a credibility space $(\Theta,2^\Theta,Cr)$, then the fuzzy entropy, $H[\xi]$, is defined by

$$H[\xi]=\int_{-\infty}^{\infty}S(Cr(\{\theta:\xi(\theta)=u\}))du \qquad (49)$$

where

$$S(t)=-t\ln t-(1-t)\ln(1-t) \qquad (50)$$

For convenience, we name $S(t)$ as entropy density at point *t*.
The maximum entropy principle provides a route such that it is possible to select the parameter(s) $\underline{\lambda}$ that maximizes the value of entropy function and satisfies certain given constraints for specifying a membership function with a given form. However, what we aim at is not obtaining parameters from the theoretical entropy function rather we must determine the parameters based on observations of the fuzzy variable, say, ξ. In other words, we need to develop a criterion to obtain data-assimilated membership function. Therefore, we suggest an *empirical* fuzzy entropy function for parameter searching since the optimal value of the data-dependent object function has to reflect the constraints specified by observational data implicitly. The data assimilated object function is the average of entropy densities evaluated at $\{z_1,z_2,\cdots,z_n\}$ respectively, i.e.,

$$J[-L_1,L_2]=\frac{1}{n}\sum_{i=1}^{N}S(Cr\{Z(\theta)=z_i;(\delta,\eta)\}) \qquad (51)$$

where a finite interval $[-L_1,L_2],\ \ L_2>L_1\ge 0$ is defined for the domain of the entropy. Note that with the finiteness of empirical entropy, $J[-L_1,L_2]\to H[Z;\lambda]$ asymptotically with parameter constrained by the data structure and $Z\in[-L_1,L_2],\ \ L_2>L_1\ge 0$ which guarantees the theoretical entropy $H\ Z$ exists and finite in general.

Then, we can estimate the parameter $\left(a_\zeta, b_\zeta, c_\zeta\right)$ of the membership of fuzzy composite error in terms of maximum entropy principle. Furthermore, we can isolate a few repair as bad-as-old regime and thus repair effect is zero for estimation parameter o for specifying ε_i, the translation error because under triangular membership assumption, the empirical membership can be defined and satisfies the asymptotical requirements.

7. Conclusion

In this paper, we argue that a differential equation motivated regression model will result in a regression model with random fuzzy error terms and thus complete our mission for solidifying a rigorous mathematical foundation for the grey modelling on system repair effects proposed by Guo [3], [4]. The maximum entropy principle facilitates a way for fuzzy parameter estimation. However, the average change distribution is also providing a way for parameter data-assimilation.

Acknowledgements

This work was supported partially by South Africa Natural Research Foundation Grant FA2006042800024. We are grateful to Professor B.D. Liu for his constant encouragements of applying his fuzzy credibility measure theory. The error frequency in *Figure 1* is provided by my MSc student, Mr Li Xiang.

References

[1] De Luca, A. & Termini, S. (1972). A Definition of Non-probabilistic Entropy in the Setting of Fuzzy Sets Theory. *Information and Control* 20, 301-312.

[2] Deng, J. L. (1985). *Grey Systems (Social · Economical).* The Publishing House of Defence Industry, Beijing (in Chinese).

[3] Guo, R. (2005[a]). Repairable System Modelling Via Grey Differential Equations. *Journal of Grey System*, 8 (1), 69-91.

[4] Guo, R. (2005[b]). A Repairable System Modelling by Combining Grey System Theory with Interval-Valued Fuzzy Set Theory. *International Journal of Reliability, Quality and Safety Engineering*, 12 (3), 241-266.

[5] Liu, B. (2004). *Uncertainty Theory: An Introduction to Its Axiomatic Foundations.* Springer-Verlag Heidelberg, Berlin.

[6] Li, P. & Liu, B. Entropy of Credibility Distributions for Fuzzy Variables. *IEEEE Transactions on Fuzzy Systems,(to be published).*