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AN ALGORITHM FOR QUADRATICALLY CONSTRAINED MULTI-OBJECTIVE QUADRATIC FRACTIONAL PROGRAMMING WITH PENTAGONAL FUZZY NUMBERS

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This study proposes a methodology to obtain an efficient solution for a programming model which is multi-objective quadratic fractional with pentagonal fuzzy numbers as coefficients in all the objective functions and constraints. The proposed approach consists of three stages. In the first stage, defuzzification of the coefficients is carried out using the mean method of α-cut. Then, in the second stage, a crisp multi-objective quadratic fractional programming model (MOQFP) is constructed to obtain a non-fractional model based on an iterative parametric approach. In the final stage, this multi--objective non-fractional model is transformed to obtain a model with a single objective by applying the ε-constraint method. This final model is then solved to get desired solution. Also, an algorithm and flowchart expressing the methodology are given to present a clear picture of the approach. Finally, a numerical example illustrating the complete approach is given.

Keywords: *multi-objective quadratic fractional programming model (MOQFPM), pentagonal fuzzy number (PFN), mean method of α-cut, parametric approach, ε-constraint method*

1. Introduction

Multi-objective quadratic fractional programming (MOQFP) is a highly successful decision-making process used for analysing practical problems where the best decisions must be made. Specific coefficients must be chosen when developing such programming models. In practice, however, these specific values are rarely known, and an approximation to these coefficients values can be made. Most of the time, the information available is uncertain and takes the form of fuzzy numbers (triangular, trapezoidal, pentagonal, pentagonal neutrosophic, intutionistic, or interval). Because such programming models have multiple quadratic objective functions that are fractional in nature, there is no single solution that meets all of the objectives at the same time.

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Thus, Pareto [46] introduced the concept of Pareto optimality, where a Pareto solution is one that satisfies all objectives at the same time.

Several approaches have been proposed by various authors over the last few decades, and much work has been done in the field of fractional programming. Charnes and Cooper [2] study linear fractional problems and devise a method for converting them to linear programming problems. Then, for dealing with linear programming problems, Martos and Whinston [9] and Jagannathan [28] introduce the parametric approach. Dinkelbach [47] later extends their approach to quadratic fractional problems and obtains optimal solutions using Newton's method in 1968. Several other authors expand this approach and create new techniques for finding optimal solutions. Almogy and Levin [48] use a parametric approach to solve fractional problems with a sum of fractional functions as the goal. Then, Falk and Palocsay [14] point out the shortcomings of their method and use the parametric approach to solve the sum and product of the linear fractional functions. Tantawy [40] also proposes a method for solving fractional problems based on the gradient method. Tammer et al. [16] employ a parametric approach to solve MOQFP problems by determining parameters. Heesterman [6] investigates quadratic problems using parametric methods as well. Salahi and Fallahi [21] also use a parametric approach to solve fractional problems. For interval coefficient fractional programming, Borza et al. [18] resort to a parametric approach. Emam [24, 25] also proposes using the ε-constraint approach to solve multi-objective integer bi-level QFP problems.

For fractional problems, Ojha and Biswal [5] suggest the ε-constraint method. For linear fractional problems, Nayak and Ojha [37, 38] also use parametric and ε-constraint methods. Arora and Gupta [26] apply the branch and bound technique to solve linear fractional problems. Valipour et al. [12] investigated a method for solving linear fractional problems by taking the distance between two solutions into account. Ehrgott et al. [19] and Kenneth Chircop et al. [15] make use of the ε-constraint method in a different way to solve multi-objective problems. Emmerich and Deutz [22] propose evolutionary methods for dealing with similar issues. Nikas et al. [4] offer the AUGMECON-R method for solving MOLP problems precisely. Goyal et al. [23, 42–45] also propose a method for solving multi-objective quadratic fractional models efficiently, using a parametric approach and the ε-constraint method. Because of flaws in structural and technical procedures, there is ambiguity and uncertainty in parameter evaluation. Zadeh et al. [17] are the first to address this issue. Later on, many researchers investigated this area, using various types of fuzzy numbers. Adhami et al. [7] investigate the problem of supplier selection in a fuzzy environment. Behera et al. [11] work with LPP with triangular fuzzy numbers. Farnam and Darehmiraki [20] deal with LFP under a fuzzy environment. Khalifa et al. [13] and Veermani et al. [10] propose working with neutrosophic numbers in QFP and transportation problems. The concept of fuzzy numbers has been expanded to include pentagonal fuzzy numbers (PFN), PFN-neutrosophic numbers, Pythagorean fuzzy numbers, and a variety of others. Pathinathan and Ponnivalavan [41] use the concept of PFN in their paper in 2014. Das et al. [29–36] work with linear fractional problems with fuzzy coefficients such as triangular, pentagonal, and neutrosophic coefficients. Following that, many researchers [3, 8, 27, 39] worked with PFN and provided their types, properties, and ranking techniques. In 2019, Chakraborty et al. [1] describe various forms of PFN and compare ranking techniques.

Motivation. The literature review shows that there is a lot of work going on in the field of QFP, but the research gap with the use of pentagonal fuzzy numbers and their defuzzification process needs to be more thoroughly explored because most of the work is done with the help of FGP using the α-cut method. Thus, the authors propose this work in QFP with the mean method of α-cut for dealing with pentagonal coefficients, but here a different approach of parametric vectors in combination with the ε-constraint method is used, which is efficient in finding the Pareto optimal set of solutions. Based on the literature, it is clear that the majority of the work done to date has been in the field of linear fractional problems and with only one objective function. Furthermore, to the best of the authors' knowledge, quadratic fractional problems with pentagonal fuzzy coefficients have not yet been addressed using a parametric approach coupled with the ε-constraint method. As a result, the authors attempt to combine both approaches to optimise MOQFP models having pentagonal fuzzy coefficients.

In this paper, we propose an approach for obtaining an optimal solution to a multi-objective quadratic fractional programming (MOQFP) model with pentagonal fuzzy numbers as coefficients in the objectives as well as constraints. The proposed approach initially converts the model with fuzzy coefficients into the crisp one with the help of the defuzzification technique, using the mean method of α-cut, and then the modified model is subjected to a parametric approach to get a non-fractional model by subjecting each fractional function equal to a vector of parameters, and this is further changed to a single-objective model, using the ε-constraint method. Here, one objective with the highest priority (or least Termination constant) is considered an objective function, and others are converted into constraints. This is all decided by the decision maker (DM).

Managerial insights. The proposed work is divided into several sections, with Section 2 providing basic definitions and related properties, Section 3 presenting the proposed model, and various approaches used in the solution procedure. Section 4 gives results and theorems on which the study is based as well as deals with ε-constraint method which helps to tackle multiple objectives problem. Section 5 then describes the complete methodology and solution procedure with the proposed approach then dealing with termination criteria and few assumptions considered in the solution procedure. Section 6 contains the algorithm for the solution procedure, the flowchart for the algorithm, and compares the proposed work to the FGP and clearly shows all of the proposed work with the help of an example. It also presents the usefulness of the proposing PFN over trapezoidal fuzzy numbers and depicts a practical application of the model and proposed work. Any organisation with multiple objectives of varying natures can benefit from the proposed work because it provides efficient solutions, and it is up to the DMs to choose any one of them as the preferred solution as all the obtained solutions are very close to each other.

2. Notations and preliminaries

R^n – space of n-dimensional real vectors
x^T – transpose of x
$\alpha^{(T)}$ – vector of parameters, T represents iteration number
t_j – termination constants
S – set of constraints
$\tilde{T}=(a,b,c,d,e),\ a\le b\le c\le d\le e$ – the pentagonal fuzzy number (PFN), is a number of the form $\tilde{T}=(a,b,c,d,e;k),\ a\le b\le c\le d\le e;\ 0<k\le 1$ with a linear membership function stated as follows:

$$\mu_{\tilde{T}}\left(\tilde{T}(x)\right)=\left\{\begin{array}{l} 0,\ e<x<a \\ k\dfrac{x-a}{b-a},\ a\le x\le b \\ 1-(1-k)\dfrac{x-b}{c-b},\ b\le x\le c \\ 1,\ x=c \\ 1-(1-k)\dfrac{d-x}{d-c},\ c\le x\le d \\ k\dfrac{d-x}{d-c},\ d\le x\le e \end{array}\right\}$$

is known as the pentagonal fuzzy number. PFN is shown in Fig. 1.

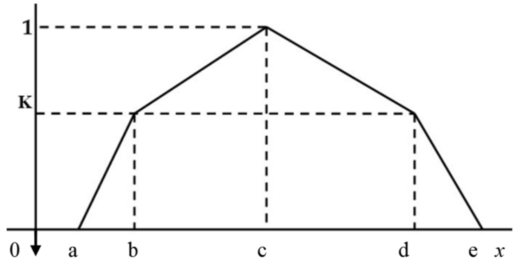

Fig. 1. Pentagonal fuzzy number

***α*-cut.** α-level set or α-cut for PFN is given as

$$\tilde{T}_\alpha = \left\{ x \in X \mid \mu_{\tilde{T}}\left(\tilde{T}(x)\right) \geq \alpha \right\} = \begin{cases} \tilde{T}_{l_1}(\alpha) = a + \dfrac{\alpha}{k}(b-a), \; 0 \leq \alpha \leq k \\ \tilde{T}_{l_2}(\alpha) = b + \dfrac{1-\alpha}{1-k}(c-b), \; k \leq \alpha \leq 1 \\ \tilde{T}_{r_1}(\alpha) = d - \dfrac{1-\alpha}{1-k}(d-c), \; k \leq \alpha \leq 1 \\ \tilde{T}_{r_2}(\alpha) = e - \dfrac{\alpha}{k}(e-d), \; 0 \leq \alpha \leq k \end{cases}$$

Pentagonal fuzzy matrix. A matrix $\tilde{M} = [\;]_{m\times m}$ in which every entry is a pentagonal fuzzy number is called a pentagonal fuzzy matrix. Its basic properties are as follows. Consider $\tilde{A} = (a_1, a_2, a_3, a_4, a_5; k_1)$ and $\tilde{B} = (b_1, b_2, b_3, b_4, b_5; k_2)$

$\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5; k), \; k = \min\{k_1, k_2\}$

$\tilde{A} - \tilde{B} = (a_1 - b_5, a_2 - b_4, a_3 - b_3, a_4 - b_2, a_5 - b_1; k); k = \min\{k_1, k_2\}$

$r\tilde{A} = (ra_1, ra_2, ra_3, ra_4, ra_5; k), r > 0 \;\; = (ra_5, ra_4, ra_3, ra_2, ra_1; k), r < 0$ for a real number a, we have, $\tilde{a} = (a, a, a, a, a)$.

Efficient solution. A point $u \in S$ is said to be an efficient solution if there is no other point $v \in S$ such that $H_j(v) \leq H_j(u)$ for all j and $H_j(v) < H_j(u)$ for at least one j.

3. Multi-objective quadratic fractional programming model with pentagonal fuzzy numbers (MOQFPM-PFN)

MOQFPM-PFN is designed to find an efficient solution to all those practical problems where the objectives are quadratic and clashing yet interrelated. In most real-life situations, parameters are not certain and hence fuzziness plays its role over here. So, PFN are considered as coefficients in the objectives as well as constraints. MOQFPM-PFN is given as follows:

$$M1: \min \tilde{H}(x) = \left\{ \tilde{H}_1(x), \tilde{H}_2(x), \tilde{H}_3(x), ..., \tilde{H}_r(x) \right\}$$

such that

$$x \in S = \left\{ x \in R^n \mid \frac{1}{2} x^T \tilde{A}_j x + \tilde{B}_j x + \tilde{c}_j \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} 0,\ x \geq 0 \right\},\ 1 \leq j \leq k$$

where

$$\tilde{H}_i(x) = \frac{\tilde{H}_{i1}(x)}{\tilde{H}_{i2}(x)},\ 1 \leq i \leq r = \frac{\frac{1}{2} x^T \tilde{D}_{i1} x + \tilde{C}_{i1} x + \tilde{d}_{i1}}{\frac{1}{2} x^T \tilde{D}_{i2} x + \tilde{C}_{i2} x + \tilde{d}_{i2}}$$

where $\tilde{D}_{i1}, \tilde{D}_{i2} = [\ ]_{n\times n}, \tilde{C}_{i1}, \tilde{C}_{i2} = [\ ]_{n\times 1}, \tilde{A}_j = [\ ]_{k\times n}, \tilde{B}_j = [\ ]_{k\times 1}$ are all pentagonal fuzzy matrices and $\tilde{d}_{i1}, \tilde{d}_{i2}, \tilde{c}_j$ are PFN.

Defuzzification of PFN with mean method of α-cut. There are several techniques of defuzzification of pentagonal fuzzy numbers but the authors worked with the mean method of α-cut of a PFN. The left and right α-cuts of PFN are as follows:

$$\tilde{T}_{l_1}(\alpha) = a + \frac{\alpha}{k}(b-a),\ 0 \leq \alpha \leq k$$

$$\tilde{T}_{l_2}(\alpha) = b + \frac{1-\alpha}{1-k}(c-b),\ k \leq \alpha \leq 1$$

$$\tilde{T}_{r_1}(\alpha) = d - \frac{1-\alpha}{1-k}(d-c),\ k \leq \alpha \leq 1$$

$$\tilde{T}_{r_2}(\alpha) = e - \frac{\alpha}{k}(e-d),\ 0 \leq \alpha \leq k$$

Since there are two left and two right values of the α-cuts of PFN as shown above, and the contribution of each of them is to be considered, thus the mean of the left and right α-cuts of PFN is taken to have the equal contribution from all of them. This method is similar to the α-cut method which is used in the case of triangular or trapezoidal fuzzy coefficients. The mean method used for the defuzzification of PFN is given as follows:

$$R(\tilde{T}) = \int_{\alpha=0}^{k} \frac{\tilde{T}_{l_1}(\alpha) + \tilde{T}_{r_2}(\alpha)}{2} + \int_{\alpha=k}^{1} \frac{\tilde{T}_{l_2}(\alpha) + \tilde{T}_{r_1}(\alpha)}{2}$$

$$R(\tilde{T}) = \frac{1}{2}\int_{\alpha=0}^{k}\left\{a + \frac{\alpha}{k}(b-a) + e - \frac{\alpha}{k}(e-d)\right\}d\alpha$$

$$+\frac{1}{2}\int_{\alpha=k}^{1}\left\{b + \frac{1-\alpha}{1-k}(c-b) + d - \frac{1-\alpha}{1-k}(d-c)\right\}d\alpha$$

$$= \frac{1}{2}\left\{(a+e)k + \frac{k}{2}(b+d-a-e)\right\}$$

$$+\frac{1}{2}\left\{(b+d)(1-k) + \frac{1-k}{2}(2c-b-d)\right\}$$

Thus, with the help of this method, our MOQFPM-PFN is converted into a crisp or deterministic MOQFPM which is easy to handle.

Parametric approach for fractional programming. In the model considered to be optimised, the objectives are not so easy to tackle as they are in the form of a fraction. To handle the fractional nature of objectives, the authors used the parametric approach as suggested by Dinkelbach [47]. In this, a vector of parameters α_j is allocated to every objective function $H_j(x)$, and is computed iteratively at each initial approximated value of the objective function. Thus, the value of the parameter α_j approaches very closely to the best optimal value of the objective function as the iterations proceed. The computation of α_j is very easy to obtain and thus this parametric aproach tackles the fractional nature of objectives very easily. With the help of this, a fractional model is reconstructed to obtain a non-fractional parametric model given as follows:

$M2$: $\min H(x) = (H_1(x), H_2(x), ..., H_r(x))$ M2:

Take each $H_j(x) = \alpha_j$

$$\frac{H_{j1}(x)}{H_{j2}(x)} = \alpha_j$$

Consider $P_j(x) = H_{j1}(x) - \alpha_j H_{j2}(x)$

The fractional model is reduced to the following non-fractional model.

M3: $\min_{x\in S} H(x) = \min_j\{P_j(x)\}$ where each $P_j(x)$ is a parametric non-fractional function.

4. Results and theorems

Result 1 [47]. A point $u \in S$ is an optimal solution of M2 iff

$$\min_{x \in S} \left\{ H_{j1}(x) - \alpha'_j H_{j2}(x) \right\} = 0$$

where $\alpha'_j = \dfrac{H_{j1}(u)}{H_{j2}(u)}$

Result 2 [47]. A point $u \in S$ is an optimal solution of M3 if for all $x \in S$, $H_{j1}(x) - \alpha'_j H_{j2}(x) = 0$ for every j or $H_{j1}(x) - \alpha'_j H_{j2}(x) \succ 0$ for at least one j.

Theorem 1. A point $u \in S$ is an efficient solution of M2 if it is an efficient solution of M3.

Proof. Let $u \in S$ is an efficient solution of M2.
Let

$$P_j(x) = H_{j1}(x) - \alpha'_j H_{j2}(x), \quad j = 1, 2, ..., r$$

Claim: $u \in S$ is an efficient solution of M3.
Let us suppose that u is an inefficient solution of M3.
So, there must exist some $v \in S$ such that $P_j(v) \le P_j(u) \forall j$ with $P_j(v) < P_j(u)$ for at least one j, i.e.,

$$H_{j1}(v) - \alpha'_j H_{j2}(v) \le H_{j1}(u) - \alpha'_j H_{j2}(u) \ \ \forall j$$

and

$$H_{j1}(v) - \alpha'_j H_{j2}(v) \prec H_{j1}(u) - \alpha'_j H_{j2}(u)$$

for at least one j, i.e.,

$$H_{j1}(v) - \alpha'_j H_{j2}(v) \le 0 \forall j \ \text{ and } \ H_{j1}(v) - \alpha'_j H_{j2}(v) < 0$$

for at least one j. Thus

$$\frac{H_{j1}(v)}{H_{j2}(v)} \le \alpha'_j \forall j \text{ and } \frac{H_{j1}(v)}{H_{j2}(v)} < \alpha'_j \text{ for at least one } j$$

$$H_j(v) \le H_j(u)\ \forall j,\ H_j(v) < H_j(u) \text{ for at least one } j$$

This is a contradiction to the fact that u is an efficient solution for M2. Thus, u is an efficient solution for model M3.

Conversely, let u be an efficient solution to M3.

Claim: u is an efficient solution to M2.

Let us suppose that $u \in S$ is an inefficient solution of M2. So, there must exist some point $v \in S$ such that $H_j(v) \le H_j(u)\ \ \forall j$ and $H_j(v) < H_j(u)$ for at least one j, i.e.,

$$\frac{H_{j1}(v)}{H_{j2}(v)} \le \alpha'_j \forall j \text{ and } \frac{H_{j1}(v)}{H_{j2}(v)} < \alpha'_j \forall j \text{ for at least one } j$$

i.e., $H_{j1}(v) - \alpha'_j H_{j2}(v) \le 0\ \ \forall j$ and $H_{j1}(v) - \alpha'_j H_{j2}(v) < 0$ for at least one j. Thus,

$$P_j(v) \le 0 \forall j \text{ and } P_j(v) < 0 \text{ for at least one } j$$

$$P_j(u) = H_{j1}(u) - \alpha'_j H_{j2}(u)$$

$$P_j(u) = H_{j1}(u) - \frac{H_{j1}(u)}{H_{j2}(u)} H_{j2}(u)$$

$$P_j(u) = 0$$

Thus

$$P_j(v) \le P_j(u)\ \ \forall j \text{ and } P_j(v) < P_j(u) \text{ for at least one } j$$

This is a contradiction to the fact that u is an efficient solution for M3. Thus, u is an efficient solution for M2.

ε-Constraint method. This is a method that helps in getting a single objective model from a multi-objective model as it becomes difficult to handle multiple objectives at a time. So, instead of solving all the functions as objectives, one objective is optimised to its best possible level and the other objectives are taken as constraints but they are also optimised to their possible acceptable levels. The ε-constraint method is as follows:

M4: $\min P_m(x),\ 1 \le m \le r$ subjected to $P_j(x) \le \varepsilon_j$ for all $1 \le j \le r,\ j \ne m$ and $x \in S$, where $\varepsilon_j \in [\varepsilon_j^l, \varepsilon_j^u]$ and ε_j^l and ε_j^u are the minimal and maximal values of $P_j(x)$. Thus, we solve this problem by substituting different values of ε_j. The obtained optimal solutions corresponding to each value of ε_j are very close to each other and all are the efficient solutions. Thus, it is completely the discretion of the DM who can choose any one of them as the preferred solution, depending upon its relative work.

5. Methodology

Model M1. $\min \tilde{H}(x) = \left\{\tilde{H}_1(x), \tilde{H}_2(x), \tilde{H}_3(x), ..., \tilde{H}_r(x)\right\}$ subject to

$$x \in S = \left\{ x \in R^n \,|\, \frac{1}{2} x^T \tilde{A}_j x + \tilde{B}_j x + \tilde{c}_j \begin{pmatrix} \le \\ \ge \\ = \end{pmatrix} 0,\ x \ge 0 \right\}, 1 \le j \le k$$

This model contains PFN as coefficients in the objectives as well as constraints. Thus, firstly we convert this fuzzy coefficient model into a crisp or deterministic one (having real coefficients), using the defuzzification by mean method of α-cut, as defined above in Section 4.

Model M2. The crisp model obtained is given as:

$\min H(x) = \{H_1(x), H_2(x), H_3(x), ..., H_r(x)\}$ subject to $x \in S$. Now, this is solved with the approach proposed by [42– 45]. Let us assume that each $H_j(x) = \alpha_j^{(t)}, 1 \le j \le r,$ t being the iteration number.

Consider $\alpha^{(t)} = (\alpha_1^{(t)}, \alpha_2^{(t)}, ..., \alpha_r^{(t)}),$ the parametric vector corresponding to $H(x)$.

Consider $P_j(\alpha^{(t)}) = H_{j1}(x) - \alpha_j^{(t)} H_{j2}(x), 1 \le j \le r.$

In this way, the above model M2 is reduced to the model M3 as shown below.

Model M3. $\min \tilde{H}(x) = \min H(x) = \min P_j(\alpha^{(t)})$ subject to $x \in S$. This is still a multi-objective model which is difficult to tackle. So, we implement an ε-constraint method which optimizes one objective, and the remaining objectives are taken as constraints. Thus, the new model M3 is shown as a next model.

Model M4. $\min P_m(\alpha^{(t)}) = H_{m1}(x) - \alpha_m^{(t)} H_{m2}(x)$ subject to $x \in S$ and $P_j(\alpha^{(t)}) \le \varepsilon_j$, $1 \le j \le r$, $j \ne m$. Suppose, X_j are the individual solutions of each $H_j(x)$ when subjected to $x \in S$. So, we construct Table 1 for the values of $H_j(X_j), 1 \le j \le r$.

Table 1. Pay-off table for $H_j(X_j)$

X_j	$H_1(X_j)$	$H_2(X_j)$	$H_3(X_j)$	...	$H_r(X_j)$
X_1	$H_1(X_1)$	$H_2(X_1)$	$H_3(X_1)$	...	$H_r(X_1)$
X_2	$H_1(X_2)$	$H_2(X_2)$	$H_3(X_2)$	...	$H_r(X_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_r	$H_1(X_r)$	$H_2(X_r)$	$H_3(X_r)$	...	$H_r(X_r)$

Next, we define ε_j^L and ε_j^U as $\varepsilon_j^L = \min P_j(X_j), 1 \le j \le r\}$

$$\varepsilon_j^U = \max\{P_j(X_j), 1 \le j \le r\}$$

Then, the initial solution $X^{(0)}$ for model M4 is calculated as $X^{(0)} = \sum_{j=1}^{r} w_j X_j$ such that summation of all the weights is a unity.

Consider a parametric vector $\alpha^{(t)}$ for $t = 1$ as

$$\alpha^{(1)} = \left(\alpha_1^{(1)}, \alpha_2^{(1)}, ..., \alpha_r^{(1)}\right) = \left(H_1(X^{(0)}), H_2(X^{(0)}), ..., H_r(X^{(0)})\right)$$

After substituting the value of $\alpha^{(1)}$ in each $P_j(\alpha^{(t)})$, we proceed with our proposed approach and test the termination conditions. If conditions are not satisfied at the end, we refine our termination constants and proceed again with the approach.

Termination constants and conditions. Termination constants (T_j) are the tolerances corresponding to the objectives $H_j(x)$ and acceptable to the DM. These T_j are decided by the DM by taking into consideration the priorities of the objectives and are taken very close to zero. Also, the termination conditions for the process to stop are given as $\left|P_j(\alpha^{(t)})\right| \le T_j, 1 \le j \le r.$

Assumptions

• Each solution of objective functions is assigned with equal weightage to obtain the initial solution to the model.

• DM determines the termination constants (nearer to zero) for each objective function.

• The initial solution to the model is obtained as: $X^{(0)} = \sum_{j=1}^{r} w_j X_j,\, w_j > 0$ and $\sum_{j=1}^{r} w_j = 1.$

6. Algorithm, flowchart, and a numerical example

Algorithm

1. For every PFN $\tilde{T} = (a, b, c, d, e; k),$ use the ranking function obtained employing the α-cut method

$$R(\tilde{T}) = \frac{1}{2}\left(k(a+e) + \frac{k}{2}(b+d-a-e) \right) + \frac{1}{2}\left((1-k)(b+d) + \frac{(1-k)}{2}(2c-b-d) \right)$$

to obtain the crisp model M2 from the fuzzy model M1.

2. Obtain $X^{(0)} = \sum_{j=1}^{r} w_j X_j.$

3. Set $t = 1$ in model M2.

4. Find $\alpha^{(1)} = \left(H_1(X^{(0)}),\, H_2(X^{(0)}),\, ...,\, H_r(X^{(0)}) \right).$

5. Put $\alpha^{(1)}$ in each $P_j(\alpha^{(t)})$ to obtain the model M3.

6. Obtain an equivalent model M4 from model M3 by selecting $P_m(\alpha^{(1)})$ as the objective to be optimised with least vale of the T_m.

7. Choose $\varepsilon_j \in [\varepsilon_j^L, \varepsilon_j^U], 1 \le j \le r,\, j \ne m$ as follows:

   a) when $[-T_j, T_j] \cap [\varepsilon_j^L, \varepsilon_j^U] = \varphi,$ select $\varepsilon_j \in [\varepsilon_j^L, \varepsilon_j^U].$

   b) otherwise, choose $\varepsilon_j \in [-T_j, T_j].$

8. Obtain a set of efficient solutions for model M4 by putting different values of ε_j.

9. Test the termination conditions $\left| P_j(\alpha^{(t)}) \right| \le T_j, 1 \le j \le r.$ If the conditions are satisfied, stop the process. Otherwise, proceed to step 9.

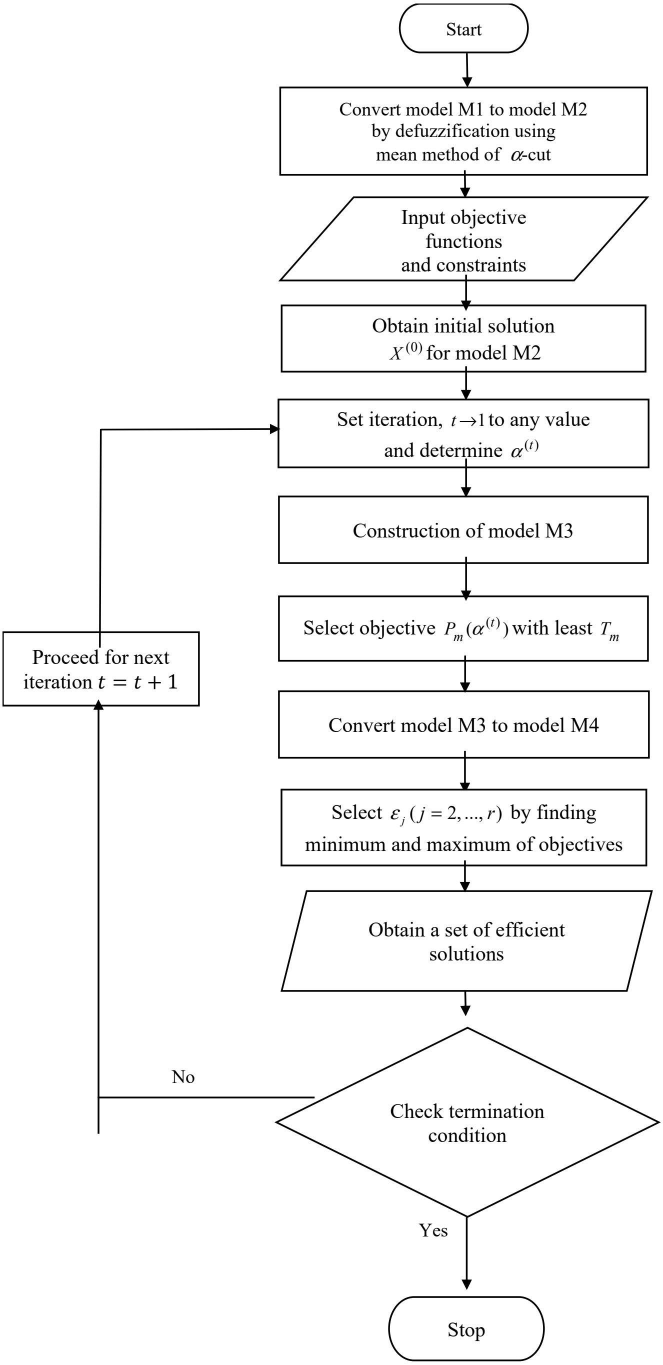

Fig. 2. Flowchart

10. Find $\min \sum_j \left(\left| P_j(\alpha^{(1)}) \right| - T_j \right)$ for that j where conditions do not get satisfied.

11. Determine $X^{(1)}$ to be the solution for which $\sum_j \left(\left| P_j(\alpha^{(1)}) \right| - T_j \right)$ has the minimum value.

12. Repeat steps 2–8 and obtain a representative set of efficient solutions for the model satisfying the termination conditions. Otherwise, DM can be asked to reset the tolerances.

13. End the process.

The proposed algorithm is expressed with the help of a flowchart which is shown in Fig. 2.

Numerical example

$$\text{M1: } \min \tilde{H}(x) = \min \left\{ \tilde{H}_1(x) = \frac{1.\tilde{7}x_1^2 + 0.\tilde{8}x_3}{0.8\tilde{5}x_2^2 + 2.\tilde{8}}, \tilde{H}_2(x) = \frac{\tilde{2}x_3^2 + 1.\tilde{1}x_2}{0.8\tilde{3}x_1^2 + 2.\tilde{9}} \right\}$$

such that

$$S = \begin{Bmatrix} \tilde{2}x_1^2 + 1.\tilde{1}x_2^2 + 0.8\tilde{5}x_3 \le 3.\tilde{9} \\ 1.\tilde{9}x_2^2 + 0.\tilde{8}x_3^2 + 1.\tilde{1}x_1 \le 5.0\tilde{1} \\ 0.8\tilde{3}x_3{}^2 + 1.\tilde{1}x_1^2 + 0.\tilde{9}x_2 \ge 2.\tilde{9} \\ 0.\tilde{9}x_1^2 + 1.0\tilde{1}x_2^2 + 0.8\tilde{3}x_3^2 \le 5.\tilde{9} \\ x_1, x_2, x_3 \ge 0 \end{Bmatrix}$$

where

$1.\tilde{7} = (1.5, 1.7, 1.9, 2.1, 2.2)$, $0.\tilde{8} = (0.8, 0.85, 0.9, 0.95, 1)$

$0.8\tilde{5} = (0.85, 0.92, 0.96, 1, 1.07)$, $2.\tilde{8} = (2.8, 2.92, 2.96, 3, 3.1)$

$\tilde{2} = (1.9, 1.95, 1.975, 2, 2.1)$, $1.\tilde{1} = (1.1, 1.13, 1.16, 1.19, 1.2)$

$0.8\tilde{3} = (0.8, 0.83, 0.86, 0.89, 0.91)$, $2.\tilde{9} = (2.83, 2.86, 2.88, 2.9, 2.92)$

$1.\tilde{9} = (1.91, 1.93, 1.95, 1.97, 1.99)$, $0.\tilde{9} = (0.87, 0.88, 0.91, 0.94, 0.96)$

$1.0\tilde{1} = (1.01, 1.04, 1.06, 1.08, 1.1)$, $3.\tilde{9} = (3.85, 3.89, 3.91, 3.93, 3.98)$

$5.0\tilde{1} = (4.87, 4.93, 4.95, 4.97, 4.99)$, $5.\tilde{9} = (5.9, 5.91, 5.93, 5.95, 5.98)$

This model is first converted into a crisp model, using the ranking function. Thus, we find the ranking function corresponding to every PFN coefficient as given below:

$$\begin{array}{lll} R(1.\tilde{7}) = 1.8825, & R(0.\tilde{8}) = 0.9, & R(0.8\tilde{5}) = 0.96 \\ R(2.\tilde{8}) = 2.9565, & R(\tilde{2}) = 1.98375, & R(1.\tilde{1}) = 1.1565 \\ R(0.8\tilde{3}) = 0.85825, & R(2.\tilde{9}) = 2.87825, & R(1.\tilde{9}) = 1.95 \\ R(0.\tilde{9}) = 0.91175, & R(1.0\tilde{1}) = 1.05825, & R(3.\tilde{9}) = 3.91175 \\ R(5.0\tilde{1}) = 4.943, & R(5.\tilde{9}) = 5.9335, & \end{array}$$

Thus, the crisp model M2 obtained from model M1 is given below

$$\text{M2: } \min\left\{ H_1(x) = \frac{1.8825x_1^2 + 0.9x_3}{0.96x_2^2 + 2.9565},\ H_2(x) = \frac{1.98375x_3^2 + 1.1565x_2}{0.85825x_1^2 + 2.87825} \right\}$$

such that

$$\begin{aligned} & 1.98375x_1^2 + 1.1565x_2^2 + 0.96x_3 \le 3.91175 \\ & 1.95x_2^2 + 0.9x_3^2 + 1.1565x_1 \le 4.943 \\ & 0.85825x_3^2 + 1.1565x_1^2 + 0.91175x_2 \ge 2.87825 \\ & 0.91175x_1^2 + 1.05825x_2^2 + 0.85825x_3^2 \le 5.9335 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

First, we find individual solutions for $H_1(x)$ and $H_2(x)$ using Lingo 15 software given as: $X_1 = (0, 1.27, 1.42)$, $X_2 = (1.17, 0.423, 1.03)$.

Table 2. Pay-off table at initial solutions

X_j	$H_1(X_j)$	$H_2(X_j)$
X_1	0.28	1.9
X_2	1.12	0.64

Table 2 shows the values of objective functions at individual optimal solutions of both functions. Here, equal weightage is given to each solution, i.e., $w_1 = w_2 = 0.5$. Thus, the initial solution to model M2 is given by $X^{(0)} = w_1X_1 + w_2X_2$.

$$X^{(0)} = (0.585,\ 0.8465,\ 1.225)$$

Next, the initial parametric vector is found as

$$\alpha^{(1)} = (H_1(X^{(0)}),\ H_2(X^{(0)})) = (0.48,\ 1.25)$$

With the help of this parametric vector, model M3, which is a non-fractional model, is obtained and given as below

$$\text{M3: min}H(x) = \left\{P_1(\alpha_1^{(1)}),\ P_2(\alpha_2^{(1)})\right\}$$

such that $x \in S$, where

$$P_1(\alpha^{(1)}) = 1.8825x_1^2 + 0.9x_3 - 0.4608x_2^2 1.41912$$

and

$$P_2(\alpha^{(1)}) = 1.98375x_3^2 + 1.1565x_2 - 1.0728x_1^2 - 3.5978$$

The initial solution to this model is given as

$$X_1 = (0, 1.27, 1.42),\quad X_2(1.18, 0.36, 1.05)$$

Table 3. Pay-off table of $P_j(X_j)$

X_j	$P_1(X_j)$	$P_2(X_j)$
X_1	–0.88	1.87
X_2	2.09	–2.49

Table 3 shows the values of $P_j(x)$ at X_j. Next, the termination constants are defined by DM and are taken as $T_1 = 0.2, T_2 = 0.3$. As $T_1 < T_2$, therefore, model M3 is reduced to model M4 (a single objective model) by ε-constraint method which is given as

$$M4: \min P_1(\alpha_1^{(1)}) = 1.8825x_1^2 + 0.9x_3 - 0.4608x_2^2 - 1.41912$$

such that

$$P_2(\alpha_2^{(1)}) \le \varepsilon_2,\ \ \varepsilon_2 \in [\varepsilon_2{}^L, \varepsilon^U]$$

$$x \in S$$

$$\varepsilon_2^L = \min\{P_2(X_j),\ j = 1, 2\} = -2.49$$

$$\varepsilon_2^U = \max\{P_2(X_j),\ j = 1, 2\} = 1.87$$

As $[-0.3, 0.3] \subseteq [-2.49, 1.87]$, we select $\varepsilon_2 \in [-0.3, 0.3]$, and by putting different values ε_2, we obtain a representative set of efficient solutions as shown in Table 4.

Table 4. Table of efficient solutions

ε_2	x_1	x_2	$(x_3$	P_1	P_2	$H_1(x)$	$H_2(x)$
–0.3	0.7699	1.193	1.135	0.06	–0.03	0.494	1.16
–0.24	0.7575	1.205	1.140	0.01	–0.024	0.484	1.17
–0.18	0.7449	1.216	1.146	–0.02	–0.018	0.474	1.19
–0.12	0.7337	1.218	1.155	–0.05	–0.012	0.468	1.21
–0.06	0.7230	1.216	1.165	–0.07	–0.006	0.464	1.23
0.06	0.7011	1.213	1.184	–0.1	0.0058	0.455	1.27
0.12	0.6899	1.212	1.194	–0.13	0.012	0.451	1.29
0.18	0.6785	1.210	1.203	–0.15	0.0178	0.446	1.3
0.24	0.6668	1.208	1.213	–0.16	0.2399	0.442	1.32
0.3	0.6549	1.207	1.222	–0.18	0.3	0.438	1.34

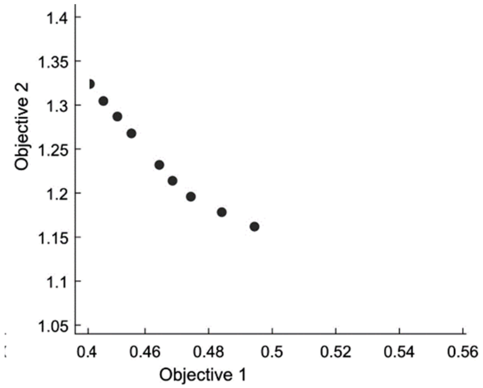

Fig. 3. Pareto front for objective functions

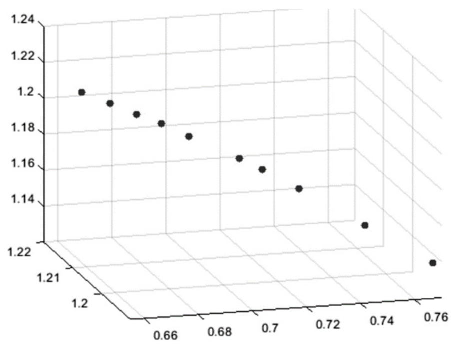

Fig. 4. Pareto front for the efficient solutions

It is clear that $\left|P_1(\alpha^{(1)})\right| \le T_1$ and $\left|P_2(\alpha^{(1)})\right| \le T_2$. Thus, we terminate our process and finally, the DM has a choice to select any one of the solutions in Table 4 as the efficient solution to the model. The Pareto front formed by the efficient solution is shown in Figs. 3 and 4.

Comparison of the proposed approach with FGP. We used FGP to solve the above numerical illustration and found that the values of objectives are given by: $H_1(x) = 0.9728$ and $H_2(x) = 0.6809$. Tables 4 clearly show that in both the approaches, one of the objectives is better optimised. Figure 5 is a better representation of it. As a result, it is concluded that both approaches are comparable, which validates our proposed methodology.

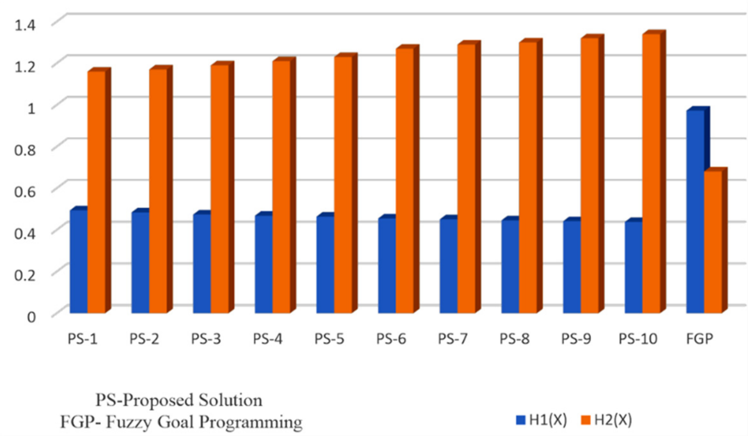

Fig. 5. Comparison of objectives with the proposed approach and FGP

The usefulness of pentagonal fuzzy numbers over trapezoidal fuzzy numbers. In the above considered numerical example, consider the coefficients in both the objectives and constraints as trapezoidal fuzzy numbers (TrFN) rather than PFN.

$$\min \tilde{H}(x) = \min\left\{ \tilde{H}_1(x) = \frac{1.\tilde{7}x_1^2 + 0.\tilde{8}x_3}{0.8\tilde{5}x_2^2 + 2.\tilde{8}},\ \tilde{H}_2(x) = \frac{\tilde{2}x_3^2 + 1.\tilde{1}x_2}{0.8\tilde{3}x_1^2 + 2.\tilde{9}} \right\}$$

subject to

$$S = \tilde{2}x_1^2 + 1.\tilde{1}x_2^2 + 0.8\tilde{5}x_3 \le 3.\tilde{9}$$
$$1.\tilde{9}x_2^2 + 0.\tilde{8}x_3^2 + 1.\tilde{1}x_1 \le 5.0\tilde{1}$$
$$0.8\tilde{3}x_3^2 + 1.\tilde{1}x_1^2 + 0.\tilde{9}x_2 \ge 2.\tilde{9}$$
$$0.\tilde{9}x_1^2 + 1.0\tilde{1}x_2^2 + 0.8\tilde{3}x_3^2 \le 5.\tilde{9}$$
$$x_1, x_2, x_3 \ge 0$$

where

$1.\tilde{7} = (1.5, 1.7, 2.1, 2.2)$, $0.\tilde{8} = (0.8, 0.85, 0.95, 1)$

$0.8\tilde{5} = (0.85, 0.92, 1, 1.07)$, $2.\tilde{8} = (2.8, 2.92, 3, 3.1)$

$\tilde{2} = (1.9, 1.95, 2, 2.1)$, $1.\tilde{1} = (1.1, 1.13, 1.19, 1.2)$

$0.8\tilde{3} = (0.8, 0.83, 0.89, 0.91)$, $2.\tilde{9} = (2.83, 2.86, 2.9, 2.92)$

$1.\tilde{9} = (1.91, 1.93, 1.97, 1.99)$, $0.\tilde{9} = (0.87, 0.88, 0.94, 0.96)$

$1.0\tilde{1} = (1.01, 1.04, 1.08, 1.1)$, $3.\tilde{9} = (3.85, 3.89, 3.93, 3.98)$

$5.0\tilde{1} = (4.87, 4.93, 4.97, 4.99)$, $5.\tilde{9} = (5.9, 5.91, 5.95, 5.98)$

This is solved with the above-proposed methodology and the values of the objective functions are shown in the Table 5.

Table 5. Objective function values with trapezoidal fuzzy numbers

$H_1(x)$	0.4978	0.4886	0.4844	0.4802	0.4758	0.4672	0.4627	0.4583	0.4538	0.4492
$H_2(x)$	1.237	1.254	1.272	1.289	1.308	1.344	1.362	1.381	1.399	1.418

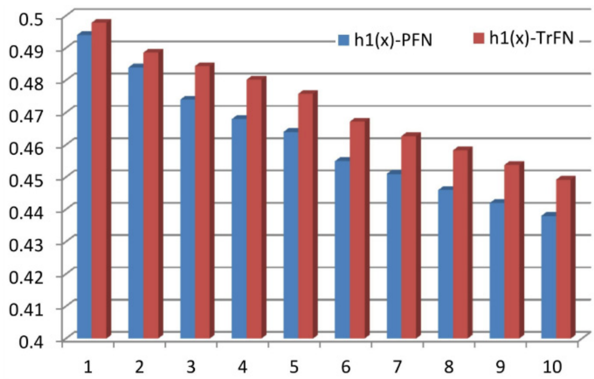

Fig. 6. Comparison of $H_1(x)$ in MOQFP models with TrFN and PFN

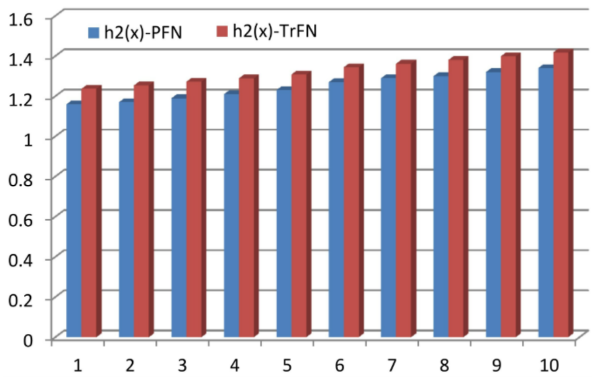

Fig. 7. Comparison of $H_2(x)$ in MOQFP models with TrFN and PFN

From Tables 4 and 5, it is clear that the values of objective functions of the problem with pentagonal fuzzy numbers as coefficients are better optimised than that of the problem with trapezoidal fuzzy numbers as coefficients. Thus, this presents the usefulness of pentagonal fuzzy numbers in those circumstances where even minor changes in the objectives play a major role in the decision-making process. This comparison is better represented in Figs. 6 and 7 which clearly depict the effectiveness of using pentagonal fuzzy numbers.

Application in the field of production. Consider an automobile unit that produces three bikes P, Q, and R each with two models: 5G and 6G. Assume that the bikes selling prices are Rs. S_P^1, S_Q^1, S_R^1 5G models and Rs. S_P^2, S_Q^2, S_R^2 for 6G models, as determined by the manufacturing unit. The unit has established a goal of creating the most B bikes. The manufacturing expenses of one model of 5G bikes are Rs. a_1, b_1, c_1 with an additional cost of Rs. l per each 5G bike of all types for time-bound completion of the target due to additional input expenditures. Similarly, one model of 6G bikes costs Rs. a_2, b_2, c_2 to manufacture, with an additional cost of Rs. m for each 6G bike. Assume the unit produces x_1, y_1, z_1 units of 5G bikes and x_2, y_{12}, z_2 units of 6G bikes.

The costs for 5G bikes are Rs. $S_P^1 x_1$, $S_Q^1 y_1$, $S_R^1 z_1$ and 6G variants are Rs. $S_P^1 x_2$, $S_Q^1 y_2$, $S_R^1 z_2$. Each 5G bike costs Rs. $a_1 + lx_1$, $b_1 + ly_1$, $c_1 + lz_1$, while each 6G bike costs Rs. $a_2 + mx_2$, $b_2 + my_2$, $c_2 + mz_2$. As a result, the overall cost of 5G automobiles is Rs. $(a_1 + lx_1)x_1$, $(b_1 + ly_1)y_1$, $(c_1 + lz_1)z_1$, and the entire cost of top model cars is Rs. $(a_2 + mx_2)x_2$, $(b_2 + my_2)y_2$, $(c_2 + mz_2)z_2$. Every manufacturing unit's goal is to maximise profit per unit cost of production. As a result, this production challenge is finally modelled as QFPP, which is written as:

$$\max H(x)$$

$$= \begin{cases} H_1(x) = \dfrac{S_P^1 x_1 + S_Q^1 y_1 + S_R^1 z_1 - \left((a_1 + lx_1)x_1 + (b_1 + ly_1)y_1 + (c_1 + lz_1)z_1\right)}{(a_1 + lx_1)x_1 + (b_1 + ly_1)y_1 + (c_1 + lz_1)z_1} \\[2ex] H_2(x) = \dfrac{S_P^2 x_2 + S_Q^2 y_2 + S_R^2 z_2 - \left((a_2 + mx_2)x_2 + (b_2 + my_2)y_2 + (c_2 + mz_2)z_2\right)}{(a_2 + mx_2)x_2 + (b_2 + my_2)y_2 + (c_2 + mz_2)z_2} \end{cases}$$

subject to

$$x_1 + y_1 + z_1 + x_2 + y_2 + z_2 \le B$$
$$x_1,\ y_1,\ z_1,\ x_2,\ y_2,\ z_2 \ge 0$$

7. Conclusion

A MOQFP model with PFN as coefficients in objectives as well as constraints is solved to get an efficient solution. The ranking function obtained by the mean method of α-cut is used to tackle PFN coefficients and to obtain a crisp model. Then, an iterative and interactive parametric approach is presented which is efficient in transforming a fractional model into a non-fractional one. Further, this approach is coupled with the ε-constraint method to tackle multiple objectives and obtain a single objective model which is easy to solve. This method changes the feasible region in such a way that we obtain a better representative set of efficient solutions. The proposed approach is very efficient in finding the solution as it converges fast towards the best optimal solution because the value of the parameter α_j decreases at each level of iteration. Also, a numerical is solved in the end along with the comparison of the approach proposed with FGP to express the feasibility of the approach. This can also be extended for finding solutions to MOQFP problems with interval valued, intuitionistic, neutrosophic fuzzy numbers as coefficients and also to Bi-level and multi-level programming problems.

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