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SPECIAL SECTION

Rotor dynamics — four open questions

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Abstract. Despite many years of development in the field of rotor dynamics, many issues still need to be resolved. This is due to the fact that turbomachines, even those with low output power, have a very complex design. The author of this article would like to signal these issues in the form of several questions, to which there are no precise answers. The questions are as follows: How can we build a coherent dynamic model of a turbomachine whose some subsystems have non-linear characteristics? How can we consider the so-called *prehistory* in our analysis, namely, the relation between future dynamic states and previous ones? Is *heuristic* modelling the future of rotor dynamics? What phenomena may occur when the stability limit of the system is exceeded? The attempt to find answers to these questions constitutes the subject of this article. There are obviously more similar questions, which encourage researchers from all over the world to further their research.

Key words: rotor dynamics; non-linear models; prehistory; heuristic modelling.

1. INTRODUCTORY REMARKS

Rotor dynamics is a rapidly developing field of knowledge. It is being developed in many centres around the world. For example, some issues related to the stable operation of a rotating machine under atypical conditions are still relevant today. It is worth noting, for example, the papers written by T. Szolc [1, 2], J. Zapoměl [3, 4], and M. Klanner [5] or papers in the fields such as model-based diagnostics and defect identification written by T. Choudhury [6], papers in the field of flutter and vibrations in turbine sets written by C. Prasad [7] and those related to the influence of gyroscopic effects written by R.S. Schittenhelm [8].

The issues related to cavitation in the oil film of slide bearings and those related to transient states are not fully investigated – papers written by C. Xing [9].

There are obviously many more of these issues. It is impossible to list them all here. In general, a few remarks can be made by grouping the poorly recognised issues in the following way: issues of nonlinear description, the so-called ‘prehistory’ problem, and heuristic modelling.

2. PROBLEMS ENCOUNTERED WHEN DESCRIBING LINEAR AND NON-LINEAR SUBASSEMBLIES OF A TURBOMACHINE

If we want to build a coherent mathematical-numerical model of a turbomachine, describing its dynamic behavior, we will immediately notice that such a machine consists of several subsystems, each of which may have different dynamic properties. The whole system of a turbomachine, including its subsystems, is shown in Fig. 1.

The main problem here is as follows: how to combine the linear and nonlinear subsystems into one coherent set of equations? The rotor line, if possible structural imperfections are omitted, can generally be treated as a linear component, i.e. the relation between the forces acting in particular rotor nodes and their displacements is linear. The situation changes if we want to consider imperfections such as a cracked rotor or anisotropic mechanical characteristics of the rotor material in our model.

Slide bearings and labyrinth seals have non-linear characteristics by nature. It is only in the case of small displacements of bearing journals that their linear dependency can be assumed, which manifests itself by small elliptical trajectories.

The supporting structure consisting of the foundation, ground, and bearing supports, as far as the vibrations of this whole subsystem are concerned, can be quite well described using linear relationships.

Let us now consider the situation when it is reasonable to assume that all subsystems of the turbomachine have linear dynamic characteristics (Fig. 2). By means of experimental measurements, we can quite easily obtain complex flexibility characteristics of the supporting structure at specific nodes. We can implement them into the complex matrix equations of the entire system, and then, by a simple operation of their inversion, elegant equations of motion are obtained for the entire system (Fig. 2). This is a commonly used method to describe the dynamic states of a turbomachine [10–26].

The use of this method is fully justified in cases where the displacements of bearing journals are small, the system has not exceeded the stability limit, and the materials used to manufacture the rotor and other components are uniform in terms of their structural properties.

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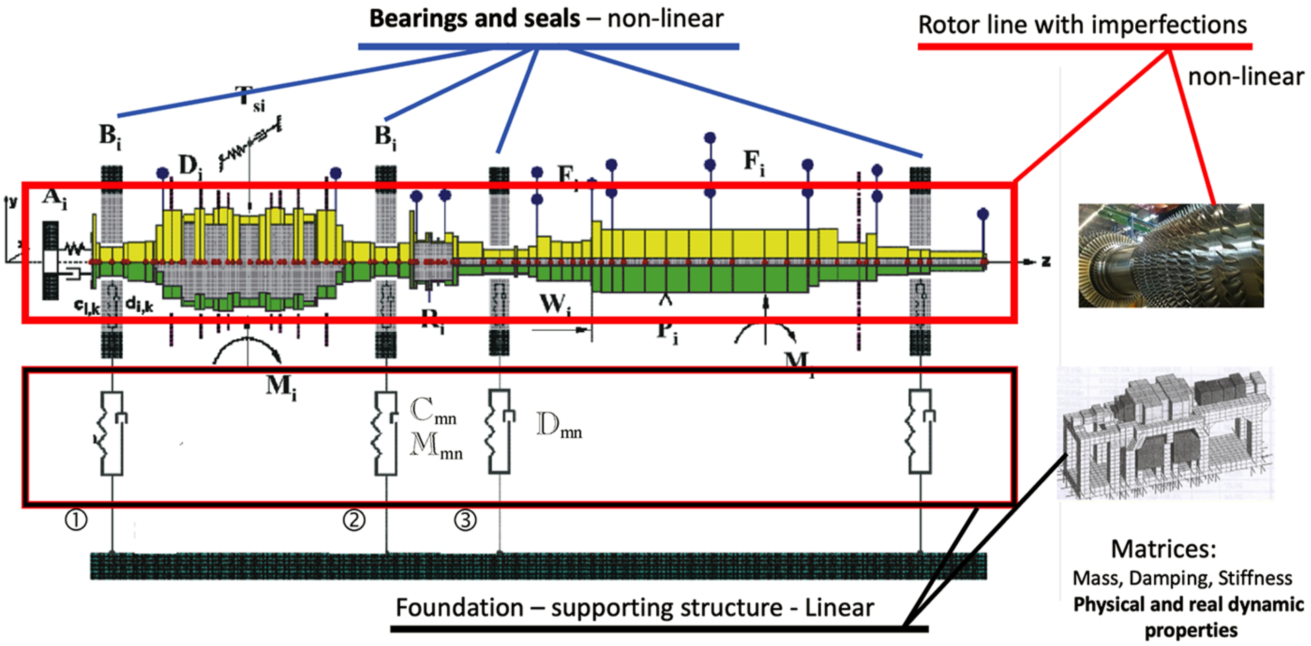


Fig. 1. Main subsystems of the turbomachine and the main questions arising from the attempt to describe them mathematically and accurately

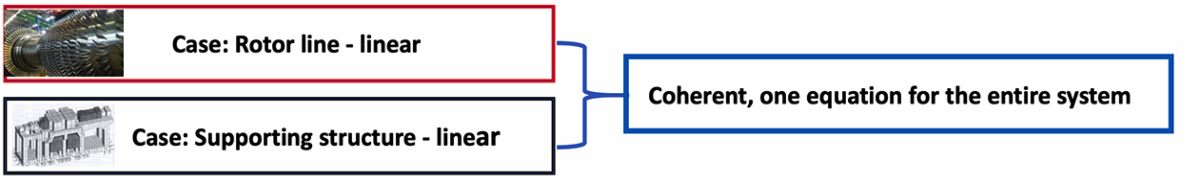


Fig. 2. The case where the dynamic characteristics of all subsystems of a turbomachine are linear (the whole system can be described consistently)

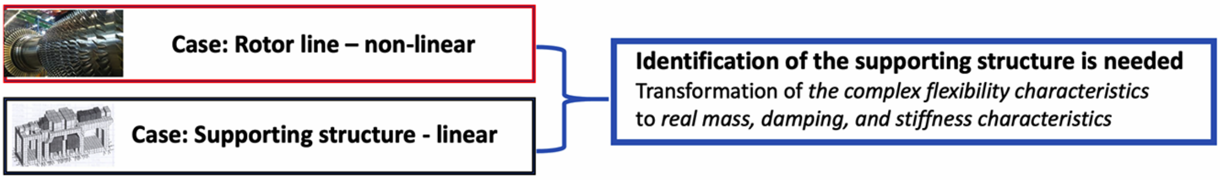


Fig. 3. Non-linear case (to build a non-linear equation of motion (equation (1)), the dynamic characteristics of the supporting structure must be transformed)

The situation becomes much more complicated when the validity of the assumptions is questionable or even unacceptable. This means that the characteristics of at least one subsystem of the entire turbomachine system are non-linear (and often even strongly non-linear). Then the questions initially indicated in Fig. 3 arise.

The non-linearity of one subsystem means that the whole system must be considered as non-linear. To build non-linear equations of motion for the entire system as in equation (1), it is necessary to convert the complex flexibility characteristics of the supporting structure into real mass, damping, and stiffness characteristics through an appropriate transformation. The following important question now arises: How can such a transformation be carried out in the case when the global stiffness and damping matrices, which occur in equation (1), are non-linear and very often strongly non-linear?

Non-linear equation for the entire system will take the following form:

$$\mathbf{M}\ddot{\boldsymbol{x}} + \mathbf{D}(\boldsymbol{x},\dot{\boldsymbol{x}})\boldsymbol{x} + \dot{\mathbf{K}}(\boldsymbol{x},\dot{\boldsymbol{x}})\boldsymbol{x} = \mathbf{P}(t), \tag{1}$$

where: $\mathbf{M}$ – global inertia matrix,
$\mathbf{D}$ – global damping matrix,
$\mathbf{K}$ – global stiffness matrix,
$\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}$ – generalized displacement, velocity, and acceleration vectors,
$\mathbf{P}$ – generalized vector of external excitations,
t – time.

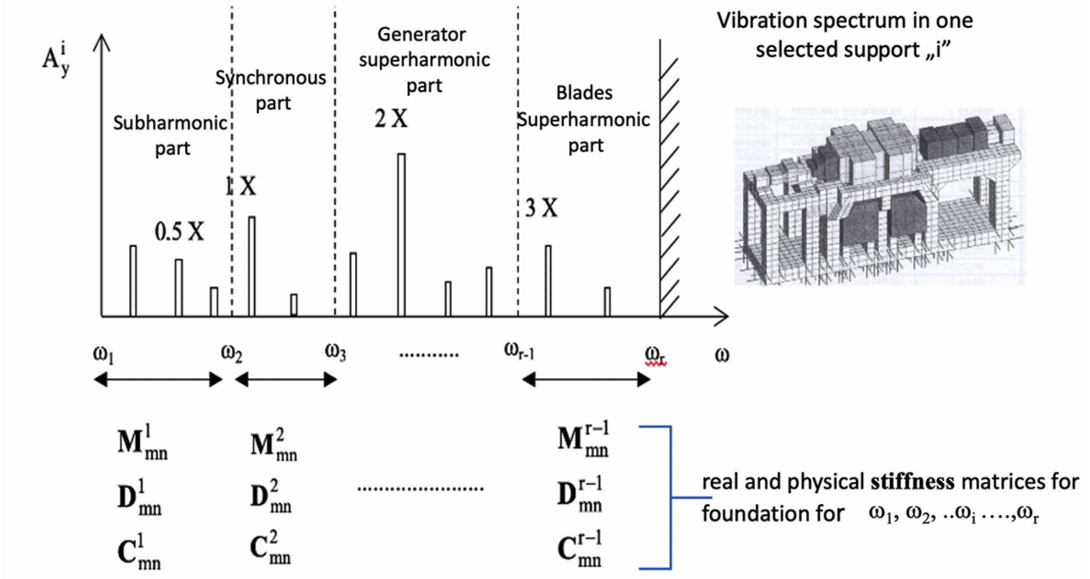


Fig. 4. First idea: adequacy intervals (a proposal to determine the adequacy intervals based on the structure of the vibration spectra; X means synchronous frequency)

The author of the article proposes two original methods to carry out such transformations, namely the adequacy intervals method and the weight function concept [27, 28]. Figure 4 explains the essence of this approach.

New ideas are needed, and, in this article, we will refer to them as adequacy intervals and weight function concepts. Now, we will try to explain the essence of the proposed method of transforming the dynamic characteristics of the supporting structure. However, the question of the right choice of adequacy intervals remains unanswered.

Formulation of the issue:

a) flexibility matrix depends on ω

For practical reasons (experimental measurements for selected frequencies of foundation excitations ω) we have only discrete values of the flexibility matrix. So, for:

$\omega_1, \omega_2, \dots \omega_i, \dots \omega_r$ we have $\mathbf{L}_{\mathrm{mn}}(\omega_1), \mathbf{L}_{\mathrm{mn}}(\omega_2), \dots \mathbf{L}_{\mathrm{mn}}(\omega_r)$.

b) Suppose now that we find such frequency interval $<\omega_1, \omega_2>$ that the determined values of $\mathbf{M}_{\mathrm{mn}}$, $\mathbf{C}_{\mathrm{mn}}$, and $\mathbf{D}_{\mathrm{mn}}$ are constant (independent of ω). And so, we have two values of flexibility matrix:

$\mathbf{L}_{\mathrm{mn}}(\omega_1), \mathbf{L}_{\mathrm{mn}}(\omega_2)$,

and one set of real and physical matrices of foundation:

$\mathbf{M}_{\mathrm{mn}}$, $\mathbf{C}_{\mathrm{mn}}$, and $\mathbf{D}_{\mathrm{mn}}$ constant in an interval $<\omega_1, \omega_2>$

c) The main problem is: how to choose the right adequacy intervals $<\omega_1, \omega_2>$?

To determine the adequacy intervals, we need the following real and physical stiffness matrices for the foundation for $\omega_1, \omega_2, \dots, \omega_r$:

$$\begin{array}{cccc}
\mathbf{M}^1_{mn} & \mathbf{M}^2_{mn} & & \mathbf{M}^{r-1}_{mn}, \\
\mathbf{D}^1_{mn} & \mathbf{D}^2_{mn} & \cdots\cdots & \mathbf{D}^{r-1}_{mn}, \\
\mathbf{C}^1_{mn} & \mathbf{C}^2_{mn} & & \mathbf{C}^{r-1}_{mn}.
\end{array}$$

Let us now take the second idea: the weight function concept. The question that now arises is as follows: if we already have adequacy intervals $<\omega_1, \omega_2>, <\omega_2, \omega_3>, \dots <\omega_{r-1}, \omega_{2r}>$. what is the contribution of $<\omega_1, \omega_2>$ in relation to the whole $<\omega_1, \omega_r>$. range?

Now let us make an important assumption: the contribution of the individual $<\omega_1, \omega_2>$ is proportional to the spectral density of the supporting structure.

The spectral lines amplitude is given by the following formula:

$$\frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \mathrm{A}_x^i(\omega) \mathrm{d}\omega. \tag{2}$$

It is not the purpose of this article to explain in detail the proposed method of transforming the characteristics, let alone to describe the entire MESWIR computer system, which was developed at IMP PAN (Institute of Fluid-Flow Machinery of the Polish Academy of Sciences) in Gdansk based on these concepts. A wide range of information is available in publications [22, 24].

To develop a reliable, consistent mathematical and numerical model of all subsystems of a turbomachine, it is also necessary, apart from the transformation of the characteristics of the supporting structure, to develop an elastic-thermo-dynamic model of a slide bearing and the FEM (finite element method) model of the rotor itself.

As the global stiffness and damping matrices—appearing in the equations of motion (equation (1)) – are non-linear functions of generalized displacement and speed, the task of building a coherent model of a turbomachine becomes very complex.

The question can now be asked why it is so important to build coherent models (i.e. describing the operation of the system under both stable and unstable conditions) of the entire system of a turbomachine. There are many reasons, but two of them seem particularly important:

- The possibility of using these models in model-based diagnostics. Only non-linear models can generate non-elliptical trajectories and vibration spectra, which are the basis for evaluating the condition of the machine in technical diagnostics.
- The possibility of tracking the machine vibrations once the stability limit is exceeded. Only coherent non-linear models make it possible to assess the degree of danger to the machine caused by, for example, oil whirls and oil whips in the bearings. The superposition of linear models of individual subsystems does not offer such possibilities, as a uniform description of the transition from a stable to unstable state is required here.

Figure 5 shows the calculation algorithm used in the MESWIR computer system, which creates the possibilities described above. It consists of two large calculation loops: the loop in which the transformation of the supporting structure characteristics is carried out and the non-linear calculation loop.

Figure 6 illustrates a calculation example of the impact of the transformation of the dynamic characteristics of the supporting structure, i.e. the impact of the presented concepts of adequacy intervals and weight functions, on the behaviour of a high-powered machine under stable and unstable operating conditions. The instability of the system was caused by a simulated crack located in the generator rotor. It can be seen here that when the system operates stably, the impact of the concept of adequacy intervals and weight functions is small. However, once the stability limit of the system is exceeded, considering the transformation of the characteristics worsens the whole situation. The subharmonic components in the vibration spectrum are clearly higher, which may mean an already dangerous situation.

The transformation method presented here certainly requires further research. The major difficulty is the proper selection of adequacy intervals and their share in the vibration spectrum structure. This is not an easy task, especially when the experimental vibration spectra at our disposal are so complex. Perhaps many other researchers are developing their own transformation methods, and these will be the subject of future publications.

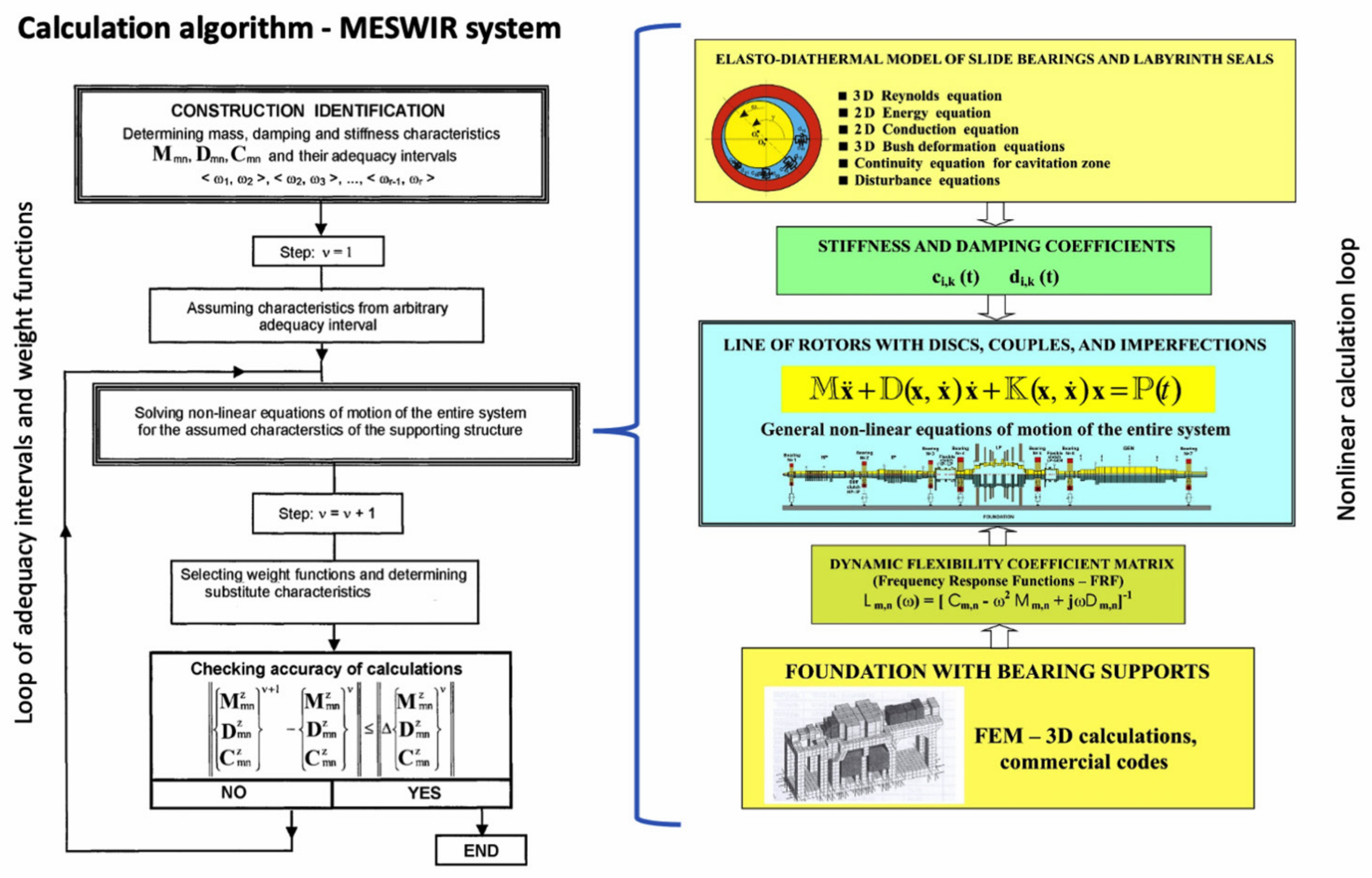
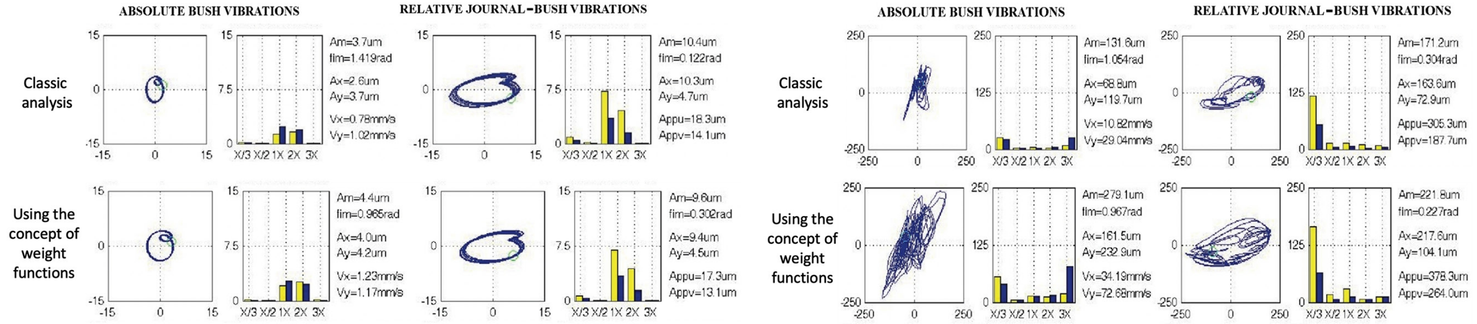


Fig. 5. MESWIR computer system developed at IMP PAN in Gdansk [22–24], calculation algorithm

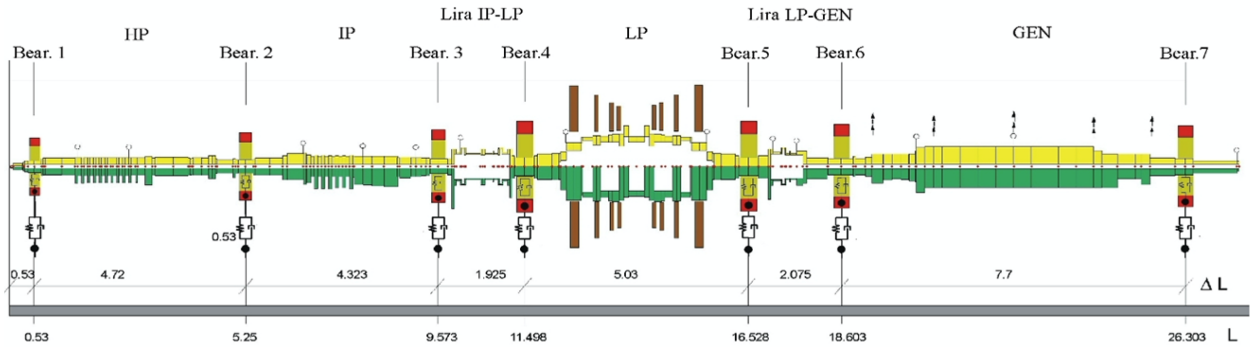
Analysis example: influence of the concept of adequacy intervals and weight functions
on the course of the rotor lines vibrations

Calculation results for stable operating conditions

Calculation results for unstable working conditions caused by the cracked generator rotor

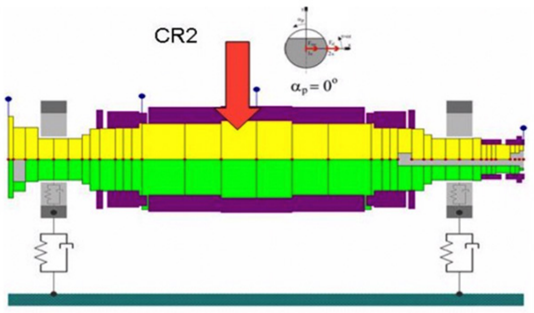


The effect of the weight function is small



Object of research - 200 MW turboset

In this case weight functions analysis indicate a dangerous situation



Simulation of the rotor crack located in the middle part of the generator.

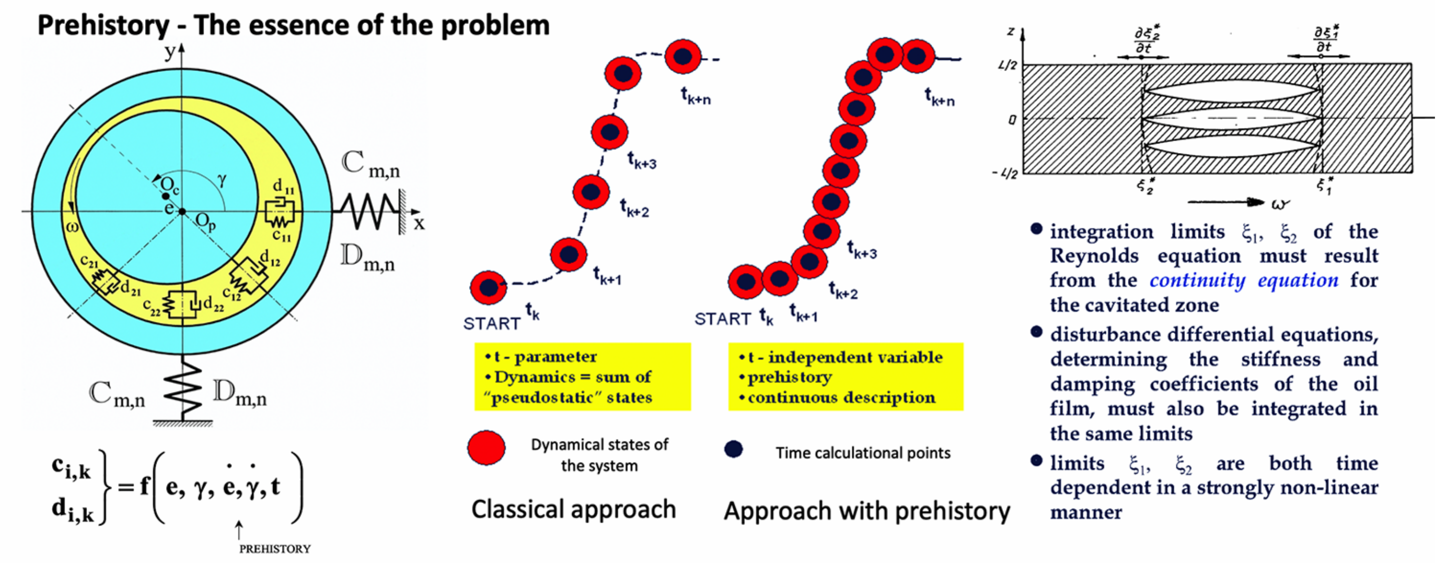
Fig. 6. Calculation results obtained from the MESWIR system (study of the impact of the transformation of the supporting structure characteristics on stable and unstable operating conditions in a turbo set with a power capacity of 200 MW)

To sum up this part of the considerations, it should be stated that the problem of a coherent description of the dynamic states of a turbomachine, both under stable operating conditions and after exceeding the stability limit, is still not fully resolved.

That is how we have come to the first group of problems and questions that pose a great challenge to researchers all over the world dealing with rotor dynamics.

3. THE PROBLEM OF *PREHISTORY*

The so-called *prehistory* issues, i.e. the impact of dynamic states occurring in the preceding moments on the future states of a turbomachine, are closely related to the way time is treated in the equation of motion (equation (1)). If time is treated as a parameter, which is common practice, then we are dealing, from a mathematical-numerical perspective, with the sum of



- integration limits ξ_1, ξ_2 of the Reynolds equation must result from the *continuity equation* for the cavitated zone
- disturbance differential equations, determining the stiffness and damping coefficients of the oil film, must also be integrated in the same limits
- limits ξ_1, ξ_2 are both time dependent in a strongly non-linear manner

Fig. 7. The essence of the "prehistory" phenomenon, explained using time-varying boundaries of a cavitation zone within the oil film of slide bearings

dynamic states that are independent of time and calculated separately at each point in time. The dynamic states of the machine are therefore the sum of the 'pseudo static' states.

Let us now assume that time is a normal independent variable in the equations of motion (equation (1)). This may mean that the external forces acting on the system may freely change over time (right side of the equation), or this may mean that the physical processes occurring, for example, in the cavitation zone are independent of the movement of the journal (left side of the equation).

This assumption complicates the situation enormously. This fact can be considered, for example, by changing the stiffness and damping coefficients of slide bearings over time, although there are, of course, other possibilities. These coefficients are now a function of the position of the journal centre, the speed of change of this position, and time. This means that the future state is linked to the previous state, which is what we have termed *prehistory* here.

What does this approach practically mean in a slide bearing? It is easy to imagine the time-varying boundaries of the cavitation zone within which the Reynolds equation must be integrated. After all, the variation of the boundaries of the cavitation zone over time depends not only on the position and speed of the bearing journal but also on the physical processes taking place in the cavitation bubbles, i.e. it depends on time. The essence of the *prehistory* phenomenon is shown graphically in Fig. 7.

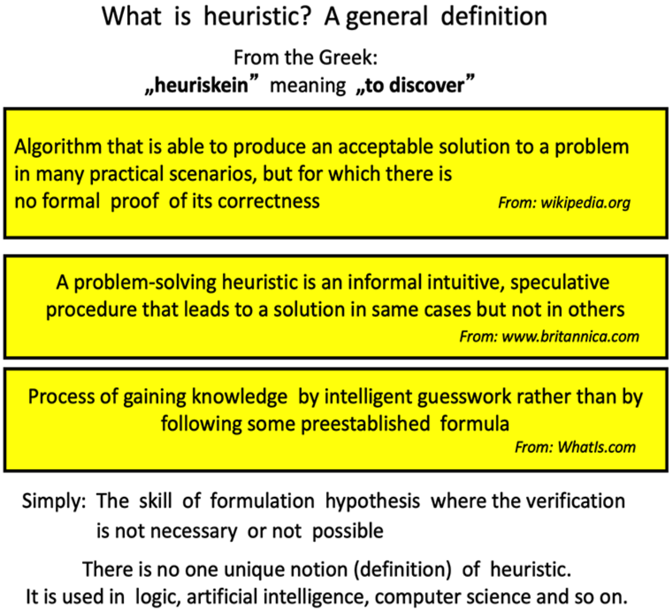
So, we have come to another still poorly diagnosed problem in rotor dynamics.

4. *HEURISTIC* MODELLING

Let us now move on to another, very interesting issue, namely to the question: Does it make sense to apply heuristic methods in rotor dynamics and do they have a future?

Let us start with the definition of *heuristic*. It has many definitions, as shown in Fig. 8. However, the general definition is important, namely that *heuristic* means, among other things, the ability to formulate hypotheses or conclusions based on uncertain data which are not always verifiable. If we look at heuristic in this way, possible fields of application in rotor dynamics immediately emerge. After all, we usually have input data for our models that may have been affected by various measurement errors, even accidental ones, and yet we expect reliable results.

The second field of application of a *heuristic* may be the operation of a turbomachine just at the limit of stability when


What is heuristic? A general definition

From the Greek:
„heuriskein" meaning **„to discover"**

Algorithm that is able to produce an acceptable solution to a problem in many practical scenarios, but for which there is no formal proof of its correctness *From: wikipedia.org*

A problem-solving heuristic is an informal intuitive, speculative procedure that leads to a solution in same cases but not in others *From: www.britannica.com*

Process of gaining knowledge by intelligent guesswork rather than by following some preestablished formula *From: WhatIs.com*

Simply: The skill of formulation hypothesis where the verification is not necessary or not possible

There is no one unique notion (definition) of heuristic.
It is used in logic, artificial intelligence, computer science and so on.

Fig. 8. Popular definitions of a heuristic

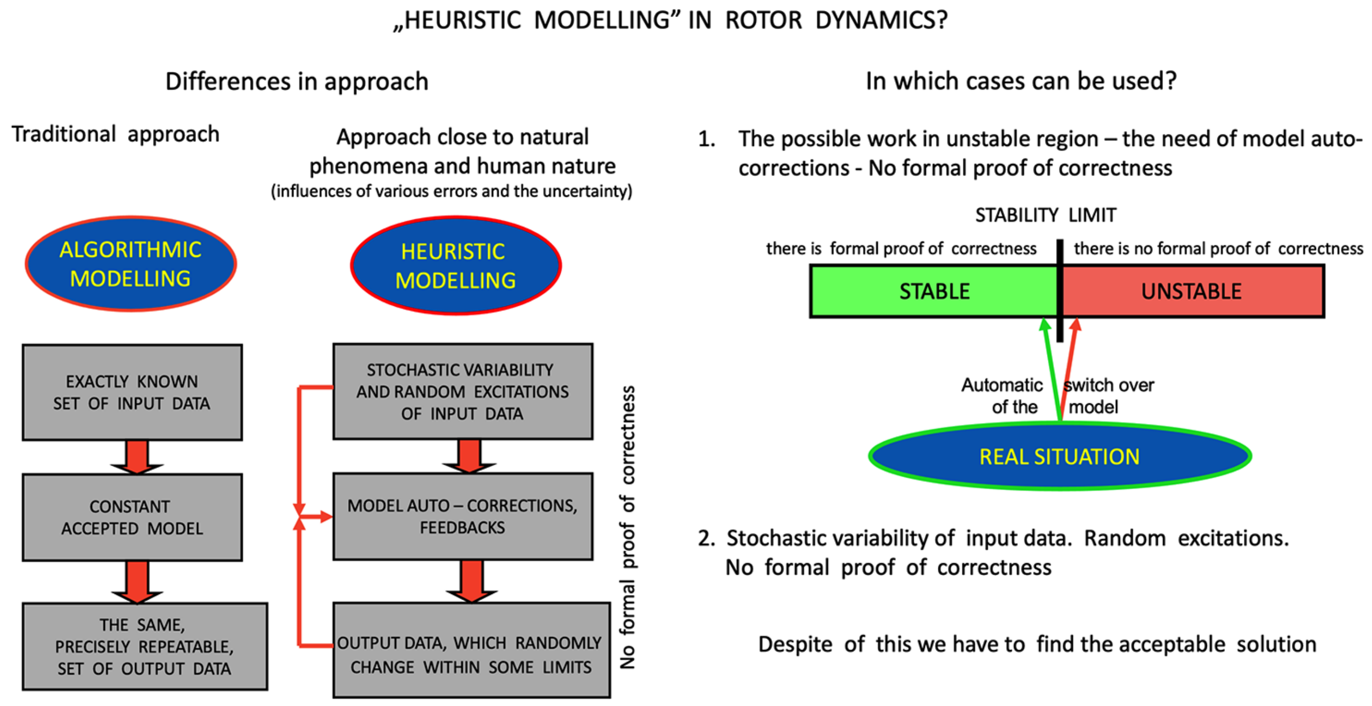


Fig. 9. Possible fields of application of heuristic methods in rotor dynamics; differences between algorithmic and heuristic modelling

it is still possible to constantly exceed it. In such situations, we would like to know how the machine behaves when the input data to our models can be chaotically variable. This is illustrated in Fig. 9.

The research team of the author of this article conducted preliminary studies, considering the stochastic and random variability of the input data for the models describing the dynamic performance of rotors.

Suppose we have a simple rotor with two supports, on which a synchronous external force P acts, which can simulate, for example, rotor unbalance. If we further assume that the rotor is subjected to an additional force +/−DP stochastically variable over time, which can simulate, for example, unbalance fluctuations, then we are dealing with a situation in which we can already apply tools appropriate for heuristic.

In the research, two cases have been analysed:

- The rotor is subjected to a constant P + DP force, and then to a P-DP force (to define the trajectory boundaries for comparison purposes).
- The rotor is subjected to a P+/−DP vector, stochastically variable in time.

In the calculation model (MESWIR system), the variable stochastic input data was obtained using a random number generator within the +/−DP limits.

Examples of calculation results for the above case are shown in Fig. 10. As shown in the figure, the vibration trajectory of the journal, considering the stochastic variation of the input data, goes slightly beyond the 'constant force band', but is still within the permitted area. Of course, things can be different in other situations. Of course, in other situations, the trajectories may look different.

As this example shows, heuristic tools can and should be used in rotor dynamics. The heuristic approach reflects reality better, but there is still a long way to go before its widespread use.

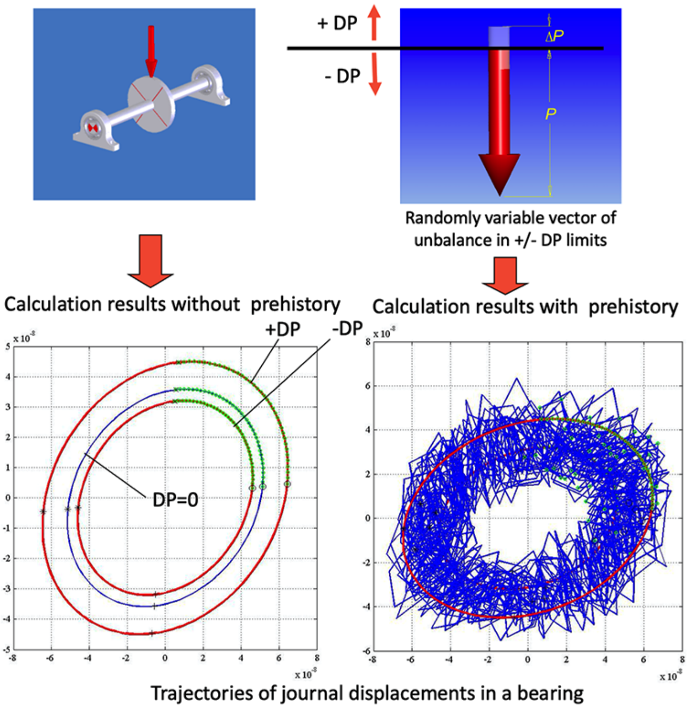


Fig. 10. Calculation results showing the trajectories of journal displacements in bearings, obtained using a random number generator for the excitation force (right side) and assuming that the excitation forces are constant (left side)

5. MULTIPLE WHIRLS PHENOMENON

Before finishing this article, an interesting phenomenon is worth mentioning, namely that of multiple whirls. The appearance of whirls in the oil film of the slide bearings (subharmonic vibrations) means that the stability limit has been exceeded. It is therefore no longer possible for the turbomachine to continue the operation stably. This is what results from known theories of the stability of mechanical systems.

However, there are situations when a turbomachine regains stability and then loses it again. This can happen when the shape of the lubrication gap changes with the rotational speed. But how can this happen in practice?

With a strong initial clamping of the bush and its partial fixation, thermal deformations of the bush and changes in the hydrodynamic pressure distribution force such a shape on the lubrication gap that a return to stable operation becomes possible, even if the rotational speed of the rotor constantly increases. Such a situation is illustrated in Fig. 11.

At IMP PAN in Gdansk, a study was carried out on a high-speed rotor with two supports, in which the bush clamp and its thermal deformations corresponded to the situation described above.

The results of the analysis are presented in Fig. 12. What is interesting is the fact that the phenomenon of multiple whirls,

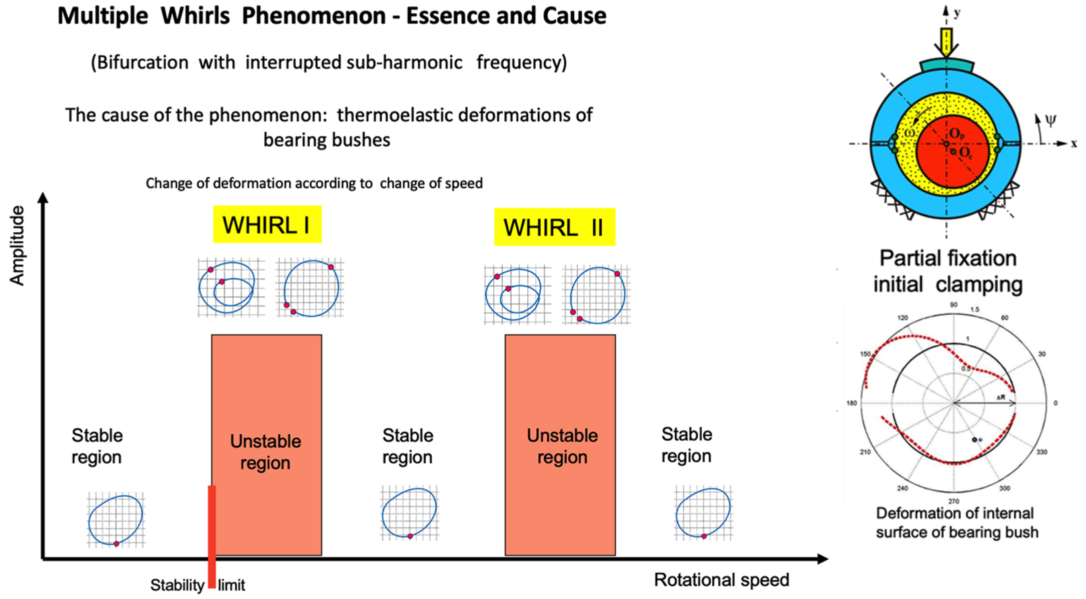


Fig. 11. The essence of the phenomenon of multiple whirls (thermal deformation of the bush and its clamp on the shaft can lead to a situation where stable operation of the turbomachine is no longer possible)

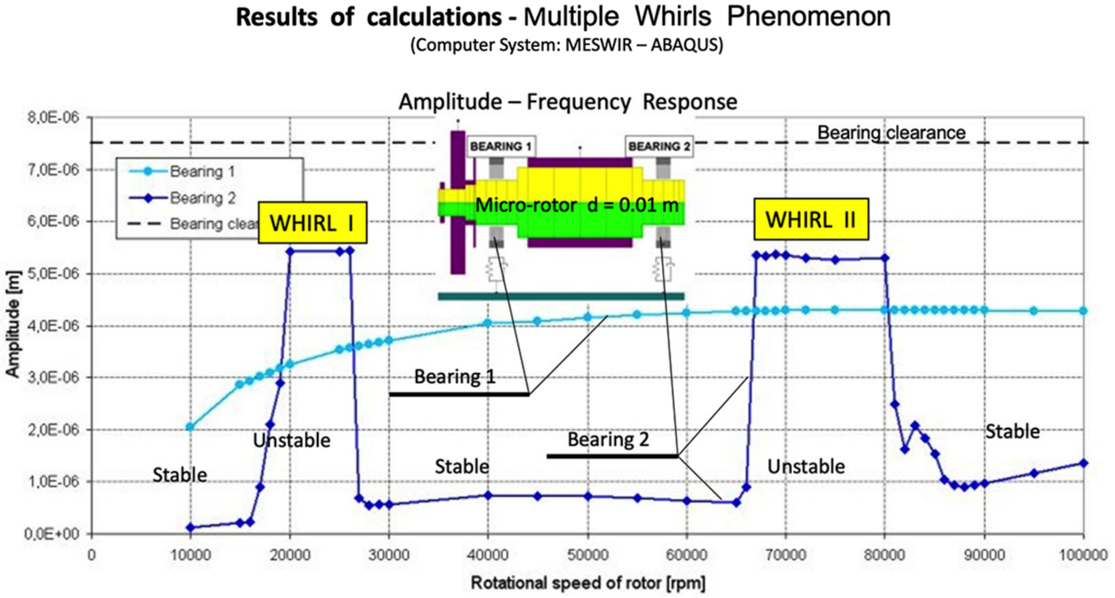


Fig. 12. Computer simulation results illustrating the occurrence of multiple whirls

that is to say, the return to stable operation, only occurs in bearing No. 2. Bearing No. 1 constantly operates under unstable conditions and the displacement trajectory of the journal reaches the limit values defined by the oil whip. The above example is the result of computer simulations conducted using the very advanced program called MESWIR (Fig. 5).

The question can now be asked as to whether such cases can occur in practice. If we deal, for example, with a high-speed compressor used in a turbocharger system of an internal combustion engine, the operating conditions of the bearings may be similar to those shown in Fig. 12.

6. CONCLUDING REMARKS

The issues discussed in this article represent only a small part of the current problems that await solutions. All the examples presented here require very advanced models and computer systems to conduct an analysis. Nowadays, we already have high-performance computers, which make the work of engineers and scientists much easier and faster.

However, it is worth answering the question: Do we need to always use such complex work tools? Of course not. In many situations where we can expect relatively stable operating conditions, we can use simpler and faster models as well as commercial programs — even when analysing a large machine. And this is a common practice.

However, reason dictates that there is a need to further develop both simple tools used for rapid engineering analysis and more complex tools that can help explain phenomena such as those mentioned in this article.

Progress in the field of rotor dynamics will be determined by the parallel development of both tools as well as by intensive experimental research. Progress is necessary because turbomachines are critical machines, that is, machines that can pose a threat to human life in the event of their failure.

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