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Engineering example of the constraint forces in non-holonomic mechanical: forklift-truck robot motion. Part I

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In the presented paper, a problem of nonholonomic constrained mechanical systems is treated. New methods in nonholonomic mechanics are applied to a problem of a Forklift-truck robot motion. This method of the geometrical theory of general nonholonomic constrained systems on fibered manifolds and their jet prolongations, based on so-called Chetaev-type constraint forces. The relevance of this theory for general types of nonholonomic constraints, not only linear or affine ones, was then verified on appropriate models. On the other hand, the equations of motion of a Forklift-truck robot are highly nonlinear and rolling without slipping condition can only be expressed by nonholonomic constraint equations. In this paper, the geometrical theory is applied to the above mentioned mechanical problem. The results of numerical solutions of constrained equations of motion, derived within the theory, are presented.

Key words: forklift-truck robot; Lagrangian systems, nonholonomic constraints, reduced equations of motion, numerical solution.

1. Introduction

Forklift-truck is a powered industrial truck used to lift and move materials on short distances. The forklift was developed in the early 20^{th} century by various companies including the transmission manufacturing company Clark and the hoist company Yale & Towne Manufacturing (Brindley, 2005). Following World War II the use and development of the forklift truck has greatly expanded worldwide. Forklifts have become an indispensable piece of equipment in manufacturing and warehousing operations. Many researchers are linearized the forklift system equations for used in open-loop and closed-loop controllers (e.g., Zimmert and Sawodny, 2010; Lee, 2014). In addition, analogous to many mechanical systems such as mobile robots (Ou et al., 2014), aerial/underwater ve-

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hicles (Zhang et al., 2014; Sun et al., 2013; Li and Wang, 2013), wheeled pendulum robots (Xu et al., 2014), underactuated robots (Xia et al., 2014; Lai et al., 2014; Xin and Liu, 2013), moving liquid containers (Zang and J. Huang, 2015), and so also on, forklifts are nonholonomic systems, whose the governing equation issues are challenging and remain open. On the other hand, in some mechanical and engineering problems one encounters different kinds of additional conditions, constraining and restricting motions of mechanical systems. Such conditions are called constraints (Swaczyna, 2011; Haddout, 2018a). Constraints may be given by algebraic equations connecting coordinates (holonomic or geometric constraints), or by differential equations, which restrict coordinates and components of velocities. Non integrable kinematic constraints, which cannot be reduced to holonomic ones, are called nonholonomic constraints (Swaczyna, 2011; Haddout, 2018b; Haddout et al., 2017). In addition, the motion of mechanical systems is frequently subjected to various constraint conditions, holonomic or nonholonomic. Nonholonomic constraints lead typically to nonlinear equations of motion of the constrained system. While theories of holonomic or some special types of linear nonholonomic constraints are already well elaborated for quite general situations, various theoretical approaches to general nonholonomic mechanics occur up to now, from the physical point of view on the one hand, and from the geometrical point of view on the other. On the other hand, in last decades numerous physical and engineering applications make necessary to profound research and complete the theory of the nonholonomic systems and numerical aspects of solutions are presented. Therefore problems of nonholonomic mechanics are intensively studied in many recent papers, e.g., Bullo and Lewis, 2004; Cardin and Favreti, 1996; Carinena and Raada, 1993; Cortes et al., 2009; de Leon et al., 1997a, 1997b; Giachetta, 1992; Janová and Musilová, 2009; Czudková, and Musilová, 2013, where modern methods and concepts of differential geometry and global analysis are used and which contribute to the essential advance in both the theoretical and application aspects. The geometrical theory used in the presented paper was applied to the first order mechanical problems in (Krupková, 1997a) and then generalized for higher order case in (Krupková, 2000). It serves as an appropriate tool for constructing certain type of equations of motion of nonholonomic mechanical systems subjected to quite general constraints (an application to typically non-linear constraint see in (Krupková and Musilová, 2001)). The theory is developed on fibered manifolds and their jet prolongations as underlying geometrical structures, naturally related to the character of physical problems. The main physical idea of the theory is based on the concept of Chetaev-type constraint forces introduced in analogy to “classical” Chetaev forces (see Chetaev, 1932–1933). Equations of corresponding unconstrained motion are related to the so-called dynamical form and they define the components of this form. Using equations of constraints, a special canonical distribution on the first jet prolongation of the underlying manifold (corresponds to the phase space) can be constructed. Then the first prolongations of admissible trajectories of the constrained motion are just integral sections of this distribution. By adding Chetaev-type forces (with Lagrange multipliers) to equations of motion, a dynamical form of the constrained problem is obtained and deformed equations of motion are constructed. These equations together with constraint conditions give the system of differential equations for unknown constrained trajectories and Lagrange multipliers. Another possible approach to the problem within the same theory starts from its description by the so-called Lepage class of forms instead the dynamical form itself. The Lepage class is, of course, closely related to the dynamical form, and it is obtained by the factorization of modules of forms by special submodules irrelevant from the point of view of the problem. This procedure leads to the so-called reduced equations of motion containing no Lagrange multipliers and giving the system of differential equations for constrained trajectories only. Nevertheless, constraint forces can be then obtained from deformed equations. In addition, on the base of the geometrical theory with Chetaev-type constraint forces, one can formulate a constraint variational principle and solve the corresponding constraint inverse variational problem (see e.g., Krupková and Musilová, 2005), as well as study symmetries of constrained systems. Symmetries and arising first integrals may then essentially simplify integration of the resulting constrained equations of motion, see e.g., Swaczyna, 2005. (Nevertheless, in the present paper no attention is paid to higher order theories, field theories and the constraint variational problem.) Of course, the calculation procedure itself is made in coordinates. Its practical advantage lies in the possibility to choose appropriate coordinates, and also in two equivalent alternatives of solving the problem. The first of them is based on the solution of reduced equations of motion free of Lagrange multipliers and additional computation of these multipliers and corresponding constraint forces from dynamical equations, or alternatively, the direct solution of dynamical equations containing Lagrange multipliers. The decision between these two procedures is influenced by the concrete physical problem. Even though the corresponding constraint is semiholonomic and thus it could be in principle treated by classical methods of Lagrange multipliers (for details concerning the method in general see e.g. the classical textbook of analytical mechanics (Brdicka and Hladík, 1987)), the direct application of Krupková's geometrical theory is very effective in this situation. On the other hand, a great interest has been devoted towards forklifts-truck modeling as it is a mechanical system characterized by nonholonomic constraints. On the other hand, the forklifts-truck have become an indispensable piece of equipment in manufacturing and warehousing operations in the word, that many researchers have tried to find proper equations to describe the dynamic of this system. Mainly, it is possible to distinguish between two different approaches: the first obtains the motion equations using the Newton's laws, while the second studies the system from the Lagrangian or Hamiltonian point of view (Neimark and Fufaev 1972; Koon and Marsden, 1997). So far, the greatest part of the existing literature has been dedicated to models with lots of simplifications, even if these have been capable to explain the dynamical characteristics of the forklift truck. For example, linearized equations of motion are commonly introduced in order to cope more easily with the problem.

The aim of this paper is to use the geometrical theory for obtaining nonlinear equations of motion of the above exposed mechanical problem, using the above mentioned Krupková approach for a practical mechanical system and find their solution in some particular cases, any simplifications are not used. This is made in the last section, where the sets of equations of motion i.e., reduced, are derived. The numerical solution of reduced equation is presented. We arrange the remaining parts as follows. In Section 2, we introduce the geometrical theory of nonholonomic mechanical systems. Further, Section 3 presents the nonlinear dynamics of forklift-truck. After that, the proposed the reduced equations for the forklift-truck and numerical solution in Section 4. Finally, the paper is wrapped up with conclusions in Section 5.

2. Geometrical theory of nonholonomic mechanical systems

In this section, we recall basic geometrical concepts of the theory we will use. For more details and proofs see (Krupková, 1997a). As underlying geometrical structures of the theory fibred manifolds and their jet prolongations are considered. Key geometrical objects adapted to the fibred structure are sections and their jet prolongations, projectile and vertical vector fields, as well as horizontal and contact differential forms. The detailed theoretical background can be noted in (Krupková, 1998).

The geometrical theory of nonholonomic mechanical systems is developed on an $(m+1)$-dimensional underlying fibred manifold (Y,π,X) with the one-dimensional base X, ($(t \in X)$ being time in non-relativistic mechanics), m-dimensional fibers (configuration space), and its jet prolongations $(J^sY,\ \pi_s,\ X)$ with $s = 1,\ 2$ for typical physical cases (a fiber of J^1Y over $(t \in X)$ represents the phase space). We denote (V,ξ), $\xi = (t,q^\sigma)$, $1 \leqslant \sigma \leqslant m$, a fibred chart on Y, (U,ζ), $U = \pi(V)$, $\zeta = (t)$, the associated chart on X and $(V_s,\ \xi_s)$, $V_s = \pi_s^{-1}(U)$, $\xi_s = (t,\ q^\sigma,\ q_s^\sigma)$ the associated fibred chart on J^sY, where $q_1^\sigma = \dot{q}^\sigma$ and $q_2^\sigma = \ddot{q}^\sigma$. Moreover, denote by $\pi_{r,s}$: $J^rY \to J^sY$, $0 \leqslant s \prec r \leqslant 2$, $J^0Y = Y$, canonical projections. A section of fibered manifold $(Y,\ \pi,\ X)$ is a smooth mapping γ: $I \to Y$, such that $\gamma \circ \pi = id_I$, $I \subset X$ being an open set. Analogously sections of $(J^rY,\ \pi_r,\ X)$

are defined. A section δ of (J^rY, π_r, X) is called holonomic if it is of the form $\delta = J^rY$, where γ is a section of (Y, π, X).

Recall, that a vector field η on J^rY is called π_r-projectile if there exists a vector field η_0 on X such that $T\pi_r\eta = \eta_0 \circ \pi_r$. A vector field η is called π_r-vertical if $T\pi_r\eta = 0$. A form ρ on J^rY is called π_r-horizontal if its contraction by an arbitrary chosen π_r-vertical vector field η vanishes, i.e. it holds $i_\eta\rho = 0$. A form ρ is called contact if $J^r\gamma^*\rho = 0$ for all sections γ of (Y, π, X). Concepts of $\pi_{r,s}$-projectile vector field, $\pi_{r,s}$-vertical vector field, and $\pi_{r,s}$-horizontal form are defined by the quite analogous way. Moreover, for every k-form ρ on J^rY there exists the unique decomposition into its q-contact components, $0 \leqslant q \leqslant k$, $\pi^*_{r+1,r}\rho = \sum_{q=0}^{k} p_q\rho$, the 0-contact components $p_q\rho = h\rho$ are called also the horizontal one.

From the point of view of physics, all possible trajectories of so-called first order unconstrained mechanical system on a fibred manifold are given just by section γ of (Y, π, X) such that they are solution of the system of m second order ordinary differential equations of motion:

$$E_\sigma \circ J^2\gamma = 0, \qquad E_\sigma = A_\sigma(t, q^\lambda, \dot{q}^\lambda) + B_{\sigma\nu}(t, q^\lambda, \dot{q}^\lambda)\,\ddot{q}^\nu, \tag{1}$$

where $1 \leqslant \lambda \leqslant m$ and Einstein summation are used. Consider the 1-contact $\pi_{2,0}$-horizontal 2-form on $J^2\gamma = 0$, $E = E_\sigma\omega^\sigma \wedge dt = 0$, called dynamical form. A solution γ of Eqs. (1) is called a path of E. We define the Lapage class $[\alpha]$ of E by the requirement $p_1\alpha = E$ (see Krupková, 1997a or 1997b).

The class $[\alpha]$ is named also the mechanical system. Every representative of this class is of the form:

$$\alpha = A_\sigma\omega^\sigma \wedge \mathrm{d}t + B_{\sigma\nu}\omega^\sigma \wedge \mathrm{d}\dot{q}^\sigma + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu, \tag{2}$$

where $\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt$ are contact 1-forms forming the basis of 1-forms $(dt, \omega^\sigma, \dot{q}^\sigma)$ on J^1Y adapted to the contact structure. So, $[\alpha] = \alpha$ mod 2-contact forms. The following proposition was proved (see Krupková, 1997a or 1997b):

Proposition 1 *A section γ of (Y, π, X) is a path of the dynamical from E if and only if*

$$J^1\gamma^* i_\eta\alpha = 0 \tag{3}$$

for every π_1-vertical vector field η on J^1Y.

2.1. Non-holonomic dynamics

A nonholonomic constrained mechanical system is defined on the $(2m+1-k)$-dimensional constrained sub-manifold $\wp \subset J^1Y$ fibered over Y and given by k equations $(1 \leqslant k \leqslant m-1)$

$$f^i(t, q^\sigma, \dot{q}^\sigma) = 0 \quad \text{such that rank} \quad \left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = k \quad 1 \leqslant i \leqslant k$$

or in the explicit normal form

$$\dot{q}^{m-k+i} - g^i(t, q^\sigma, \dot{q}^l), \qquad 1 \leqslant l \leqslant m-k. \tag{4}$$

It is evident that only admissible trajectories for a nonholonomic mechanical system are such sections $\gamma:\ I \ni t \to Y$ for which $J^1\gamma(t) \in \wp$ for all $t \in I$, i.e, $f^i \circ J^1\gamma = 0$ for $1 \leqslant i \leqslant k$ (the so-called $\wp$-admissible sections). The constraint (4) leads to the canonical distribution $\Im$ of codimension k on $\wp$. Its annihilator is of the form

$$\Im^0 = span\{\varphi^i\}, \qquad \varphi^i = -\frac{\partial g^i}{\partial \dot{q}^l}\omega^l + \iota^*\omega^{m-k+i}, \tag{5}$$

where ι: $\wp \to J^1Y$ is the canonical embedding. The canonical distribution is closely related to the constraint ideal $\Theta(\Im^0)$

$$\Theta(\Im^0) = \left\{\varphi^i \wedge \chi_i \middle| \chi_i \text{ is a form on } \wp\right\}, \tag{6}$$

where φ^i are 1-forms on $\wp$ called canonical constraint 1-forms. The importance of the canonical distribution is evident from its following property (see Swaczyna, 2005; Krupková, 1997b):

A section γ of Y is $\wp$-admissible if and only if $J^1\gamma$ is an integral section of the canonical distribution.

We have already mentioned in the first part that there are two possible equivalent approaches to the description of nonholonomic mechanical system-one of them, called physical, is based on deformed equations with constraint forces and Lagrange multipliers and the other, geometrical one, uses reduced equations.

Geometrical approach introduces the constrained mechanical system related to the mechanical system $[\alpha]$ by the equivalence relation:

$$[\alpha_\wp] = [\iota^*\alpha] \ \text{mod}\ \Theta(\Im^0). \tag{7}$$

A $\wp$-admissible section γ of (Y, π, X) is called a path constrained system $[\alpha_\wp]$ if for every π_1-vertical vector field η belonging to the canonical distribution it holds

$$J^1\gamma^* i_\eta \alpha = 0. \tag{8}$$

The following proposition can be formulated (see again Krupková, 1997b):

Proposition 2 *A section γ of (Y,π,X) is a path of the deformed system $[\alpha_\Phi]$ if and only if for every π_1-vertical vector field η belonging to $\Im$ holds*

$$f^i \circ J^1\gamma = 0, \qquad (A'_l + B'_{ls}\ddot{q}^s)\circ J^2\gamma = 0, \tag{9}$$

$$A'_l = \left(A_l + \sum_{j=1}^{k} A_{m-k+j}\frac{\partial g^j}{\partial \dot{q}^l} + \sum_{i=1}^{k}\left(B_{l,m-k+i} + \sum_{j=1}^{k} B_{m-k+j,m-k+i}\frac{\partial g^j}{\partial \dot{q}^l}\right)\right. \\ \left. \times\left(\frac{\partial g^i}{\partial t} + \frac{\partial g^i}{\partial q^\sigma}\dot{q}^\sigma\right)\right)\circ \iota, \tag{10}$$

$$B'_{ls} = \left(B_{ls} + \sum_{i=1}^{k}\left[B_{l,m-k+i}\frac{\partial g^i}{\partial \dot{q}^s} + B_{m-k+i,s}\frac{\partial g^i}{\partial \dot{q}^l}\right]\right. \\ \left. + \left(B_{m-k+j,m-k+i}\frac{\partial g^j}{\partial \dot{q}^l}\frac{\partial g^i}{\partial \dot{q}^s}\right)\right)\circ \iota. \tag{11}$$

Relations (9) represent the system of reduced equations for m unknown functions $q^\sigma\gamma$ (k of them are first order and $(m-k)$ second order ordinary differential equations).

Physical approach is based on Chetaev-type constraint forces. Such a force is given by the constraint itself, in analogy with holonomic situations. It is expressed by the dynamical form:

$$\Phi = \Phi_\sigma \omega^\sigma \wedge dt = \mu_i \frac{\partial f^i}{\partial \dot{q}^\sigma}\wedge dt, \qquad 1 \leqslant i \leqslant k, \tag{12}$$

where functions $\mu_i(t, q^\lambda, \dot{q}^\lambda)$ are Lagrange multipliers.

Note that such dynamical form satisfies the generalized principle of virtual work $i_\eta \Phi|_U = 0$ for every π_1-vectical vector field η belonging to the constraint distribution $\Im_U$, $\Im_U^0 = span\left\{\varphi^i, df^i,\ 1 \leqslant i \leqslant k\right\}, U,\ U \cap Q \neq \emptyset$ being an open set of a chart on J^1Y, see [36].

Denote

$[\alpha_\Phi] = [\alpha - \Phi]$,

$$\alpha_\Phi = \left[A_\sigma - \mu_i \frac{\partial f^i}{\partial \dot{q}^\sigma}\right]\omega^\sigma \wedge \mathrm{d}t + B_{\sigma\nu}\omega^\sigma \wedge \mathrm{d}\dot{q}^\sigma + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu. \tag{13}$$

The equivalence class $[\alpha_\Phi]$ is called the deformed mechanical system.

A $\wp$-admissible section γ of (Y,π,X) is called a path of $[\alpha_\Phi]$ if $(E_\sigma - \Phi_\sigma)\circ J^2\gamma$. The following proposition holds (see Krupková, 1997b; Krupková, 1998):

Proposition 3 *A section γ of Y is a path of the deformed system $[\alpha_\Phi]$ if and only if for every π_1-vectical vector field η on J^1Y it holds*

$$J^1\gamma^* i_\eta \alpha_\Phi = 0 \text{ or equivalently } A_\sigma + B_{\sigma\nu}\ddot{q}^\nu = \mu_i \frac{\partial f^i}{\partial \dot{q}^\sigma} \text{ and } f^i \circ J^1Y = 0. \quad (14)$$

System (14) is given by k first order and m second order ordinary differential equations for unknown functions μ_i and $q^\sigma\gamma$ and it represents the *deformed equation*.

2.2. Semiholonomic constraints

Let us now describe a special type of nonholonomic constraints, called semiholonomic. Such conditions usually take place for rolling of rigid bodies without slipping. A system of constraints (3) is called semiholonomic if the constraint ideal (5) is differential, i.e the canonical distribution (4) is completely integrable (see e.g. Krupková, 1997a). This means that $d\varphi \in \Theta(\Im^0)$ and thus following conditions hold:

$$\frac{\partial_c g^i}{\partial q^l} - \frac{d'_c}{dt}\left(\frac{\partial g^i}{\partial \dot{q}^l}\right) = 0, \qquad \frac{\partial^2 g^i}{\partial \dot{q}^s \partial \dot{q}^l} = 0, \quad 1 \leqslant l, \quad s \leqslant m-k, \quad 1 \leqslant i \leqslant k, \quad (15)$$

where

$$\frac{\partial_c}{\partial q^l} = \frac{\partial_c}{\partial q^l} + \left(\frac{\partial g^j}{\partial \dot{q}^l}\right)\frac{\partial}{\partial q^{m-k+j}} = 0, \qquad \frac{d'_c}{dt} = \frac{\partial_c}{\partial t} + \dot{q}^l \frac{\partial}{\partial q^l} + g^j \frac{\partial}{\partial q^{m-k+j}}.$$

In the following section we apply the obtained equations (8 and 13) obtained for general nonholonomic mechanical system to the example of forklift robot system.

3. Lagrange's equation of forklift-truck robot

We will consider the forklift-truck shown in Figure 1, consists of a vehicle (forklift) and a forks-mast support mechanism.

The forklift-truck enables four degrees of freedom. The Lagrange equation determination can be determined by defining total potential and kinetic energy of the system as a function of generalized coordinates: swing angle of mast θ, swing angle of the vehicle about (OZ) axis φ (or forklift body rotation), vehicle position r (or the centre of the mass of the vehicle translation displacement on the (XOY) plane) and z centre of the mass projection of the load on the (OZ) axis (see Fig. 2). On the other hand, the total kinetic and the potential energies of the

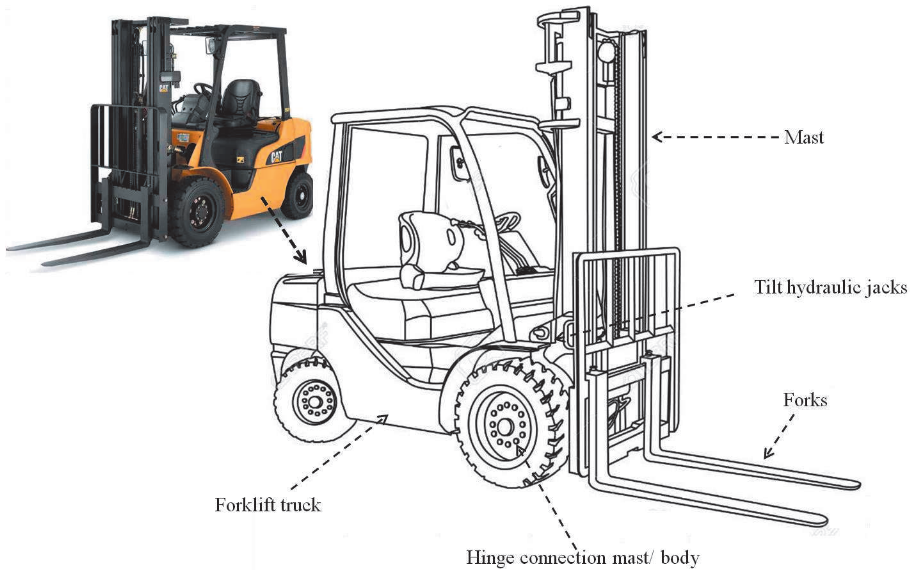

Figure 1: Schematic geometry of forklift-truck structure

system (i.e., forklift robot) in function of generalized coordinates are written as follows:

$$
\begin{aligned}
2T = {} & m_1\dot{r}^2 + m_1 r^2\dot{\varphi}^2 + \dot{\varphi}^2 J_1 + \frac{4(\dot{r}^2 + r^2\dot{\varphi}^2)}{R^2} J_{wheel} + \\
& ..m_2(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi - d\dot{\theta}\sin\theta\cos\varphi.. \\
& .. - d\dot{\varphi}\cos\theta\sin\varphi)^2 + m_2(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi - \\
& ..d\dot{\theta}\sin\theta\sin\varphi + d\dot{\varphi}\cos\theta\cos\varphi)^2 + m_2(d\dot{\theta}\sin\theta)^2 + (\dot{\varphi}\cos\theta)^2(J_2' + J_3') + \\
& (\dot{\varphi}\sin\theta)^2(J_2 + J_3) + \dot{\theta}^2(J_2'' + J_3'') + m_3(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \\
& \dot{z}\tan\theta\cos\varphi + z\frac{\dot{\theta}}{\cos^2\theta}\cos\varphi - z\dot{\varphi}\tan\theta\sin\varphi)^2 + m_3(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \\
& + s\dot{\varphi}\cos\varphi + \dot{z}\tan\theta\sin\varphi + z\frac{\dot{\theta}}{\cos^2\theta}\sin\varphi + z\dot{\varphi}\tan\theta\cos\varphi)^2 + m_3\dot{z}^2, \qquad (16)
\end{aligned}
$$

$$V = m_1 g p + m_2 g d\cos\theta + m_3 g z, \qquad (17)$$

where g is the gravitational acceleration; m_1, m_2 and m_3 are the masses of the mast, vehicle, and the load, respectively; J_1, $J_2 - J_2' - J_2''$, $J_3 - J_3' - J_3''$ and J_{wheel} are the mass moment of inertia of the forklift, mast, load and forklift-wheel respectively. R is the radius wheel and s is the distance between mast-forklift mass centre. The Lagrange function of unconstrained mechanical system is given by

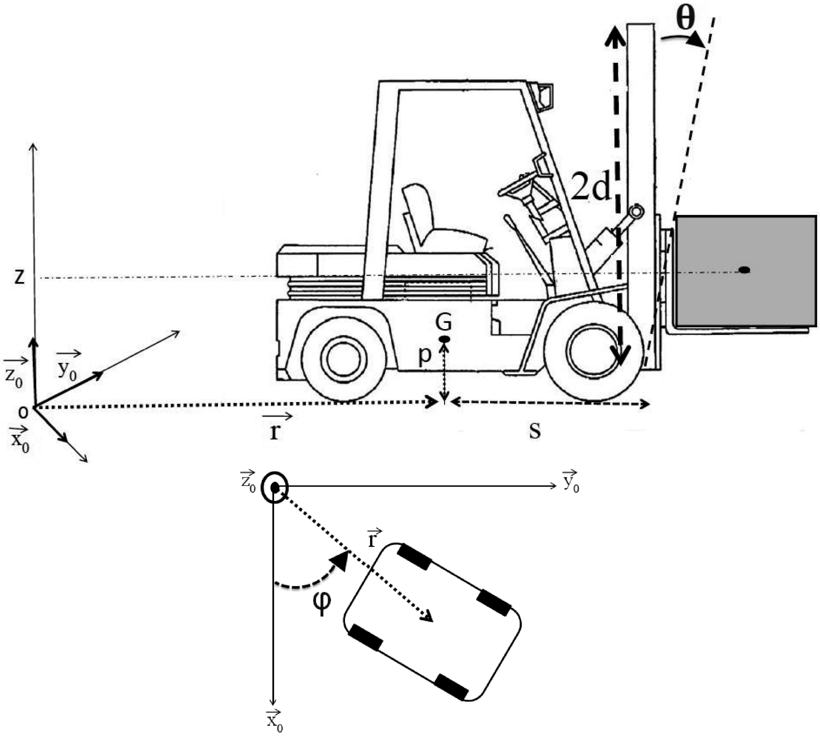

Figure 2: Coordinate systems of the forklift-truck robot

relation:

$$L = T - V,$$

$$\begin{aligned}
L = {}& \frac{1}{2}m_1\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\varphi}^2 + \frac{1}{2}\dot{\varphi}^2 J_1 + \frac{2(\dot{r}^2 + r^2\dot{\varphi}^2)}{R^2}J_{wheel} + \frac{1}{2}m_2(\dot{r}\cos\varphi \\
& - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi - d\dot{\theta}\sin\theta\cos\varphi - d\dot{\varphi}\cos\theta\sin\varphi)^2 + \frac{1}{2}m_2(\dot{r}\sin\varphi \\
& + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi - d\dot{\theta}\sin\theta\sin\varphi + d\dot{\varphi}\cos\theta\cos\varphi)^2 + \frac{1}{2}m_2(d\dot{\theta}\sin\theta)^2 \\
& + \frac{1}{2}(\dot{\varphi}\cos\theta)^2(J_2' + J_3') + \frac{1}{2}(\dot{\varphi}\sin\theta)^2(J_2 + J_3) + \frac{1}{2}\dot{\theta}^2(J_2'' + J_3'') \\
& + \frac{1}{2}m_3(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\tan\theta\cos\varphi + z\frac{\dot{\theta}}{\cos^2\theta}\cos\varphi \\
& - z\dot{\varphi}\tan\theta\sin\varphi)^2 + \frac{1}{2}m_3(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\tan\theta\sin\varphi \\
& + z\frac{\dot{\theta}}{\cos^2\theta}\sin\varphi + z\dot{\varphi}\tan\theta\cos\varphi)^2 + \frac{1}{2}m_3\dot{z}^2 - m_1 g p - m_2 g d\cos\theta - m_3 g z. \qquad (18)
\end{aligned}$$

For small swing, $\sin\theta \simeq \theta$; $\sin\varphi \simeq \varphi$; $\cos\theta \simeq 1$ and $\cos\varphi \simeq 1$. In this case, with the trigonometric functions approximated, the mast-forklift angles terms in the Lagrange equation can be simplified. Then the Lagrange equation is simplified to the following equation:

$$\begin{aligned}
L = &\frac{1}{2}m_1\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\varphi}^2 + \frac{1}{2}\dot{\varphi}^2 J_1 + \frac{2(\dot{r}^2 + r^2\dot{\varphi}^2)}{R^2}J_{wheel} \\
&+ \frac{1}{2}m_2(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi)^2 \\
&+ \frac{1}{2}m_2(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi - d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi)^2 \\
&+ \frac{1}{2}m_2(d\dot{\theta}\theta)^2 + \frac{1}{2}m_3(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi \\
&+ z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi)^2 \\
&+ \frac{1}{2}m_3(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi)^2 \\
&+ \frac{1}{2}m_3\dot{z}^2 + \frac{1}{2}\dot{\varphi}^2(J_2' + J_3') + \frac{1}{2}(\dot{\varphi}\theta)^2(J_2 + J_3) \\
&+ \frac{1}{2}\dot{\theta}^2(J_2'' + J_3'') - m_1 gp - m_2 gd - m_3 gz. \qquad (19)
\end{aligned}$$

4. Forklift-truck dynamic motion

4.1. Formulation of a problem

Using the geometrical theory of nonholonomic system mentioned in the second section we define the structure of the mechanical system as follows. In cases where the number of degrees of freedom is greater than the number of generalized coordinates, additionally defined coordinates are not independent of the present generalized coordinates. Equations with terms in the time derivatives of the generalized coordinates and which cannot be integrated are called non-holonomic constraint equations. There are four degrees of freedom of the corresponding unconstrained mechanical system i.e. Thus, the fibered manifold of the problem is $\left(\mathbb{R}\times\mathbb{R}^4, pr_1, \mathbb{R}\right)$ where pr_1 is the cartesian projection on the first factor. We choose the fibered chart on Y as (V,ξ) where V is an open set $V \subset Y$ and $\xi_1 = \left(t, q^1, q^2, q^3, q^4\right) = (t, r, z, \theta, \varphi)$. The associated chart on the base is (pr_1, Φ), $\Phi = (t)$ where t is the time coordinate, and associated fibered chart on $J^1Y = \mathbb{R}\times\mathbb{R}^4\times\mathbb{R}^4$ is $(V_1,\, \xi_1)$, $V_1 = pr^{-1}(V,\xi)$, $\xi_1 = (t, q^\sigma, \dot{q}^\sigma)$, $1 \leqslant \sigma \leqslant 4$, i.e. $\xi_1 = \left(t, r, z, \theta, \varphi, \dot{r}, \dot{z}, \dot{\theta}, \dot{\varphi}\right)$. The basic parameters used to specify the forklift-truck geometry are illustrated in Figure 2. On the other hand, the Euler–Lagrange equations of system motion are:

$$
\begin{aligned}
E_1 \equiv\; & m_1 r\dot{\varphi}^2 + 4\frac{r\dot{\varphi}^2}{R^2}J_{wheel} - m_2\dot{\varphi}\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi \\
& - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi) \\
& + m_2\dot{\varphi}\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi - d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi) \\
& - m_3\dot{\varphi}\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi + z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi) \\
& + m_3\dot{\varphi}\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi) \\
& - m_1\ddot{r} - \frac{4\ddot{r}}{R^2}J_{\text{whell}} + m_2\dot{\varphi}\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi \\
& - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi) \\
& - m_2\cos\varphi\{\ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi - s\ddot{\varphi}\sin\varphi - s\dot{\varphi}^2\cos\varphi \\
& - d\ddot{\theta}\theta\cos\varphi - d\dot{\theta}^2\cos\varphi + d\dot{\theta}\theta\dot{\varphi}\sin\varphi - d\ddot{\varphi}\sin\varphi - d\dot{\varphi}^2\cos\varphi\} \\
& - m_2\dot{\varphi}\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi - d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi) \\
& - m_2\sin\varphi\{\ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi - \dot{r}\dot{\varphi}^2\sin\varphi + s\ddot{\varphi}\cos\varphi - s\dot{\varphi}^2\sin\varphi \\
& - d\ddot{\theta}\theta\sin\varphi - d\dot{\theta}^2\sin\varphi - d\dot{\theta}\theta\dot{\varphi}\cos\varphi + d\ddot{\varphi}\cos\varphi - d\dot{\varphi}^2\sin\varphi\} \\
& + m_3\dot{\varphi}\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi + z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi) \\
& - m_3\cos\varphi\{\ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi - s\ddot{\varphi}\sin\varphi - s\dot{\varphi}^2\cos\varphi \\
& + \ddot{z}\theta\cos\varphi + 2\dot{z}\dot{\theta}\cos\varphi - 2\dot{z}\theta\dot{\varphi}\sin\varphi \\
& + z\ddot{\theta}\cos\varphi - 2z\dot{\theta}\dot{\varphi}\sin\varphi - z\ddot{\varphi}\theta\sin\varphi - z\dot{\varphi}^2\theta\cos\varphi\} \\
& - m_3\dot{\varphi}\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi) \\
& - m_3\sin\varphi\{\ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi - \dot{r}\dot{\varphi}^2\sin\varphi + s\ddot{\varphi}\cos\varphi - s\dot{\varphi}^2\sin\varphi \\
& + \ddot{z}\theta\sin\varphi + 2\dot{z}\dot{\theta}\sin\varphi \\
& + 2\dot{z}\theta\dot{\varphi}\cos\varphi + z\ddot{\theta}\sin\varphi + 2z\dot{\theta}\dot{\varphi}\cos\varphi + z\ddot{\varphi}\theta\cos\varphi - z\dot{\varphi}^2\theta\sin\varphi\} = 0, \quad (20)
\end{aligned}
$$

$$
\begin{aligned}
E_2 \equiv\; & -m_2 d\dot{\theta}\cos\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi) \\
& - m_2 d\dot{\theta}\sin\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi - d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi) \\
& + m_2 d^2\dot{\theta}^2\theta + m_3(\dot{z}\cos\varphi - z\dot{\varphi}\sin\varphi)(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi \\
& + \dot{z}\theta\cos\varphi + z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi) + m_3(\dot{z}\sin\varphi + z\dot{\varphi}\cos\varphi) \\
& \cdot (\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi) \\
& + \dot{\varphi}^2\theta(J_2 + J_3) + m_2 d\dot{\theta}\cos\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi \\
& - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi)
\end{aligned}
$$

$$
\begin{aligned}
&- m_2 d\theta\dot{\varphi}\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi)\\
&+ m_2 d\theta\cos\varphi\{\ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi - s\ddot{\varphi}\sin\varphi\\
&- s\dot{\varphi}^2\cos\varphi - d\ddot{\theta}\theta\cos\varphi - d\dot{\theta}^2\cos\varphi + d\dot{\theta}\theta\dot{\varphi}\sin\varphi - d\ddot{\varphi}\sin\varphi - d\dot{\varphi}^2\cos\varphi\}\\
&+ m_2 d\dot{\theta}\sin\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi - d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi)\\
&+ m_2 d\theta\dot{\varphi}\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi - d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi)\\
&+ m_2 d\theta\sin\varphi\{\ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi - \dot{r}\dot{\varphi}^2\sin\varphi + s\ddot{\varphi}\cos\varphi\\
&- s\dot{\varphi}^2\sin\varphi - d\ddot{\theta}\theta\sin\varphi - d\dot{\theta}^2\sin\varphi - d\dot{\theta}\theta\dot{\varphi}\cos\varphi + d\ddot{\varphi}\cos\varphi - d\dot{\varphi}^2\sin\varphi\}\\
&- m_2\ddot{\theta}\theta^2 - 2m_2\dot{\theta}^2\theta\\
&- m_3\dot{z}\cos\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi + z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi)\\
&+ m_3 z\dot{\varphi}\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi + z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi)\\
&- m_3 z\cos\varphi\{\ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi - s\ddot{\varphi}\sin\varphi - s\dot{\varphi}^2\cos\varphi\\
&+ \ddot{z}\theta\cos\varphi + 2\dot{z}\dot{\theta}\cos\varphi - 2\dot{z}\theta\dot{\varphi}\sin\varphi\\
&+ z\ddot{\theta}\cos\varphi - 2z\dot{\theta}\dot{\varphi}\sin\varphi - z\ddot{\varphi}\theta\sin\varphi - z\dot{\varphi}^2\theta\cos\varphi\}\\
&- m_3\dot{z}\sin\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi)\\
&- m_3 z\dot{\varphi}\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi)\\
&- m_3 z\sin\varphi\{\ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi - \dot{r}\dot{\varphi}^2\sin\varphi + s\ddot{\varphi}\cos\varphi - s\dot{\varphi}^2\sin\varphi\\
&+ \ddot{z}\theta\sin\varphi + 2\dot{z}\dot{\theta}\sin\varphi + 2\dot{z}\theta\dot{\varphi}\cos\varphi + z\ddot{\theta}\sin\varphi\\
&+ 2z\dot{\theta}\dot{\varphi}\cos\varphi + z\ddot{\varphi}\theta\cos\varphi - z\dot{\varphi}^2\theta\sin\varphi\} - \ddot{\theta}(J_2'' + J_3'') = 0, \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
E_3 \equiv\ & m_2(-\dot{r}\sin\varphi - r\dot{\varphi}\cos\varphi - s\dot{\varphi}\cos\varphi + d\dot{\theta}\theta\sin\varphi - d\dot{\varphi}\cos\varphi)(\dot{r}\cos\varphi\\
&- r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi)\\
&+ m_2(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi)(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi\\
&+ s\dot{\varphi}\cos\varphi - d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi)\\
&+ m_3(-\dot{r}\sin\varphi - r\dot{\varphi}\cos\varphi - s\dot{\varphi}\cos\varphi - \dot{z}\theta\sin\varphi - z\dot{\theta}\sin\varphi - z\dot{\varphi}\theta\cos\varphi)(\dot{r}\cos\varphi\\
&- r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi + z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi)\\
&+ m_3(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi + z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi)(\dot{r}\sin\varphi\\
&+ r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi)\\
&- 2m_1\dot{r}r\dot{\varphi} - m_1 r^2\ddot{\varphi} - \ddot{\varphi}J_1 - 4\frac{2\dot{r}r\dot{\varphi} + r^2\ddot{\varphi}}{R^2}J_{wheel}\\
&- m_2(-\dot{r}\sin\varphi - r\dot{\varphi}\cos\varphi - s\dot{\varphi}\cos\varphi - d\dot{\varphi}\cos\varphi)(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi\\
&- s\dot{\varphi}\sin\varphi - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi)
\end{aligned}
$$

$$
\begin{aligned}
&-(-r\sin\varphi - s\sin\varphi - d\sin\varphi)\{\ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi \\
&- s\ddot{\varphi}\sin\varphi - s\dot{\varphi}^2\cos\varphi \\
&- d\ddot{\theta}\theta\cos\varphi - d\dot{\theta}^2\cos\varphi + d\dot{\theta}\theta\dot{\varphi}\sin\varphi - d\ddot{\varphi}\sin\varphi - d\dot{\varphi}^2\cos\varphi\} \\
&- m_2(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi - d\dot{\varphi}\sin\varphi)(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi \\
&- d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi) \\
&- m_3(r\cos\varphi + s\cos\varphi + d\cos\varphi)\{\ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi - \dot{r}\dot{\varphi}^2\sin\varphi \\
&+ s\ddot{\varphi}\cos\varphi - s\dot{\varphi}^2\sin\varphi \\
&- d\ddot{\theta}\theta\sin\varphi - d\dot{\theta}^2\sin\varphi - d\dot{\theta}\theta\dot{\varphi}\cos\varphi + d\ddot{\varphi}\cos\varphi - d\dot{\varphi}^2\sin\varphi\} \\
&- m_3(-\dot{r}\sin\varphi - r\dot{\varphi}\cos\varphi - s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\theta\dot{\varphi}\cos\varphi)(\dot{r}\cos\varphi \\
&- r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi + z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi) \\
&- m_3(-r\sin\varphi - s\sin\varphi + z\theta\sin\varphi)\{\ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi \\
&- s\ddot{\varphi}\sin\varphi - s\dot{\varphi}^2\cos\varphi + \ddot{z}\theta\cos\varphi + 2\dot{z}\dot{\theta}\cos\varphi - 2\dot{z}\theta\dot{\varphi}\sin\varphi \\
&+ z\ddot{\theta}\cos\varphi - 2z\dot{\theta}\dot{\varphi}\sin\varphi - z\ddot{\varphi}\theta\sin\varphi - z\dot{\varphi}^2\theta\cos\varphi\} \\
&- m_3(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi - z\dot{\theta}\cos\varphi - z\theta\dot{\varphi}\sin\varphi)(\dot{r}\sin\varphi \\
&+ r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi) \\
&- m_3(r\cos\varphi + s\cos\varphi + z\theta\cos\varphi)\{\ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi \\
&- \dot{r}\dot{\varphi}^2\sin\varphi + s\ddot{\varphi}\cos\varphi - s\dot{\varphi}^2\sin\varphi + \ddot{z}\theta\sin\varphi + 2\dot{z}\dot{\theta}\sin\varphi \\
&+ 2\dot{z}\theta\dot{\varphi}\cos\varphi + z\ddot{\theta}\sin\varphi + 2z\dot{\theta}\dot{\varphi}\cos\varphi + z\ddot{\varphi}\theta\cos\varphi - z\dot{\varphi}^2\theta\sin\varphi\} \\
&- \ddot{\varphi}(J_2' + J_3') - (\ddot{\varphi}\theta^2 + 2\dot{\varphi}\dot{\theta}\theta)(J_2 + J_3) = 0, \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
E_4 \equiv\ & m_3(\dot{\theta}\cos\varphi - \dot{\varphi}\theta\sin\varphi)(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi \\
&+ z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi) \\
&+ m_3(\dot{\theta}\sin\varphi + \dot{\varphi}\theta\cos\varphi)(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi \\
&+ z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi) - m_3 g \\
&- m_3(\dot{\theta}\cos\varphi - \theta\dot{\varphi}\sin\varphi)(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi \\
&+ z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi) \\
&- m_3\theta\cos\varphi\{\ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi - s\ddot{\varphi}\sin\varphi - s\dot{\varphi}^2\cos\varphi \\
&+ \ddot{z}\theta\cos\varphi + 2\dot{z}\dot{\theta}\cos\varphi - 2\dot{z}\theta\dot{\varphi}\sin\varphi + z\ddot{\theta}\cos\varphi - 2z\dot{\theta}\dot{\varphi}\sin\varphi - z\ddot{\varphi}\theta\sin\varphi \\
&- z\dot{\varphi}^2\theta\cos\varphi\} - m_3(\dot{\theta}\sin\varphi + \theta\dot{\varphi}\cos\varphi)(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \\
&+ s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi) \\
&- m_3\theta\sin\varphi\{\ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi - \dot{r}\dot{\varphi}^2\sin\varphi
\end{aligned}
$$

$$+s\ddot{\varphi}\cos\varphi - s\dot{\varphi}^2\sin\varphi + \ddot{z}\theta\sin\varphi + 2\dot{z}\dot{\theta}\sin\varphi + 2\dot{z}\theta\dot{\varphi}\cos\varphi$$
$$+z\ddot{\theta}\sin\varphi + 2z\dot{\theta}\dot{\varphi}\cos\varphi + z\ddot{\varphi}\theta\cos\varphi - z\dot{\varphi}^2\theta\sin\varphi\} - m_3\ddot{z} = 0. \quad (23)$$

The Lepage class of the unconstrained mechanical system is thus given by the representative:

$$\alpha = A_1\omega^1\wedge dt + A_2\omega^2\wedge dt + A_3\omega^3\wedge dt + A_4\omega^4\wedge dt + ..$$
$$..\left[-m_1 - \frac{J_{wheel}}{R^2} - m_2\right]\omega^1\wedge d\dot{r}..$$
$$+\left[-m_3\theta^2 - m_3\right]\omega^2\wedge d\dot{z}..$$
$$..+\left[-m_2d^2\theta^2 - m_2\theta^2 - m_3z^2 - J_3'' - J_2''\right]\omega^3\wedge d\dot{\theta}..$$
$$..+\begin{bmatrix}-m_1r^2 - J_1 - 4\frac{r^2}{R^2}J_{wheel} - m_3(r\sin\varphi + s\sin\varphi)^2 + z^2\theta^2\sin^2\varphi\\ -m_3(r\cos\varphi + s\cos\varphi + z\theta\cos\varphi)^2 - J_3' - J_2' - \theta^2(J_3+J_2)\end{bmatrix}\omega^4\wedge d\dot{\varphi}, \quad (24)$$

where:

$$\omega^1 = dr - \dot{r}dt, \qquad \omega^2 = dz - \dot{z}dt, \quad \omega^3 = d\theta - \dot{\theta}dt, \quad \omega^4 = d\varphi - \dot{\varphi}dt$$

and

$$A_1 = m_1r\dot{\varphi}^2 + 4\frac{r\dot{\varphi}^2}{R^2}J_{wheel} - m_2\dot{\varphi}\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi$$
$$-d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi) + m_2\dot{\varphi}\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi$$
$$-d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi) - m_3\dot{\varphi}\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi$$
$$+\dot{z}\theta\cos\varphi + z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi) + m_3\dot{\varphi}\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$$
$$+s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi), \quad (25)$$

$$A_2 = m_3(\dot{\theta}\cos\varphi - \dot{\varphi}\theta\sin\varphi)(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi$$
$$+z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi)$$
$$+m_3(\dot{\theta}\sin\varphi + \dot{\varphi}\theta\cos\varphi)(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi$$
$$+z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi) - m_3g - m_3\dot{\theta}\theta\dot{z}, \quad (26)$$

$$A_3 = -m_2d\dot{\theta}\cos\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi)$$
$$-m_2d\dot{\theta}\sin\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi - d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi)$$

$$
\begin{aligned}
&+m_2d^2\dot\theta^2\theta+m_3(\dot z\cos\varphi-z\dot\varphi\sin\varphi)(\dot r\cos\varphi-r\dot\varphi\sin\varphi-s\dot\varphi\sin\varphi\\
&+\dot z\theta\cos\varphi+z\dot\theta\cos\varphi-z\dot\varphi\theta\sin\varphi)\\
&+m_3(\dot z\sin\varphi+z\dot\varphi\cos\varphi)(\dot r\sin\varphi+r\dot\varphi\cos\varphi+s\dot\varphi\cos\varphi+\dot z\theta\sin\varphi\\
&+z\dot\theta\sin\varphi+z\dot\varphi\theta\cos\varphi)\\
&+\dot\varphi^2\theta(J_3+J_2)+m_2d\dot\theta\cos\varphi(\dot r\cos\varphi-r\dot\varphi\sin\varphi-s\dot\varphi\sin\varphi\\
&-d\dot\theta\theta\cos\varphi-d\dot\varphi\sin\varphi)-m_2d^2\theta\dot\theta^2\\
&+m_2d\dot\theta\sin\varphi(\dot r\sin\varphi+r\dot\varphi\cos\varphi+s\dot\varphi\cos\varphi-d\dot\theta\theta\sin\varphi+d\dot\varphi\cos\varphi)\\
&-2m_2\dot\theta^2\theta-m_3\dot\theta z\dot z,
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
A_4=\;&m_2(-\dot r\sin\varphi-r\dot\varphi\cos\varphi-s\dot\varphi\cos\varphi+d\dot\theta\theta\sin\varphi-d\dot\varphi\cos\varphi)(\dot r\cos\varphi\\
&-r\dot\varphi\sin\varphi-s\dot\varphi\sin\varphi-d\dot\theta\theta\cos\varphi-d\dot\varphi\sin\varphi)\\
&+m_2(\dot r\cos\varphi-r\dot\varphi\sin\varphi-s\dot\varphi\sin\varphi-d\dot\theta\theta\cos\varphi-d\dot\varphi\sin\varphi)(\dot r\sin\varphi\\
&+r\dot\varphi\cos\varphi+s\dot\varphi\cos\varphi-d\dot\theta\theta\sin\varphi+d\dot\varphi\cos\varphi)\\
&+m_3(-\dot r\sin\varphi-r\dot\varphi\cos\varphi-s\dot\varphi\cos\varphi-\dot z\theta\sin\varphi-z\dot\theta\sin\varphi\\
&-z\dot\varphi\theta\cos\varphi)(\dot r\cos\varphi-r\dot\varphi\sin\varphi-s\dot\varphi\sin\varphi+\dot z\theta\cos\varphi+z\dot\theta\cos\varphi-z\dot\varphi\theta\sin\varphi)\\
&+m_3(\dot r\cos\varphi-r\dot\varphi\sin\varphi-s\dot\varphi\sin\varphi+\dot z\theta\cos\varphi+z\dot\theta\cos\varphi\\
&-z\dot\varphi\theta\sin\varphi)(\dot r\sin\varphi+r\dot\varphi\cos\varphi+s\dot\varphi\cos\varphi+\dot z\theta\sin\varphi\\
&+z\dot\theta\sin\varphi+z\dot\varphi\theta\cos\varphi)\psi^2\theta).
\end{aligned}
\tag{28}
$$

4.2. The constraint

The condition that the forklift rolls without sliding on the plane means that the instantaneous velocity of the point of contact of the forklift-wheel is equal to zero at all times. This gives rise to the following nonholonomic constraints:

$$
\begin{aligned}
&f^1\equiv\dot r(\cos\varphi+\sin\varphi)+r\dot\varphi(\cos\varphi-\sin\varphi)=0,\\
&f^2\equiv\dot z+z\dot\theta\tan\theta=0,
\end{aligned}
\tag{29}
$$

or in normal form

$$
\begin{aligned}
&\dot r=g_1\equiv-r\dot\varphi\frac{(\cos\varphi-\sin\varphi)}{(\cos\varphi+\sin\varphi)},\\
&\dot z=g_2\equiv-z\dot\theta tan\theta.
\end{aligned}
\tag{30}
$$

These two nonholonomic conditions define the constraint submanifold $\wp$. Constraint (29) obeys condition, i.e. it is semiholonomic. The geometric theory allows us to solve such a problem immediately, without integrating the constraint.

$$rank\left[\frac{\partial f^i}{\partial \dot{q}^\sigma}\right] = rank\begin{bmatrix} 1 & 0 & 0 & r\dfrac{(\cos\varphi-\sin\varphi)}{(\cos\varphi+\sin\varphi)} \\ 0 & 1 & z\tan\theta & 0 \end{bmatrix} = 2. \tag{31}$$

4.3. Constrained mechanical system-reduced equations

The geometrical approach described in the first section applied to our problem leads to the constrained mechanical system $[\alpha_{\wp}]$ related to the unconstrained mechanical system $[\alpha]$. The class $[\alpha_{\wp}]$ is generated e.g. by the following representative:

$$\begin{aligned}\alpha_{\wp} &= A_1'\omega^1\wedge dt + A_2'\omega^2\wedge dt + A_3'\omega^3\wedge dt \\ &\quad + \sum_{l=1}^{3} B_{l1}'\omega^1\wedge d\dot{z} + B_{l2}'\omega^2\wedge d\dot{\theta} + B_{l3}'\omega^3\wedge d\dot{\varphi},\end{aligned} \tag{32}$$

where:

$$\omega^1 = dz - \dot{z}dt, \qquad \omega^2 = d\theta - \dot{\theta}dt, \qquad \omega^3 = d\varphi - \dot{\varphi}dt.$$

Computing the coefficients A_l'according to equation (10) we obtain following expressions:

$$\begin{aligned}A_1' &= m_1 r\dot{\varphi}^2 + 4\frac{r\dot{\varphi}^2}{R^2}J_{wheel} - m_2\dot{\varphi}\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \\ &\quad - s\dot{\varphi}\sin\varphi - d\dot{\theta}\theta\cos\varphi - d\dot{\varphi}\sin\varphi) \\ &\quad + m_2\dot{\varphi}\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi - d\dot{\theta}\theta\sin\varphi + d\dot{\varphi}\cos\varphi) \\ &\quad - m_3\dot{\varphi}\sin\varphi(\dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi - s\dot{\varphi}\sin\varphi + \dot{z}\theta\cos\varphi + z\dot{\theta}\cos\varphi - z\dot{\varphi}\theta\sin\varphi) \\ &\quad + m_3\dot{\varphi}\cos\varphi(\dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi + s\dot{\varphi}\cos\varphi + \dot{z}\theta\sin\varphi + z\dot{\theta}\sin\varphi + z\dot{\varphi}\theta\cos\varphi),\end{aligned} \tag{33}$$

$$A_2' = A_2 - A_3 z\tan\theta, \tag{34}$$

$$A_3' = A_3 - A_2 r\frac{(\cos\varphi-\sin\varphi)}{(\cos\varphi+\sin\varphi)} \tag{35}$$

and coefficients B_{ls}' according to equation (11) are:

$$B_{11}' = -m_3\theta^2 - m_3\,, \tag{36}$$

$$B_{12}' = B_{21}' = -m_3 z\theta - 2m_3 z^2\theta^2\sin\varphi\cos\varphi\tan\theta, \tag{37}$$

$$B'_{13} = B'_{31} = 2m_3z\theta^2 \sin\varphi\cos\varphi + m_3z\theta r\frac{(\cos\varphi - \sin\varphi)}{(\cos\varphi + \sin\varphi)}, \tag{38}$$

$$B'_{22} = -m_2d^2\theta^2 - m_2\theta^2 - m_3z^2 - J''_3 - J''_2 + 4m_3z^3\theta \sin\varphi\cos\varphi\tan\theta, \tag{39}$$

$$\begin{aligned}B'_{23} = B'_{32} &= -2m_3z^2\theta\sin\varphi\cos\varphi \\ &+ (m_2d^2\theta^2 - m_2\theta^2 - m_3z^2 - J''_3 - J''_2)r\frac{(\cos\varphi - \sin\varphi)}{(\cos\varphi + \sin\varphi)} \\ &- \Bigg(-m_1r^2 - J_1 - 4\frac{r^2}{R^2}J_{wheel} - m_3(r\sin\varphi + s\sin\varphi)^2 + z^2\theta^2\sin^2\varphi \\ &- m_3(r\cos\varphi + s\cos\varphi + z\theta\cos\varphi)^2 - J'_3 - J'_2 - \theta^2(J_3 + J_2)z\tan\theta,\end{aligned} \tag{40}$$

$$\begin{aligned}B'_{33} &= -m_1r^2 - J_1 - 4\frac{r^2}{R^2}J_{wheel} - m_3(r\sin\varphi + s\sin\varphi)^2 + z^2\theta^2\sin^2\varphi \\ &- m_3(r\cos\varphi + s\cos\varphi + z\theta\cos\varphi)^2 - J'_3 - J'_2 - \theta^2(J_3 + J_2) \\ &+ 2m_3z^2\theta\sin\varphi\cos\varphi 2r\frac{(\cos\varphi - \sin\varphi)}{(\cos\varphi + \sin\varphi)} \\ &+ (-m_2d^2\theta^2 - m_2\theta^2 - m_3z^2 - J''_3 - J''_2)r^2\left(\frac{\cos\varphi - \sin\varphi}{\cos\varphi + \sin\varphi}\right)^2.\end{aligned} \tag{41}$$

The reduced equations are of the form:

$$\begin{aligned}A'_1 + B'_{11}\ddot{\psi} + B'_{12}\ddot{\theta} + B'_{13}\ddot{\varphi} &= 0, \\ A'_2 + B'_{22}\ddot{\psi} + B'_{21}\ddot{\theta} + B'_{23}\ddot{\varphi} &= 0, \\ A'_3 + B'_{33}\ddot{\psi} + B'_{31}\ddot{\theta} + B'_{32}\ddot{\varphi} &= 0,\end{aligned} \tag{42}$$

$$\begin{aligned}\dot{r} + r\dot{\varphi}\frac{(\cos\varphi - \sin\varphi)}{(\cos\varphi + \sin\varphi)} &= 0, \\ \dot{z} + z\dot{\theta}\tan\theta &= 0.\end{aligned} \tag{43}$$

There is no analytical solution of reduced equations of motion, in general situation.

4.4. Numerical solution of reduced equations of motion

The reduced equations of the forklift motion derived in Section 4.3 (Eqs. (42)) were numerically solved for following example values of parameters characterizing the forklift: $m_1 = 280$ kg, $m_2 = 40$ kg and $m_3 = 100$ kg; $R = 0.15$ m; $s = 0.5$ m; $J_1 = 160$ kg·m^2, $J_2 = 0.018$ kg·m^2, $J'_2 = 13.36$ kg·m^2, $J''_2 = 0.327$ kg·m^2, $J_3 = 1.67$ kg·m^2, $J'_3 = 1.67$ kg·m^2, $J''_3 = 1.67$ kg·m^2 and

$J_{wheel} = 0.045$ kg·m^2; $g = 9.81$ m/s^2, $d = 1$ m. A numerical solution was made with the help of the program Maple.13. In Figs. 3–6, the graphical outputs $\theta(t)$, $\varphi(t)$, $r(t)$ and $z(t)$ of calculations are presented.

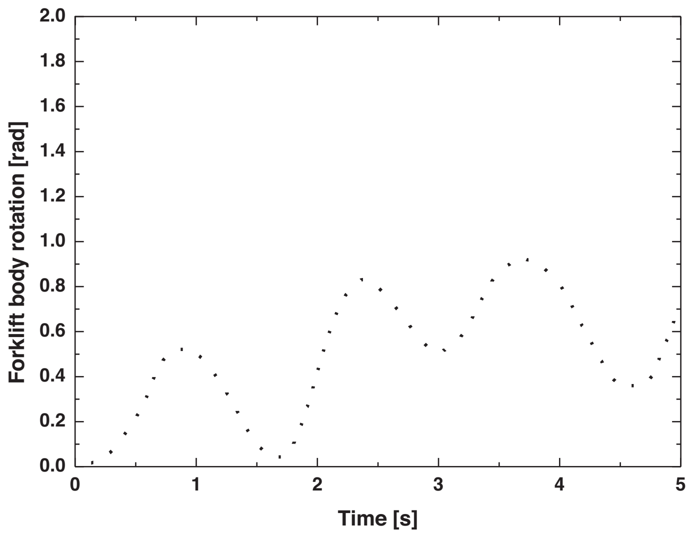

Figure 3: Solution of forklift body rotation $\varphi(t) \cdot (10^{-1})$

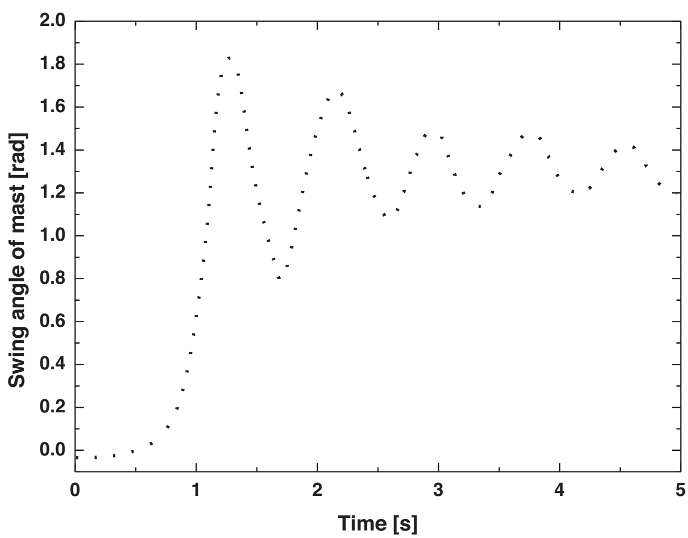

Figure 4: Solution of tangential swing angle $\theta(t) \cdot (10^{-1})$

A real forklift truck is a complex space structure. The governing equations for the dynamic response of a forklift-truck (i.e., reduced equations) are derived

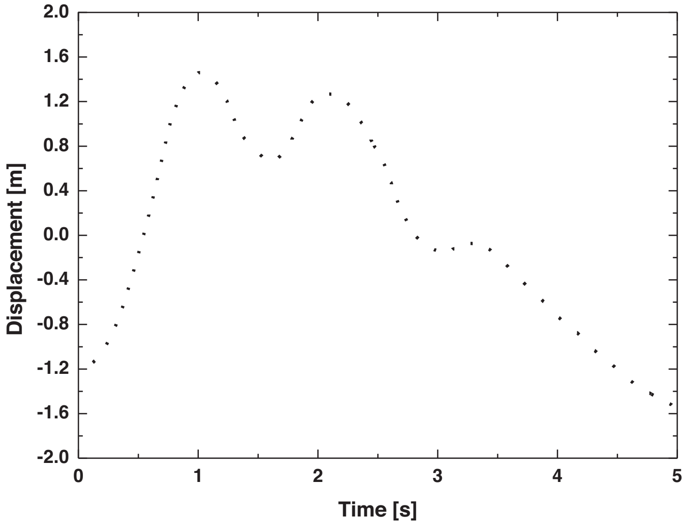

Figure 5: Solution of forklift translation displacement $r(t)$

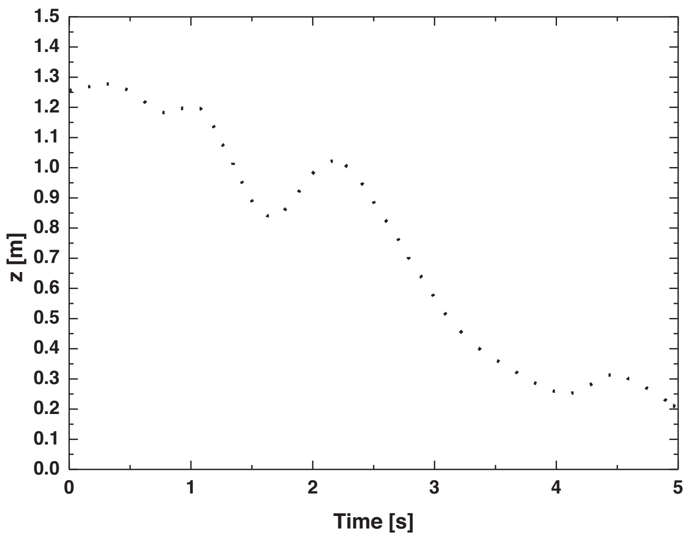

Figure 6: Solution of center of the mass projection of the load on the (OZ) axis $z\ (t)$

based on Krupková approach. These equations are essentially representing the coupled engineering problem of structural dynamics and multi-body dynamics are difficult to solve analytically. The numerical studies of reduced equations of motion are presented and we find it as effective and applicable for problems in physics and engineering for preliminary visualization.

5. Conclusions

The presented results formulation indicate the effectiveness of the geometrical theory of nonholonomic constraints for formulating of motion of concrete nonholonmoic constraints systems with constraints based on the assumption of rolling without slipping. In perspective of this research (Part II), actual work of the authors focuses on experimental model of forklift truck for Krupkova approach validation are detailed.

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