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THE METRIC DIMENSION OF CIRCULANT GRAPHS

Tapendra BC and Shonda Dueck

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Abstract. A pair of vertices x and y in a graph G are said to be *resolved* by a vertex w if the distance from x to w is not equal to the distance from y to w. We say that G is resolved by a subset of its vertices W if every pair of vertices in G is resolved by some vertex in W. The minimum cardinality of a resolving set for G is called the *metric dimension* of G, denoted by $\dim(G)$. The circulant graph $C_n(1,2,\ldots,t)$ is the Cayley graph $Cay(\mathbb{Z}_n : \{\pm 1, \pm 2, \ldots, \pm t\})$. In this note we prove that, for $n = 2kt + 2t$, $\dim(C_n(1,2,\ldots,t)) \geq t+2$, confirming Conjecture 4.1.2 in [K. Chau, S. Gosselin, *The metric dimension of circulant graphs and their Cartesian products*, Opuscula Math. 37 (2017), 509–534].

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1. INTRODUCTION

1.1. DEFINITIONS

Let $G = (V, E)$ be a connected graph, and let $d(x, y)$ denote the distance between vertices $x, y \in V(G)$. A subset $W \subseteq V(G)$ is called a resolving set for G if for every pair of distinct vertices $x, y \in V(G)$, there is $w \in W$ such that $d(x, w) \neq d(y, w)$. The minimum cardinality of a resolving set for G is called the metric dimension of G, denoted by $\dim(G)$. A set W of vertices of G is said to *distinguish* a set $S \subseteq V(G)$ if every pair of vertices of S is resolved by some vertex of W.

For positive integers t and n, the *circulant graph* $C_n(1,2,\ldots,t)$ is the simple graph with vertex set $\mathbb{Z}_n = \{v_0, v_1, \ldots, v_{n-1}\}$, the integers modulo n, in which vertex v_i is adjacent to the vertices $v_{i-t}, v_{i-t+1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+t-1}, v_{i+t}$ $(\mathrm{mod}\ n)$ in $C_n(1,2,\ldots,t)$. Observe that the distance between two vertices v_i and v_j in $G = C_n(1,2,\ldots,t)$ is given by

$$d_G(v_i, v_j) = \begin{cases} \left\lceil \frac{|i-j|}{t} \right\rceil & \text{if } |i-j| < \lceil \frac{n}{2} \rceil, \\ \left\lceil \frac{n-|i-j|}{t} \right\rceil & \text{if } |i-j| \geq \lceil \frac{n}{2} \rceil. \end{cases}$$

The *outer cycle* of the circulant graph $G = C_n(1, 2, \dots, t)$ is a spanning subgraph of G in which the vertex v_i is adjacent to exactly the vertices v_{i+1} and v_{i-1}. We denote this outer cycle by C. A set of vertices $v_i, v_{i+1}, \dots, v_{i+s}$ with consecutive indices are called *consecutive vertices* of G. For a vertex v_i of $G = C_n(1, 2, \dots, t)$ where n is even, we define the *right side of* v_i by

$$R(v_i) = \{v_{i+1}, v_{i+2}, \dots, v_{i+\frac{n}{2}-1}\}.$$

Similarly, we define the *left side of* v_i by

$$L(v_i) = \{v_{i-1}, v_{i-2}, \dots, v_{i-\frac{n}{2}+1}\}.$$

For a vertex $v_i \in G$, the *opposite vertex* of v_i is the vertex $v_{i+\frac{n}{2}}$.

In the graph of Figure 1, $R(0) = \{1, 2, 3, 4, 5\}$, $L(0) = \{7, 8, 9, 10, 11\}$, $R(7) = \{8, 9, 10, 11, 0\}$ and $L(7) = \{2, 3, 4, 5, 6\}$. The opposite vertex of 0 is 6. Notice that $|R(u)| = |L(v)|$ for all $u, v \in V(G)$. Also,

$$L(u) = R\left(u + \frac{n}{2} (\text{mod } n)\right) \quad \text{and} \quad R(u) = L\left(u + \frac{n}{2} (\text{mod } n)\right).$$

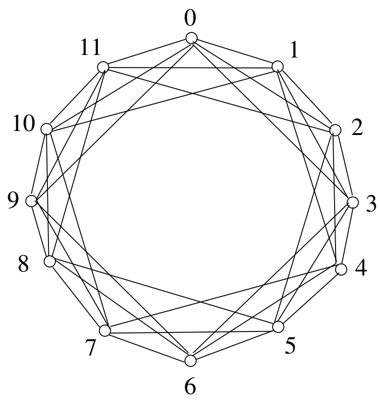

Fig. 1. G = C_{12}(1,2,3)

For a vertex u of $C_n(1, 2, \dots, t)$ and an integer i $(0 \le i \le k)$, the *right* i^{th} *neighborhood of* u, denoted $N_i^+(u)$, is the set of consecutive vertices in $R(u)$ at distance i from u. Similarly, the *left* i^{th} *neighbourhood of* u, denoted $N_i^-(u)$, is the set of consecutive vertices in $L(u)$ at distance i from u. The $(k+1)^{th}$ *neighbourhood of* u, denoted $N_{k+1}(u)$, is the set of consecutive vertices at distance $k+1$ from u. In the graph G of Figure 1, $N_0^+(0) = N_0^-(0) = \{0\}$, $N_1^+(0) = \{1, 2, 3\}$, $N_1^-(0) = \{9, 10, 11\}$, and $N_2(0) = \{4, 5, 6, 7, 8\}$.

Note that if $n = 2tk + r$, where $r \in \{2, 3, \dots, 2t, 2t+1\}$, then the circulant graph $C_n(1, 2, \dots, t)$ has diameter $k+1$, and for any vertex u,

$$|N_i^+(u)| = |N_i^-(u)| = \begin{cases} 1 & \text{if } i = 0, \\ t & \text{if } 1 \le i \le k \end{cases} \quad \text{and} \quad |N_{k+1}(u)| = r - 1.$$

1.2. HISTORY AND LAYOUT OF THE PAPER

The concept of the metric dimension of a graph was first introduced by Slater [15, 16], and independently by Harary and Melter [9]. Their introduction of this invariant was motivated by its application to the placement of a minimum number of sonar/loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. Khuller *et al.* [12] later studied the metric dimension as an application to the navigation of robots in a graph space, and showed that the problem of determining the metric dimension of a given graph is NP-hard, and they determined the metric dimension of trees. An alternate proof of the formula for the metric dimension of trees was given by Chartrand *et al.* in [5], and they characterized the graphs of order n with metric dimension 1 (paths), $n-1$ (complete graphs) and $n-2$. Their study of the metric dimension was motivated by its applications to a problem in pharmaceutical chemistry. The metric dimension of a graph is related to several other well studied graph invariants such as the *determining number* (the base size of its automorphism group), and a good survey of these invariants and their relation to one another was written by Bailey and Cameron in 2011 [1].

Due to the fact that metric dimension has applications in network discovery and verification, combinatorial optimization, chemistry, and many other areas, researchers focus on computing or bounding the metric dimension of certain classes of graphs. In particular, there is great interest in finding classes of graphs whose metric dimension does not increase with the number of vertices. Such classes of graphs are said to have *bounded metric dimension.* Circulant graphs are an important class of graphs that can be used in the design of local area networks. They have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities. Javaid *et al.* [11] initiated a study of the metric dimension of circulants as some classes of these graphs had been shown to have bounded metric dimension. Imran *et al.* [10] later bounded the metric dimension of $C_n(1,2)$ and $C_n(1,2,3)$, and then Borchert and Gosselin [2] extended their results and determined the exact metric dimension of these two families of circulants for all n.

Proposition 1.1 ([2]).

(1) *For* $n \geq 6$,

$$\dim(C_n(1,2)) = \begin{cases} 4 & \text{if } n \equiv 1 \mod 4, \\ 3 & \text{otherwise.} \end{cases}$$

(2) *For* $n \geq 8$,

$$\dim(C_n(1,2,3)) = \begin{cases} 5 & \text{if } n \equiv 1 \mod 6, \\ 4 & \text{otherwise.} \end{cases}$$

In 2014, Grigorious *et al.* [8] bounded the metric dimension of the circulant graph $C_n(1,2,\dots,t)$ for all n and t, as stated in the following result.

Proposition 1.2. *Suppose* $n \equiv r \mod 2t$ *where* $2 \le r \le 2t+1$. *Then*

$$\dim(C_n(1,2,\dots,t)) \le \begin{cases} t+1 & \textit{if } 2 \le r \le t+1, \\ r-1 & \textit{if } t+2 \le r \le 2t+1. \end{cases}$$

The bounds in Propositions 1.2 were obtained from resolving sets consisting of consecutive vertices on the outer cycle of $C_n(1,2,\dots,t)$. In 2017, Vetrík [17] improved on these upper bounds for some values of r, and proved some lower bounds on metric dimension of these graphs.

Proposition 1.3 ([17])**.**

(1) *If* $n = 2tk + t$ *where* $t \ge 4$ *is even and* $k \ge 2$, *then*

$$\dim(C_n(1,2,\dots,t)) \le t.$$

(2) *If* $n = 2tk + t + p$ *where* t *and* p *are even,* $t \ge 4$, $2 \le p \le t$ *and* $k \ge 1$, *then*

$$\dim(C_n(1,2,\dots,t)) \le t + \frac{p}{2}.$$

In addition, Vetrík gave the following lower bounds on $\dim(C_n(1,2,\dots,t))$.

Proposition 1.4 ([17])**.**

(1) *If* $n \ge t^2 + 1$ *where* $t \ge 2$, *then*

$$\dim(C_n(1,2,\dots,t)) \ge t.$$

(2) *If* $n = 2tk + r$ *where* $t \ge 2$, *and* $t+2 \le r \le 2t+1$, *then*

$$\dim(C_n(1,2,\dots,t)) \ge t+1.$$

In 2017, Chau and Gosselin [6] showed that, for large enough n,

$$\dim(C_n(1,2,\dots,t)) = \dim(C_{n+2t}(1,2,\dots,t)),$$

which implies that the metric dimension of these circulants is completely determined by the congruence class of n modulo $2t$, and they improved on the known bounds on the metric dimension of $C_n(1,2,\dots,t)$ for some congruence classes. Their results are summarized below.

Proposition 1.5 ([6])**.** *Let* $G = C_n(1,2,\dots,t)$ *where* $n = 2tk + r$, $t \ge 4$ *and* $k \ge 2$.

(1) *If* $n \equiv r \mod 2t$ *where* $t+1 \le r \le 2t+2$, *then* $\dim(G) \ge t+1$.
(2) *If* $n \equiv r \mod 2t$ *where* $r = 2, t+1, t+2$, *then* $\dim(G) = t+1$.
(3) *If* $n \equiv 0 \mod 2t$ *where* t *is odd, then* $t+1 \le \dim(G) \le 2t-2$.
(4) *If* $n \equiv 1 \mod 2t$, *then* $t+2 \le \dim(G) \le 2t-2$.
(5) *If* $n \equiv 1 \mod 8$ *where* $t = 4$, *then* $\dim(G) = 6$.
(6) *If* $n \equiv (t+3) \mod 2t$ *where* t *is odd then* $\dim(G) = t+1$.

Chau and Gosselin had also asserted that for $3 \le r \le t$, $\dim(G) \ge t$ whenever $n \ge 2t+1$, but this result has since been shown to be incorrect by Vetrík *et al.* in [18], who proved that the metric dimension of $G = C_n(1, 2, \dots, t)$ can be less than t, when $n < t^2 + 1$. Chau and Gosselin [6] made the following conjectures, based on data in their appendix computed by Robert Bailey using the program GAP.

Conjecture 1.6. *Let $G = C_n(1, 2, \dots, t)$ where $n = 2tk + r$, $t \ge 4$ and $k \ge 2$.*

(1) *If $n \equiv t \mod 2t$ where t is odd, then $\dim(G) = t + 1$ (proved in [7]).*
(2) *If $n \equiv 0 \mod 2t$, then $\dim(G) \ge t + 2$.*
(3) *If $n \equiv (t + 3) \mod 2t$ where t is even, then $\dim(G) = t + 2$.*
(4) *If $n \equiv r \mod 2t$ where t is even, $3 \le r \le t - 1$, and $k = 1$ then $\dim(G) = t$.*

In 2023, Gao *et al.* [7] proved that Conjecture 1.6 (1) is true. In this paper, we will prove that Conjecture 1.6(2) also holds. Conjecture (4) was disproved by Vetrík *et al.* in 2023 [18], but Conjecture (3) is still open.

To prove Conjecture 1.6(2), we will make use of the following lemma which was proved in [7].

Lemma 1.7 ([7])**.** *Let $G = C_n(1, 2, \dots, t)$ where $n = 2tk + r$, $t \le r \le 2t + 1$ and $2 \le m \le t$. If a vertex set W can distinguish m consecutive vertices of G, then $|W| \ge m - 1$.*

In Section 2, we prove some preliminary results about $\dim(G = C_n(1, 2, \dots, t))$ for $n = 2tk + 2t$, toward a proof of Conjecture 1.6(2). In Section 3, we prove our main result, Theorem 3.1.

2. PRELIMINARY RESULTS

The first result we prove is a useful lemma about distinguishing consecutive sets of vertices in $C_n(1, 2, \dots, t)$. Lemma 1.7 is a generalization of Lemma 2.2 in [7].

Lemma 2.1. *Let $n = 2tk + r$ such that $t + 1 \le r \le 2t + 1$. If a vertex set $W \subseteq V(C_n(1, 2, \dots, t))$ can distinguish $t + 1$ consecutive vertices, then $|W| \ge t$.*

Proof. Since $C_n(1, 2, \dots, t)$ is vertex-transitive, we may assume without loss of generality that W distinguishes $V_1 = \{v_1, v_2, \dots, v_t, v_{t+1}\}$. Let $W_1 = W \cap V_1$, where $|W_1| = p$. If $p \ge t$ then $|W| \ge p \ge t$ and we are done. So assume $p \le t - 1$. Let $V_1 \setminus W_1 = \{v_{i_1}, v_{i_2}, \dots, v_{i_{t+1-p}}\}$. Notice, $d(u, v) = 1$ for all $u, v \in V_1$ and $u \ne v$. Thus, no vertex in W_1 resolves any pair of vertices in $V_1 \setminus W_1$. However, since W is a distinguishing set for V_1, $W \setminus W_1$ must distinguish $V_1 \setminus W_1$. Thus, W must resolve the pairs $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \dots, (v_{i_{t-p}}, v_{i_{t+1-p}})$. Since $|N_j^+(u)| = |N_j^-(t)| \ge t$ for all $1 \le j \le k$, and $|N_{k+1}(t)| \ge t$, any vertex in $W \setminus W_1$ can resolve at most one pair of vertices from $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \dots, (v_{i_{t-p}}, v_{i_{t+1-p}})$. Therefore, $|W \setminus W_1| \ge t - p$. Hence,

$$|W| = |W_1| + |W \setminus W_1| \ge p + t - p = t. \qquad \square$$

The following lemmas are specifically about $\dim(C_n(1,2,\dots,t))$ where $n = 2kt+2t$ (i.e., $r = 2t$). Since n is divisible by t, it is useful to consider the congruence classes modulo t of the indices of the vertices in a resolving set. For each $i \in \{0,1,\dots,t\}$, we define $[i]_t = \{v_j \in C_n(1,2,\dots,t) \ : \ j \equiv i \bmod \ t\}$. The first result applies when two vertices in a resolving set for $C_n(1,2,\dots,t)$ have indices from the same congruence class modulo t.

Lemma 2.2. *Let a graph $G = C_n(1,2,\dots,t)$ where $n = 2kt+2t$. If W is the resolving set of G containing two vertices $u, v \in [i]_t$ and a vertex $w \in W$ such that $u, v \in L(w)$ or $u, v \in R(w)$, then $|W| \geq t+2$.*

Proof. Let $V(G) = \{0,1,2,\dots,n-1\}$. We know that

$$N_{k+1}(w) = \Big\{w+kt+1, w+kt+2, \dots, w+\frac{n}{2} = w+kt+t, \dots, w+kt+2t-1\Big\}$$

where vertices in N_{k+1} are taken modulo n.

Case 1. $w \in [i]_t$.

If $u, v \in R(w)$, then u, v do not resolve any pair of vertices in

$$V_1 = \{w+kt+1, w+kt+2, \dots, w+kt+t\}.$$

Since $V_1 \subseteq N_{k+1}(w)$, w does not resolve any pair of vertices in V_1. Because W is a resolving set of G, $W \backslash \{u,v,w\}$ must distinguish V_1. Therefore, by Lemma 1.7, $|W \backslash \{w,u,v\}| \geq t-1$. Finally,

$$|W| = |W \backslash \{w,u,v\}| + |\{w,u,v\}| \geq t-1+3 = t+2.$$

On the other hand, if $u, v \in L(w)$, then u, v do not resolve any pair of vertices in

$$V_1 = \{w+kt+t, w+kt+t+1, \dots, w+kt+2t-1\}.$$

Since $V_1 \subseteq N_{k+1}(w)$, w does not resolve any pair of vertices in V_1. Because W is a resolving set of G, $W \backslash \{u,v,w\}$ must distinguish V_1. Therefore, by Lemma 1.7, $|W \backslash \{w,u,v\}| \geq t-1$. Finally,

$$|W| = |W \backslash \{w,u,v\}| + |\{w,u,v\}| \geq t-1+3 = t+2.$$

Case 2. $w \notin [i]_t$.

Since $w \notin [i]_t$, $[w]_t \neq [i]_t$ therefore $u \notin [w]_t = [w+kt+t]_t$ and $v \notin [w]_t = [w+kt+t]_t$.

Subcase 2.1. $u, v \in R(w)$.

We know $[w+kt+1]_t, [w+kt+2]_t, \dots, [w+kt+t]_t$ are all the congruence classes modulo t. Since u and v are not in $[w+kt+t]_t$, they must be in one of $[w+kt+j]_t$ where $1 \leq j \leq t-1$. Now Let

$$V_1 = \{w+kt+j+1, w+kt+j+2, \dots, w+kt+j+t\}.$$

Since $j \leq t-1$,

$$w + kt + j + t \leq w + kt + t - 1 + t = w + kt + 2t - 1.$$

Therefore, $V_1 \subseteq N_{k+1}(w)$. Furthermore, vertex w, u, v do not resolve any pair of vertices in V_1. However, since W is a resolving set of G, $W \backslash \{w, u, v\}$ must distinguish V_1. Therefore, by Lemma 1.7, $|W \backslash \{w, u, v\}| \geq t - 1$. Finally,

$$|W| = |W \backslash \{w, u, v\}| + |\{w, u, v\}| \geq t - 1 + 3 = t + 2.$$

Subcase 2.2. $u, v \in L(w)$.

We know $[w + kt + t]_t, [w + kt + t + 1]_t, \ldots, [w + kt + 2t - 1]_t$ are all the congruence classes modulo t. Since u and v are not in $[w + kt + t]_t$, they must be in one of $[w + kt + t + j]_t$ where $1 \leq j \leq t - 1$. Now Let

$$V_1 = \{w + kt + t + j - 1, w + kt + t + j - 2, \ldots, w + kt + j\}.$$

Since $j \geq 1$, $w + kt + j \geq w + kt + 1$. Therefore, $V_1 \subseteq N_{k+1}(w)$. Furthermore, vertex w, u, v do not resolve any pair of vertices in V_1. However, since W is a resolving set of G, $W \backslash \{w, u, v\}$ must distinguish V_1. Therefore, by Lemma 1.7, $|W \backslash \{w, u, v\}| \geq t - 1$. Finally,

$$|W| = |W \backslash \{w, u, v\}| + |\{w, u, v\}| \geq t - 1 + 3 = t + 2. \qquad \square$$

The next result applies when three vertices of a resolving set for $C_n(1, 2, \ldots, t)$ have indices from the same congruence class modulo t.

Lemma 2.3. *Let a graph $G = C_n(1, 2, \ldots, t)$ where $n = 2kt + 2t$, and $t \geq 4$. If W is the resolving set of G containing three vertices $u, v, w \in [i]_t$, then $|W| \geq t + 2$.*

Proof. Assume $V(G) = \{0, 1, 2, \ldots, n-1\}$. Since, $t \geq 4$, $|W| \geq t + 1 = 4 + 1 = 5$, by Proposition 1.5.

Case 1. There is a vertex $v_1 \in W$ such that $v_1 \notin [i]_t$.

Since $v_1 \notin [i]_t$, u, v, w are neither v_1 nor the opposite vertex of v_1. Therefore, $x \in \{u, v, w\}$ then $x \in L(v_1)$ or $x \in R(v_1)$. So two of the vertices in $\{u, v, w\}$, must be on the same side (right or left) of v_1. Therefore, $|W| \geq t + 2$ by Lemma 2.2.

Case 2. $w \in W$ implies $w \in [i]_t$.

Let $V_1 = \{a, b, c, d, e\}$ be a set of five vertices in W. We know, at most one vertex from V_1 can be the opposite vertex of a. Therefore, there are at least 3 vertices in V_1 which are neither a nor an opposite vertex of a. Assume, without loss of generality, that $\{b, c, d\}$ are such vertices. Now, $x \in \{b, c, d\}$ then $x \in L(a)$ or $x \in R(a)$. So two of the vertices in $\{b, c, d\}$, must be on the same side (right or left) of a. Therefore, $|W| \geq t + 2$ by Lemma 2.2. $\square$

The next lemma applies when two vertices of a resolving set for $C_n(1, 2, \ldots, t)$ with indices from consecutive congruence classes modulo t lie on opposite sides (left and right) of another resolving vertex w.

Lemma 2.4. *Let a graph $G = C_n(1, 2, \dots, t)$ where $n = 2kt+2t$. If W is the resolving set containing two vertices $u \in [i]_t, v \in [i+1]_t$ and a vertex $w \in W$ such that $u \in R(w)$ and $v \in L(w)$, then $|W| \geq t+2$.*

Proof. Let $V(G) = \{0, 1, 2, \dots, n-1\}$.

Case 1. $u \in [w]_t$.

Since $u \in R(w)$, vertex u is neither w nor the opposite vertex of w. We know, $u \in [w]_t = [w+kt]_t$. Now u does not resolve any pair of vertices in $V_1 = \{w+kt+1, w+kt+2, \dots, w+kt+t\}$ (where vertices in V_1 are taken modulo n). Similarly, since $v \in L(w)$ and $w+kt+t+1 \in [w+1]_t = [i+1]_t$, v does not resolve any pair of vertices in V_1. Additionally, $V_1 \subseteq N_{k+1}(w)$ so w does not resolve any pair of vertices in V_1. Hence, $W \backslash \{u, v, w\}$ must distinguish V_1. Therefore, $|W\backslash\{u, v, w\}| \geq t-1$ by Lemma 1.7. Finally,

$$|W| = |W\backslash\{u, v, w\}| + |\{u, v, w\}| \geq t-1+3 = t+2.$$

Case 2. $u \notin [w]_t$.

We know $[w+kt+1]_t, [w+kt+2]_t, \dots, [w+kt+t]_t$ are all the congruence classes modulo t. Since u is not in $[w]_t = [w+kt+t]_t$, it must be in one of $[w+kt+j]_t$ where $1 \leq j \leq t-1$. Now let

$$V_1 = \{w+kt+j+1, w+kt+j+2, \dots, w+kt+j+t\}.$$

Since $j \leq t-1$,

$$w+kt+j+t \leq w+kt+t-1+t = w+kt+2t-1.$$

Therefore, $V_1 \subseteq N_{k+1}(w)$. Similarly, since $v \in L(w)$ and $w+kt+j+t+1 \in [i+1]_t$, v does not resolve any pair of vertices in V_1. Additionally, $V_1 \subseteq N_{k+1}(w)$ so w does not resolve any pair of vertices in V_1. Hence, $W\backslash\{u, v, w\}$ must distinguish V_1. Therefore, $|W\backslash\{u, v, w\}| \geq t-1$ by Lemma 1.7. Finally, we observe that

$$|W| = |W\backslash\{u, v, w\}| + |\{u, v, w\}| \geq t-1+3 = t+2. \qquad \square$$

The next result applies when two vertices of a resolving set for $C_n(1, 2, \dots, t)$ are at distance at most $t-2$ from each other on the outer cycle C. We let $d_C(u, v)$ denote the distance between u and v on the outer cycle C.

Lemma 2.5. *Let $n = 2tk+2t$. If W is the resolving set for $G = C_n(1, 2, \dots, t)$ containing two vertices $u, v \in W$ such that $d_C(u, v) \leq t-2$, then $|W| \geq t+2$.*

Proof. Since $|N_{k+1}(u)| = 2t-1$ and $d_C(u, v) \leq t-2$, $|N_{k+1}(u) \cap N_{k+1}(v)| \geq t+1$. Let V_1 be formed by taking $t+1$ consecutive vertices from $N_{k+1}(u) \cap N_{k+1}(v)$. Now neither u nor v resolves any pair of vertices in V_1, hence $W\backslash\{u, v\}$ must distinguish V_1. Therefore, $|W\backslash\{u, v\}| \geq t$ by Lemma 2.1. Therefore,

$$|W| = |W\backslash\{u, v\}| + |\{u, v\}| \geq t+2. \qquad \square$$

Our last lemma applies when a resolving set for $C_n(1, 2, \ldots, t)$ contains no vertices with indices from a given congruence class modulo t.

Lemma 2.6. *Let $G = C_n(1, 2, \ldots, t)$ where $n = 2kt + 2t$ and $t \geq 4$. If W is the resolving set that does not contain any vertex from some congruence class $[i]_t$, then W must have at least two vertices from congruence class $[i-1]_t$ and at least two vertices from congruence class $[i+1]_t$. Furthermore, if W contains exactly two vertices of congruence class $[i-1]_t$ (or $[i+1]_t$) then they should be opposite vertices.*

Proof. Let $V(G) = \{0, 1, 2, \ldots, n-1\}$. Since W is a resolving set, it must have a vertex to resolve vertex pairs $(i-1, i)$ and $(i+\frac{n}{2}-1, i+\frac{n}{2})$ (where vertices are taken modulo n). Notice that only vertex in congruence class $[i-1]_t$ or $[i]_t$ can resolve pairs $(i-1, i)$ and $(i+\frac{n}{2}-1, i+\frac{n}{2})$. However, W does not have any vertex of congruence class $[i]_t$ therefore W must have a vertex of congruence class $[i-1]_t$. Let $a \in W$ be the vertex that resolves the pair $(i-1, i)$, then a must be in $L(i)$. However, since $a \in R(i+\frac{n}{2})$, a does not resolve the pair $(i+\frac{n}{2}-1, i+\frac{n}{2})$ so W must have another vertex, say b, in congruence class $[i-1]_t$ to resolve the pair $(i+\frac{n}{2}-1, i+\frac{n}{2})$. Therefore, W has at least 2 vertex from congruence class $[i-1]_t$. Now, using the same argument as above on vertex pairs $(i, i+1)$ and $(i+\frac{n}{2}, i+\frac{n}{2}+1)$ (where vertices are taken modulo n) we can show that W has at least 2 vertices of congruence class $[i+1]_t$.

Now we will prove if W contains exactly 2 vertices from congruence class $[i-1]_t$ (or $[i+1]_t$) then they should be opposite vertex. Assume a and b are the only vertex of congruence class $[i-1]_t$ in W. Assume, for the sake of contradiction, that a and b are not opposite. Then either $b \in R(a)$ or $b \in L(a)$. Assume, without loss of generality, that $b \in R(a)$. Since a and b are in the same congruence class and are not opposite, $d_C(a, b) = b - a \bmod n \leq kt$. Now $b + 1 - a \bmod n \leq kt + 1 < kt + t$ for $t \geq 4$. Therefore, $b + 1 - a (\bmod n) < kt + t = \frac{n}{2}$. Hence, vertex $b + 1 \in R(a)$. Consequently, $a \in L(b+1)$. Clearly, $b \in L(b+1)$. We know the vertex in W that can resolve the pair $(b+\frac{n}{2}, b+\frac{n}{2}+1)$ must be in $R(b+1)$ and congruence class $[i-1]_t$, as W does not contain a vertex from congruence class $[i]_t$. Therefore, neither a nor b resolves the pair $(b+\frac{n}{2}, b+\frac{n}{2}+1)$, contradicting the fact that W is a resolving set for G. Therefore, a and b must be opposite. We can use a similar argument for the $[i+1]_t$ case. □

3. PROOF OF THE MAIN RESULT

Theorem 3.1. *Let a graph $G = C_n(1, 2, \ldots, t)$ where $n = 2kt + 2t$, $t \geq 4$ and $k \geq 1$, then $\dim(G) \geq t + 2$.*

Proof. Let $V(G) = \{0, 1, 2, \ldots, n-1\}$ and W be any resolving set of G.

Case 1. W contains vertices from each congruence class modulo t.

Assume, for the sake of contradiction, that $|W| = t + 1$. Then W must have two vertices from the same congruence class modulo t. Let $u, v \in [i]_t$ be vertices in W that are in the same congruence class modulo t. From the case we are in there must be exactly one vertex from each congruence class $[i+j]_t$ where $1 \leq j \leq t-1$. Let us define v_j be the vertex in W that is in congruence class $[i+j]_t$, for $1 \leq j \leq t-1$.

Since $v_j \notin [i]_t$, v_j can neither be u nor the opposite vertex of u. Similarly, v_j is neither v nor the opposite vertex of v. Therefore, each v_j is in $R(u)$ or $L(u)$. Similarly, each v_j is in $R(v)$ or $L(v)$. Now we will prove that every v_j lies on the shortest $u - v$ *path* in the outer cycle of G.

Subcase 1.1. $d_C(u, v) = \frac{n}{2}$.

In this case, there are two shortest paths from u to v of length $\frac{n}{2}$ in the outer cycle of G. Now, either $v_1 \in R(u)$ or $v_1 \in L(u)$. Assume, without loss of generality, that $v_1 \in R(u)$. (Note that if $v_1 \in L(u)$, then $v_1 \in R(v)$, so we can switch the names of u and v.) Since $v_2 \in [i+2]_t = [v_1 + 1]_t$, v_2 must be in $R(u)$; otherwise $|W| \geq t + 2$ by Lemma 2.4, which is a contradiction. Now, since $v_3 \in [i+3]_t = [v_2 + 1]_t$, v_3 must be in $R(u)$. Inductively, we must have $v_4, v_5, \ldots, v_{t-1} \in R(u)$. Therefore, every v_j lies on $R(u)$ which is in one of the shortest $u - v$ paths in the outer cycle of G.

Subcase 1.2. $d_C(u, v) < \frac{n}{2}$.

In this case, there is a unique shortest path from u to v in the outer cycle of G. In the case we are in, the vertex v is neither u nor the opposite vertex of u, therefore either $v \in L(u)$ or $v \in R(u)$. Assume, without loss of generality, that $v \in R(u)$. (Note that if $v \in L(u)$, then $u \in R(v)$, so we can switch the names of u and v.) Since $v_1 \in [i+1]_t = [v+1]_t$, v_1 must be in $R(u)$; otherwise $|W| \geq t + 2$ by Lemma 2.4, which is a contradiction. Now, since $v_2 \in [i+2]_t = [v_1 + 1]_t$, v_2 must be in $R(u)$. Inductively, we must have $v_4, v_5, \ldots, v_{t-1} \in R(u)$. Therefore, every v_j lies on $R(u)$. Additionally, since $v \in R(u)$, $u \in L(v)$. Since $u \in [i]_t = [v_{t-1} + 1]_t$, v_{t-1} must be in $L(v)$; otherwise $|W| \geq t + 2$ by Lemma 2.4, which is a contradiction. Now, since $v_{t-1} \in [i + t - 1]_t = [v_{t-2} + 1]_t$, v_{t-2} must be in $L(v)$. Inductively, we must have $v_{t-3}, v_{t-4}, \ldots, v_1 \in L(v)$. Therefore, all v_j are in $R(u) \cap L(v)$ which is in the shortest path between u and v in the outer cycle C of G.

Now, since every v_j lies on the shortest path between u and v in the outer cycle of G, $u \in L(v_j)$ and $v \in R(v_j)$. If there is any v_j between v_1 and v in the outer cycle of G, then $v \in R(v_j)$ and $v_1 \in L(v_j)$, so $|W| \geq t+2$ by Lemma 2.4, which is a contradiction. Therefore, all v_j other than v_1 must lie between u and v_1 in the outer cycle of G. Now v_2 lies between u and v_1. If any v_j lies between v_1 and v_2, then $v_1 \in R(v_j)$ and $v_2 \in L(v_j)$, so $|W| \geq t + 2$ by Lemma 2.4, which is a contradiction. Therefore, all v_j other than v_1 and v_2 must lie between u and v_2. By induction, we can see that $v_{j-1} \in R(v_j)$ and $v_{j+1} \in L(v_j)$ for all $1 \leq j \leq t - 1$ (see Figure 2).

Now, vertex v_2 does not resolve its neighbours $(v_2 - 1, v_2 + 1)$(where vertices are taken modulo n). Since $t \geq 4$, $\{v_1, v_2, v_3\} \subseteq \{v_j \ : \ 1 \leq j \leq t - 1\}$ (as defined above). Therefore, W does not have another vertex in the same congruence class as v_2. Now, the pair $(v_2 - 1, v_2 + 1)$ is either resolved some vertex x such that $x \in L(v_2)$ and $x \in [v_1]$ or $x \in R(v_2)$ and $x \in [v_3]$. However, we do not have such vertex x in W. Therefore, W is not a resolving set of G, which is a contradiction. Hence, G does not have a resolving set of cardinality t+1. Therefore, $\dim(G) \geq t + 2$.

Case 2. W does not contain vertices from some congruence class $[i]_t$.

In this case, W has at least 2 vertex of congruence class $[i-1]_t$ and at least 2 vertex of congruence class $[i+1]_t$ by Lemma 2.3. If W has three or more vertex of congruence class $[i-1]_t$ or $[i+1]_t$ then $|W| \geq t+2$ by Lemma 2.3. Now, let us assume that W has exactly two vertex from each congruence class $[i-1]_t$ and $[i+1]_t$. Let a, b be vertex in W that are in congruence class $[i-1]_t$ and c, d be vertex in W that are in congruence class $[i+]_t$. By Lemma 2.6, a and b must be opposite, and c and d must be opposite. Now, a does not resolve the pair $(a-1, a+1)$ (where vertices are taken modulo n). Similarly, b does not resolve $(a-1, a+1)$ because both $a-1$ and $a+1$ are in $N_{k+1}(b)$. Now, the vertex that resolves the pair $(a-1, a+1)$ either is in congruence class $[i-2]_t$ or $[i]_t$. However, W does not contain any vertex from congruence class $[i]_t$, so W must have some vertex from congruence class $[i-2]_t$. Let x be the vertex in W that is in congruence class $[i-2]_t$. Since $t \geq 4$ congruence classes $[i-2]_t, [i-1]_t, [i]_t, [i+1]_t$ are distinct. Therefore, $x \neq c$ and $x \neq d$. Now either $x \in L(c) = R(d)$ or $x \in R(c) = L(d)$. Similarly, $a \in L(c) or a \in R(c)$. Assume, without loss of generality, that $a \in L(c) = R(d)$, then $b \in R(c) = L(d)$. If $x \in L(c) = R(d)$ then we know $b \in L(d)$ and $b \in [x+1]_t$. Therefore, by Lemma 2.4 $|W| \geq t+2$. On the other hand, if $x \in R(c) = L(d)$, then we know $a \in L(c) = R(d)$ and $a \in [x+1]_t$. Therefore, by Lemma 2.4, $|W| \geq t+2$. □

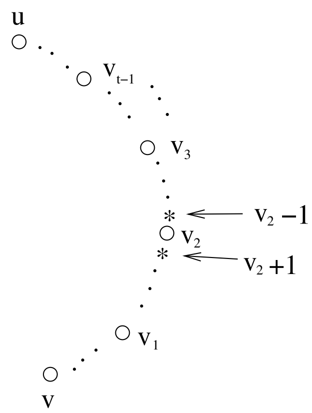

Fig. 2. Orientation of v_j $(1 \leq j \leq t)$

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Tapendra BC
tapendra.320@gmail.com

Department of Mathematics and Statistics
University of Winnipeg
515 Portage Avenue, Winnipeg, Manitoba, R3B 2E9, Canada

Shonda Dueck (corresponding author)
sm.dueck@uwinnipeg.ca

Department of Mathematics and Statistics
University of Winnipeg
515 Portage Avenue, Winnipeg, Manitoba, R3B 2E9, Canada

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