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ELEMENTS OF MATHEMATICAL EPISTEMOLOGY – ELEMENTS OF THE PHILOSOPHY OF TEACHING MATHEMATICS

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Abstract

In the paper we present the main idea of the concept which we have called confrontational concept of mathematical epistemology. We refer it to philosophy of mathematics (in the context of epistemology of research) as well as to didactic problems (in the context of teacher preparation). Although we tried not to involve our discussion directly with any existing concepts of the philosophy of mathematics, however, in the paper one can notice some elements of modern formalism as well as Lakatos quasi-empiricism or a modern approach to structuralism.

1. Introduction

> **Any philosophy of mathematics has powerful implications for social and educational issues and many didactic consequences.**
>
> **P. Ernest**

Let us start first by considering the idea of E. P. Wigner, described by R. Murawski in the introduction to the article *Inconceivable effectiveness of mathematics in the natural sciences* ([13]):

Wigner states that mathematics has no content and it is only a "formal fun". A mathematician therefore does not have any real knowledge, but only some specific skills of deft concept manipulation. A prime example, quoted by the author, are complex numbers. They have no analogue in the

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real world. They are only a very useful tool. They have no ontological background.

If we are surprised by such an attitude to mathematics, then we unfortunately must swallow a bitter pill – it is a very popular point of view, even among highly educated people. Why? Because they have encountered such mathematics at school, college and university, even in books that aim to popularize maths. Such mathematics is given for applications to engineers, physicists, chemists and... future maths teachers.

Is it dangerous? Definitely, YES. It leads to the opinion that it is impossible or even unnecessary to understand mathematics: that it is sufficient to know which formulas are necessary in a particular situation and apply them. Those formulas are useful, but only by accident... A hammer is also useful, but frequently one can manage without it replacing it, for example, with a stone... Why do such opinions arise? One cannot escape these opinions even when formulas usefulness is demonstrated or when it is shown that they are the best methods for solving a problem. They are caused by a lack of understanding of "the spirit of mathematics" and a lack of basic reference to the philosophy of mathematics.

The aim of this paper is not to present detailed characterizations of different concepts within the scope of the philosophy of mathematics. We completely omit, for example, ontological problems. One can find detailed information about historical and modern approaches to particular branches in books e.g. [11], [12], [13]. More information, in particular referring to intuitionism and formalism can be found in other sources as well as on Internet. Presenting them or referring such literature related to these theories here would be an unnecessary lengthening of this article. Our aim is to show a new concept which although rooted in old ideas of the philosophy of maths, will shed new light on contemporary problems of mathematical epistemology.

2. Epistemological basis and the concept of confrontation

Let us start by specifying the meaning of notions which will be used in the article.

By a *mathematical theory* we will mean a theory built on a specified axiomatic system, preserving inference rules. We will not touch on logicism[1]

[1] More contemporary approach to logicism can be found in the studies of H. Weyl ([21], [22]) and E. T. Bell ([1]). Currently, this concept has been rather absorbed by the formalism.

and formalism too much here[2]. Thus analysis of formal theories is not essential in our considerations. Roughly speaking: Mathematical theory is the (mathematical) universe in which we currently operate. We can change or modify it (for example by adding or eliminating axioms), but we "function" within it, and our activities have to be focused on discovering it as effectively as possible.

A *mathematical structure* is a set of known mathematical objects (definitions, theorems, local assumptions, examples etc.) creating a connected system within the scope of a specified mathematical theory. It is a very broad notion. We can consider, for example, topological structures, but we can also distinguish some specific structures: the structure of general topology or the structure of metric topology (the second one, is in a natural way, contained in the first one). One can also talk (using mathematical language) about, for example, structure of algebraic topology. Sometimes it is difficult to distinguish and name a mathematical structure.

A system of specified theory and the mathematical structures complemented by the genesis (history) of a problem[3] will be called an **epistemological basis**.

Let us consider some examples.

The first one is connected with the notion of **entropy** in discrete dynamical systems.

Let us start with a mathematical theory. Obviously, the basis of all analysis contains, for example, the axioms of the set theory, although it is not always emphasized by researchers. However, if "continuum hypothesis" is employed, then reference to axiomatic basis is essential[4].

Yet, this example very consciously takes into account the history of the problem and the *mathematical structure*, in which considerations are carried out.

Now we will quote (with slight editorial modifications) a fragment of the article by E. Korczak-Kubiak, A. Loranty, R. J. Pawlak ([10]):

First, we present some intuitive description of problems connected with information system and information flow. Assume that we have a set X of elements (information) divided into a finite number of disjoint subsets $\{A_1, A_2, \ldots, A_k\}$, which are distinguished on the basis of fixed attributes (this partition is denoted by P). Suppose also that we have a probability

[2] We do not touch on means: we do not create derivative interpretations even in such contemporary works as [16].

[3] It is worth noting here the relationship of epistemological basis with Principle of parallelism mentioned, among others, by R. Duda ([6]).

[4] In the context of this issue in relation to discrete dynamical systems, prof. P. Szuca drew my attention to the work of A. Blass ([2]).

measure μ on X, so $\sum_{i=1}^{k} \mu(A_i) = 1$. Then we may assign to the partition P the number (the entropy of partition) defined in the following way:

$$H(P) := -\sum_{i=1}^{k} \mu(A_i) \cdot \log \mu(A_i).$$

Roughly speaking, if partition P describes a state of information flow, the number $H(P)$ may be regarded as a "measure of uncertainty". If $H(P) = 0$, then situation is defined precisely – measure is focused on some set A_{i_0} from the partition P (i.e. $\mu(A_{i_0}) = 1$). Moreover, we can say that the higher the entropy of partition is, the greater uncertainty is (in this case, the measure is more evenly distributed over the different sets of the partition).

After a given period of time, elements of X change the values of their attributes and thereby they "move" to the other sets. Perhaps a new partition of X (onto sets measurable with respect to μ) is created. These changes are described by a certain (invariant) function – let us denote it by ϕ. After the next unit of time, the elements "move" again and we obtain a new partition of X. The changes are described by the function ϕ. It means that in comparison to initial state these changes are described by the function $\phi^2 = \phi \circ \phi$. Going further in this way, we obtain the dynamics of the function ϕ. The entropy of ϕ with respect to the measure μ is defined in the following way. Let $\mathcal{P} = \{A_i : i = 1, ..., m\}$ be a decomposition of X such that A_i are μ-measurable for $i \in \{1, ..., m\}$. The metric entropy of ϕ with respect to the measure μ is the number $h_\mu(\phi) = \sup_{\mathcal{P}} h_\mu(\phi, \mathcal{P})$, where

$$h_\mu(\phi, \mathcal{P}) = -\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{B \in R_{n-1}(\mathcal{P})} \mu(B) \cdot \log \mu(B)$$

and $R_{n-1}(\mathcal{P})$ is the set containing all intersections of the form

$$A_{i_1} \cap \phi^{-1}(A_{i_2}) \cap ... \cap \phi^{-(n-1)}(A_{i_n}).$$

The entropy of this function determines the level of uncertainty of dynamics of function ϕ. If it is 0, then we can talk about a certain stability of this dynamics. If it is greater than 0, we can say that this dynamics is chaotic and the number qualified as the entropy can be considered as a certain kind of "measure of chaos".

An important way of "describing chaos" of certain actions directly related to the structures that could be conventionally called "metrical" was created in this manner. This, however, was insufficient. Many considerations are based on topological structures, so that these achievements were *confronted* with them.

In the sixties of the twentieth century R. L. Adler, A. G. Konheim and M. H. McAndrew introduced the notion of the topological entropy of a continuous function $f : X \to X$ defined on a compact space X.

However, introduction of this notion could not be isolated from its historical context, so that there were attempts of *confronting* it with metrical approach. In 1971 T. Goodman proved the variational principle determining the relationship between the topological entropy and the entropy with respect to measure, which in the most general form, taking into account later research, one can write in the following way:

Theorem 1. *For any function f we have:*

$$h(f) = \sup\{h_\mu(f) : \mu \textit{ is a probability } f\textit{-invariant Borel measure on } X\}.$$

One should mention in passing that the changes mentioned above, were also the consequence of *confrontation* with different structures. The starting points were structures based on continuous functions, and later analogues were searched for, in the case of less obvious structures of Darboux-like functions: [19], [4]. Simultaneously, studies have been carried out, where existing structures connected with examinations of discrete dynamical systems are *confronted* with algebraic structures (see [7] or [8]) or with generalized topological spaces (see e.g. [15]).

It is worth noting here the emergence of the generalized topological spaces.

In many issues, the assumptions associated with topological spaces are very difficult to achieve. Indeed, suppose that we have a finite or infinite set X and a dynamics on the elements of this set described by a function or multifunction T. To be able to apply mathematical tools, we need some (topological) structure on this set. How to define it? If we examine the dynamics of the function, it is natural to distinguish such sets A for which $A \subset T(A)$ (in the notation which is typical for the theory of discrete dynamical systems: $A \underset{T}{\to} A$). Thus it would be convenient to consider such a family of sets, but it will not be a topology. We can *confront* this situation with generalized topological spaces introduced by Á. Császár ([5]).

The problems we have mentioned need not be referred only to new notions, but also to the properties of existing objects. Brouwer Theorem on a fixed point does not lead to new notions (in opposite to Banach Theorem which gives the possibility to distinguish a new class of functions: contraction). Simultaneously, it is frequently implied that new notions necessitate expansion of the existing structures. Then we return to the logic (and fixed inference rules), understood as the specified mathematical theory. However, sometimes we need to modify our theory. In particular, note that our problem may be unprovable within the theory (Gödel discoveries). A meaningful example of such a situation is the continuum hypothesis. Obviously, then we can add a new axiom or simply assume it locally. It is in accordance with Russell concept (following R. Murawski [13]):

Speaking about the difficulties faced by the doctrine of logicism, Carnap first mentioned of problems related to the fact that in the proofs of many mathematical theorems it is necessary to use certain principles that are not purely logical, for example, the axiom of infinity and the axiom of choice. Russell, however, has found a simple solution to this problem i.e. considering in this case not the theorem T, but the implication $A \Rightarrow T$, where A is, for example, the axiom of infinity. By the theorem of deduction, this implication is then a logical theorem.

In order to illustrate this problem let us present a piece of research led, among others, by prof. T. Natkaniec (see [14]).

Problem. *Is every function $f \in D^*(\mathbb{R}, \mathbb{R})$ (i.e. function having a dense level) a composition of (two) almost continuous functions?*

Let us adopt the following denotations:
$A(c)$ – the union of less than 2^ω many first category subsets of $\mathbb{R}$ is of the first category again.
$A(m)$ – the union of less than 2^ω many Lebesgue measure zero subsets of $\mathbb{R}$ is of the Lebesgue measure zero again.

It is well known that these two conditions follow from Martin's axiom and also from Continuum Hypothesis.

A general answer to the problem of ZFC, presented above has not yet been found. Nevertheless, it is possible to give some answers that "enrich" the theory (speaking less scientifically: by introducing some new assumptions to our mathematical universe).

Theorem 2. ([14]). *Assume $A(c)$. Then each $f \in D^*(\mathbb{R}, \mathbb{R})$ can be expressed as a composition of two almost continuous functions.*

It is clear that we have here an application of Russell's concept. The last theorem is a certain solution to this problem. Is it the ultimate one? Time will tell. For now, the solution has been included into the structures of real analysis and topology.

There is one more important element which should be emphasized: joining the structures. A proper example here is famous proof of Fermat Theorem[5]. Sometimes we definitely have to go beyond our structure.

At the fringe of our considerations we should mention that during the necessary *confrontation* within the scope of a mathematical theory as well

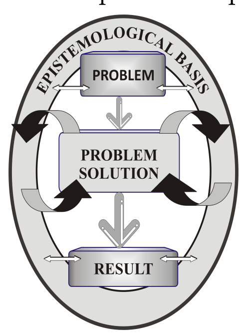
[5] An interesting description of the problem as well as a sketch of solution connected with elliptic curves can be found for example in [20].

as mathematical structures, one can meet epistemological obstacles[6] that may make the process of discovering mathematical facts slower.

What is the main idea of the concept we have called the **confrontational concept of mathematical epistemology**?

First we have the PROBLEM, which sometimes results from the *confrontations* of existing solutions, and of existing necessities with respect to different epistemological basis – for example problem of entropy with respect to measure and topological entropy. The basic element is *confrontation* of the problem with an epistemological basis. Further, the process of SOLVING THE PROBLEM begins. Here, we can refer to rich literature within the scope of the philosophy of mathematics, as well as didactics of mathematics. The solution to the problem appears as a result of continuous *confrontation* between intuition and formalism, supported by motivating factors, with continuous *confrontation* of obtained (partial) results with epistemological basis (such idea is wider than formalistic approach). Finally, THE RESULT appears. We avoid using the phrase "solution to the problem" intentionally, as it is frequently only a partial solution or statement that the problem is unsolvable. The result is also *confronted* with epistemological basis.

The graphical illustration of the process is presented below.



Although we have declared we should not touch on logicism and formalism, it is clear that in the concept described above one can easily find elements of modern formalism[7] as well as Lakatos quasi-empiricism[8] and a modern approach to structuralism[9]. However, our concept does not contain the whole scope of the philosophy of mathematics. It is, in a sense,

[6]This notion is due to G. Brusseau (see [17]).

[7]A particular role is played here by ZFC axiom system mentioned earlier. Moreover, iterative concepts and problem of infinity seems to be interesting.

[8]The wider context of quasi-empiricism would arise in the analysis of the process of solving problem. Particular importance in the context of these issues would be critical rationalism and rational falsification.

[9]Studies of C. H. Parsons are particularly interesting here.

its power. For example, we do not analyse whether the infinity should be treated only hypothetically and we do not analyse the sense of existence of infinity axiom. If we examine an issue where we need such assumptions, then we *confront* these needs with existing knowledge of them. We check in which structures the analysed problems exist and how we can or should change the scope of these structures. Finally, we are interested in the history of the problem: how to cope with this particular or similar problem(s).

3. Didactic issues

> **In the framework of the range of important topics in the theory-practice relation, I shall concentrate on certain aspects of the mathematical knowledge negotiated and mediated in this relationship. Theoretical perspective will not be curricular, historical, or mathematical, but an attempt to use the epistemological basis[10] of mathematics. If it is accepted that epistemology is the scientific enterprise of investigating the status, structure, and meaning of knowledge, then this perspective becomes indispensable for the analysis of such indirect modes of cooperation between scientific didactics and everyday teaching practice that aim a communication as a reciprocal dialogue searching for possibilities off constructing and enhancing meaning and not simply conveying knowledge matter. The intention is not to describe the mediation of a coherent didactical theory named "mathematical epistemology" to the practice of mathematics teaching, but to stress and to use epistemological considerations of mathematical knowledge, because this is an essential characteristic of every process of mediating knowledge between teacher and students as well as between researcher and teacher.**
>
> **H. Steinbring** ([18])

Mathematical education is a complex issue which is examined and analysed in various ways. It is no wonder that within the scope of didactics of mathematics there are many papers whose parts deal with mathematical philosophy and the philosophy of didactics of mathematics (e.g. articles by P. Ernest, M. Otte and F. Seeger, M. Niss, or R. Noss contained in monograph edited by: R. Biehler, R. W. Scolz, R. Strässer, B. Winkelmann Didactics of mathematics as a scientific discipline, Kluwer Academic

[10]The meaning of the notion "epistemological basis" in the H. Steinbring's article (1994) is a little different and narrower than in our paper. However this quotation may constitute some kind of introduction to this part of our article.

Publishers, Math. Education Library; and in Poland, in a very interesting article by E. Jagoda, M. Pytlak, E. Swoboda, S. Turnau, A. Urbańska ([9]).[11]

What do we mean by the **philosophy of didactics of mathematics**? First of all we should refer directly to mathematical epistemology, so that we understand it as a *science examining general cognitive processes within mathematics*. Of course we do not refer to the whole issue, but we concentrate on 'teaching philosophy' (in some coherence of learning, but also with a certain distance to the student, as referred to a teachers' approach). Before presenting our concept, we will formulate a general "axiom"[12] concerning this issue:

Nowadays, building mathematical knowledge of individuals must contain elements of confrontation with the modern educational and epistemological basis.

Leaving the above statement without any comment may cause a lot of misunderstandings.

Let us start with a simple statement: The above axiom refers mainly to a teacher[13]. It concerns a student only indirectly: through the impact of a teacher (not through requirements!). Those involved in the process of acquiring knowledge are "getting used to" *confronting* their activities with a contemporary educational and epistemological basis (within the scope of the availability of these activities).

It is also important to join modern educational and epistemological basis by a conjunction. An essential problem for a teacher is the capability to include innovative didactic and educational solutions into the teaching process, and *confronting* the effects with an epistemological basis. It is directly connected with the following observation:

If a teacher does not have any knowledge of or reflections on the philosophy of mathematics, he or she will "create" his own philosophy, often imperfect and highly confined, and sometimes burdened with many errors. His/her attitude and seemingly unimportant comments will create in students' minds a completely false view of mathematics – a view which often turns into a reluctance to pursue the subject. Let us look at some examples.

[11]Of course, these are only examples. In fact, it would be possible to mention more authors here.

[12]The use of the word "axiom" is to show the relationship with the general concept of Maths. One can use words like: principle, suggestion, etc., but in our opinion it would not show the heart of the matter. This thesis is set to be an axiom (in the absolute understanding of this word).

[13]We mean here also someone preparing e-learning materials.

Negative numbers.
Let us start with the historical background. Negative numbers were introduced into mathematics quite late. The Babylonians did not know them (although they certainly knew the concept of debt). The Chinese in II century BC made some observations connected with them, and in Greece only Diofantos from Alexandria (III century) used some of their properties. If we follow more closely the history of these numbers, we will see that practical motivation was not sufficient to develop their wider theory. The introduction of these numbers to mathematics is due to research by Descartes and I. Newton. Modern approach to arithmetic of integers appeared in XIX century.

By the didactic principle of parallelism we already have the first *confrontations* with the epistemological basis (history of the problem). Let us repeat: the notion of a debt was not enough motivation to consider such numbers! When was the breakthrough? The first step forward came as a natural need "to extend the axis" and from Descartes' work. This is the first observation.

Now, let us consider further *confrontations*: one of the basic constructions of these numbers in the modern theoretical arithmetic is the method using equivalence classes of pairs of positive integers. Thus we have our next observation: a negative number should be related to an (ordered) pair of numbers.

Let us perform one more *confrontation*. Pairs of numbers related to integers should be connected with operations on these pairs. What is more, these operations have to agree with the basis of ring structure and they can not "change results of operations for positive integers".

All these things constitute the <u>knowledge of a teacher</u>, which should be used while developing definitive didactic solutions aiming to introduce these numbers.

How do we translate this knowledge into examples of a methodical solution? We will be avoiding detailed solutions here. We will only present a general scheme.

Students know the number line (semi-number line) "with beginning at 0". We draw the number line and try to interpret operations like $2 + 3$, $8 - 2$, $6 - 4$, etc. with respect to specific issues such as, for example, inflow and outflow of money. In this way we realize a request of one of *confrontations*: we operate with pairs of numbers. At some point, we meet the operation $2 - 3$. The point is not to give a result but to interpret the operation on the number line (corollary from the principle of parallelism). May be it would be advantageous to give such a problem to students several times. At some point a natural solution will appear: "extension of the number line into the left". In such a way, new numbers will appear. How should they be denoted? Again, it is worth to *confront* it with the epistemological basis in the interpretation of our needs. Numbers obtained from operations $2-3$ and $0-1$ are the same. Similarly the same are the numbers derived from operations $3-5$ and $0-2$. We could therefore denote these numbers by $0-1$, $0-2$, $\ldots$. As 0 would always repeat, we can skip it and we obtain -1, -2, $\ldots$ And now the problem of operations. Here we come to the heart of the *confrontational method*! We should not create the conviction that we deal with the following situation: the operations are "somehow" defined and "accidentally" they have the same properties as they have for positive integers.

We definitely need to emphasize that the operations should be defined in such a way to have respective properties (associativity, commutativeness, a neutral element, etc.)

We do not have to name these properties. We can discover the computation ways with students, but we should "convince them" that this is not accidental, that the power of mathematics lies in its structures. Definitely, the matter is not learning structures, or even through the structures, but within the *confrontation* to these structures[14].

Let us come back for a moment to the idea of E. P. Wigner (at the beginning of the paper). The power of the application of complex numbers does not come from the fact that someone has discovered, during sleepless nights, that we can solve the equation $x^2+1=0$. Then, it would be enough simply to join a symbol to the set of real numbers and to make the agreement that this is a solution to this equation. In fact, it would not give us anything because, for example, the equation $x^2+2=0$ would still not have a solution. "The power of complex numbers" lies in the fact that the considerations for that problem have always been connected with certain structures, and what is more, the isomorphism of Gauss and Hamilton solutions was obtained.

Similar solutions can be shown with respect to a derivative of a function. If you do not *confront* didactic solutions with an epistemological basis, then the notion will be very difficult. Its usefulness will seem completely accidental, and it may discourage rather than motivate the student to work. The sources of the concept of derivative can be found in physics. So let us start with the simplest problem[15], for example:

Within the first five seconds of a motion, the dependence of distance on the duration x is described by the function $f(x) = 5x^2$.

[14] Meaningful examples here are quaternions and Cayley octaves. We are forced to restrict the structures here (for example, multiplication is not commutative).

[15] Obviously one can relate the problem for example to falling objects. The presented function is "close" to the description of a free fall.

Now, we give a free hand to students, simultaneously directing them to the question: how was the object velocity, for example, in the third second of the motion? The problem is that we do not even know how to say: what is a velocity at a moment? However, we can compute the average velocities in successive intervals with an excess (between the 3-rd and the 4-th second) and with insufficiency (between the 2-nd and the 3-rd second). Thus the object velocity at the 3-rd second of the motion is within the range of 25 m/s and 35 m/s. By decreasing the intervals (for example $(3; 3.5)$ and $(2.5; 3)$) we obtain more accurate estimations: the range between 27.5 m/s and 32.5 m/s. Obviously one can further decrease the intervals (calculations using a calculator or computer), but very soon one can find that the obtained ranges "will converge" (whatever it means) to the value of 30 m/s. This is a derivative $f'(3) = 30$. Now we must consciously *confront* our discovery with an epistemological basis. As a result of this *confrontation*, the average velocity will change into the difference quotient and converging of the obtained ranges into the limit of difference quotient. It may also become possible to find an answer to the question: what does instantaneous velocity mean?

We have presented the examples of *confrontational method* with respect to introducing new notions, because it is particularly easily seen in such a case.

We should once more emphasize: the presented "axiom" is related mainly to a teacher and to his/her didactic and methodical choices. Just as the mathematical maturity of students grows, the style of *confrontation with epistemological basis* should be included into their direct actions.

In conclusion, it should be emphasized that if we do not apply this "axiom", it may lead not only to a lack of understanding of mathematics as a science, but also to a lack of understanding of the mathematical tools we use. The classic example is the ancient method of analysis (analisis antiquorum) for solving equations.

4. Conclusion

We are aware of the fact that the article has an introductory character, in the sense of "inducing the discussion". Mathematics is surprisingly useful. To fully understand why the moth flying into to a bulb, continues to approach it until its death, you need to know the logarithmic spiral (Bernoulli)[16]. In order to investigate certain sociological or economic issues we apply a Gaussian normal distribution using the number π, whose genesis has nothing to do with these issues. The solutions of logistical problems are often based on difference and differential equations, which arose

[16]Following K. Ciesielski, K. Pogoda [3]).

when the scale of the logistical problems did not require too sophisticated mathematical tools. Finally, climate change, some aspects of biology and environmental protection, or even elements of art, are examined and analysed using mathematical rules that were created without any connection with these applications.

It all indicates that in the modern world mathematics is needed more than ever before. Many mathematics educators indicate that one of the most important tasks of modern teaching of mathematics is the problem of the relationship of theory and practice[17]. This is a very important observation! However, a one-sided approach to this issue may in the long term bring about the opposite effect. Simple mathematical issues may be applied for simple problems only. This is needed but at school level we can not go further. However, to understand many of the issues "the modern world" needs higher mathematics, what's more, using mathematics, not previously met. One of the students of doctoral studies talked about her practice at the Ministry of Foreign Affairs. She had to perform analysis using difference equations, which she did not know. Moreover, she could not apply directly the found facts – they had to be modified. That is the example of situation people in modern world must be prepared for. In our opinion, one of the elements of such preparation will be awareness of the need to *confront* one's thoughts with epistemological basis. Such an awareness must be built (gradually and systematically) and maintained by a teacher, but first a teacher must have knowledge in this field and be convinced of such a need.

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