Opuscula Math. 37, no. 6 (2017), 821–827
http://dx.doi.org/10.7494/OpMath.2017.37.6.821 Opuscula Mathematica

A DIRECT APPROACH TO LINEAR-QUADRATIC STOCHASTIC CONTROL

Tyrone E. Duncan and Bozenna Pasik-Duncan

Communicated by Marek Galewski

Abstract. A direct approach is used to solve some linear-quadratic stochastic control problems for Brownian motion and other noise processes. This direct method does not require solving Hamilton–Jacobi–Bellman partial differential equations or backward stochastic differential equations with a stochastic maximum principle or the use of a dynamic programming principle. The appropriate Riccati equation is obtained as part of the optimization problem. The noise processes can be fairly general including the family of fractional Brownian motions.

Keywords: linear-quadratic Gaussian control, Riccati equation for optimization, stochastic control.

Mathematics Subject Classification: 93E20, 93C05.

1. INTRODUCTION

The control of a linear stochastic differential equation with an additive control, an additive Brownian motion, and a quadratic cost in the state and the control is probably the most well known control problem and has a simple, explicit solution e.g. [6]. The typical approaches to the solution of this control problem are either to use the Hamilton–Jacobi–Bellman (HJB) equation or the stochastic maximum principle with the solution of a backward stochastic differential equation. An important feature concerning the HJB equation approach for this problem is that the solution of this equation is basically the same as the solution of the Hamilton–Jacobi partial differential equation for the corresponding deterministic linear-quadratic control problem that arises by eliminating the noise term in the stochastic system though the former partial differential equation is of second order and the latter partial differential equation is of first order. In [4] an approach to the linear-quadratic control problem with a general noise process provides evidence that the stochastic control problem can be considered as an affine translation of the corresponding deterministic control problem. However this relation between optimization problems assumed that the Riccati equation is given and it can be used to solve the HJB equation. In this paper motivated by the algebraic method of completion of squares the form of the Riccati equation is derived and its geometric justification is noted from a flow in the Lagrangian Grassmannian [1,2]. The HJB equation is a second order nonlinear partial differential equation which is clearly very difficult to solve explicitly in any generality. Furthermore it is typically very difficult to obtain explicit solutions to backward stochastic differential equations which are stochastic equations that are solved backward in time but having a forward measurability. An aim of this paper is to solve the linear-quadratic stochastic control problem in a direct and natural way which should provide more insight into the fundamental features of this problem.

2. CONTROL PROBLEM FORMULATION WITH BROWNIAN MOTION

The controlled linear stochastic system is described by the following stochastic differential equation:

$$dX(t) = A(t)X(t)dt + B(t)U(t)dt + C(t)dW(t), \tag{2.1}$$

$$X(0) = X_0, \tag{2.2}$$

where $X_0 \in \mathbb{R}^n$ is random, $X(t) \in \mathbb{R}^n, A(t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B(t) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $C(t) \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^n)$, for each $t \in [0,T]$, and these deterministic, linear transformations are continuous and thus uniformly bounded, $U(t) \in \mathbb{R}^m$, for each $t \in [0,T]$, where $U \in \mathcal{U}$, is the control and $\mathcal{U}$ is the family of admissible controls, $(W(t), t \in [0,T])$ is an $\mathbb{R}^k$-valued standard Brownian motion that is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0,T])$ is the filtration for the Brownian motion W. The random elements X_0 and W are assumed to be independent and X_0 is defined on the same probability space as W. The positive integers (k, m, n) are arbitrary.

The family of admissible controls, $\mathcal{U}$, is

$$\mathcal{U} = \{U : U \text{ is an } \mathbb{R}^m\text{-valued } (\mathcal{F}(t))\text{-adapted process such that } U \in L^2([0,T]) \text{ a.s.}\}.$$

The cost functional J is a quadratic functional of X and U that is given by

$$J_0(U) = \frac{1}{2}\int_0^T (\langle Q(s)X(s), X(s)\rangle + \langle R(s)U(s), U(s)\rangle)ds + \frac{1}{2}\langle MX(T), X(T)\rangle, \tag{2.3}$$

$$J(U) = \mathbb{E}[J_0(U)] \tag{2.4}$$

where $Q(t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), R(t) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m), M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $Q(t) > 0$, $R(t) > 0$ for each $t \in [0,T]$ and $M > 0$ are symmetric linear transformations and Q, R are continuous and deterministic, $\langle \cdot, \cdot \rangle$ denotes the canonical Euclidean inner product on the Euclidean space of the appropriate dimension and $\mathbb{E}$ is expectation with respect to the probability measure $\mathbb{P}$. The dependence of J on X_0 is suppressed for notational simplicity.

An important idea to understanding the solution of this control problem is to note some geometry associated with it. The natural geometric setting for the problem (2.1) and (2.3) is the Lagrangian Grassmannian that is usually denoted by $\Lambda(n)$ for an appropriate positive integer n. It is a Grassmannian of n planes in $2n$ dimensional Euclidean space with a closed, nondegenerate two form denoted ω. Arnold [1] used these Grassmannians particularly for his study of classical mechanics [2]. The space $\Lambda(n)$ has dimension $\frac{n(n+1)}{2}$ and it can also be described as the homogeneous space $U(n)/O(n)$ where $U(n)$ is the group of unitary transformations on $\mathbb{C}^n = \mathbb{R}^{2n}$ and $O(n)$ is the orthogonal group on $\mathbb{R}^n$. Let $V = \mathbb{R}^{2n}$ and $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ be a symplectic basis of (V, ω), that is,

$$\omega(e_i, e_j) = 0 \quad \text{for all } i, j \in \{1, \ldots, n\}, \tag{2.5}$$

$$\omega(e_i, f_j) = \delta_{ij} \quad \text{for all } i, j \in \{1, \ldots, n\}, \tag{2.6}$$

$$\omega(u, v) = u^T \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} v \tag{2.7}$$

for $u, v \in \mathbb{R}^{2n}$. Similar to a line through the origin in $\mathbb{R}^2$ being described as the pair (x, ax) where the line has finite slope a, a (nonsingular) plane in $\Lambda(n)$ can be described as (x, Px) where P is a symmetric matrix. This description of $\Lambda(n)$ shows that $dim\Lambda(n) = \frac{n(n+1)}{2}$ and $\Lambda(n)$ is a natural description to determine optimal controls from the solution of a Riccati equation.

The following theorem is a solution to the well known linear-quadratic stochastic control problem.

Theorem 2.1. *An optimal control, U^*, for the stochastic control problem given by* (2.1) *and* (2.3) *is given by*

$$U^*(t) = -R^{-1}(t)B^T(t)P(t)X(t), \tag{2.8}$$

where P is the unique, symmetric, positive definite solution of the following Riccati equation

$$\frac{dP}{dt} = -PA - A^T P + PBR^{-1}B^T P - Q, \tag{2.9}$$

$$P(T) = M, \tag{2.10}$$

and the optimal cost is

$$J(U^*) = \int_0^T tr(PCC^T P)dt + \mathbb{E}\langle P(0)X(0), X(0)\rangle, \tag{2.11}$$

Proof. In this proof, the Riccati equation is obtained from a requirement that the optimal control is obtained from a suitable squared expression. From the properties of the Lagrangian Grassmannian, it is natural to describe the evolution of the optimization problem by a curve in symmetric elements of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. Let $Y(t) = \langle P(t)X(t), X(t)\rangle$

for $t \in [0,T]$ for a symmetric $P(t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. Apply the change of variables formula (Ito formula) to the process $(Y(t), t \in [0,T])$ to obtain a quadratic expression so that an optimal control becomes clear. It is assumed that $(P(t), t \in [0,T])$ satisfies an ordinary differential equation that has the following form.

$$\frac{dP}{dt} = K_1 P + P K_1^T + K_2 P K_3 P K_2^T + K_4, \tag{2.12}$$

$$P(t) = M, \tag{2.13}$$

where K_1, K_2, K_3, K_4 are $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$-valued and K_3, K_4 are also symmetric. These four linear transformations are determined from the description of the control problem by requiring that the optimal control is identified by a squared expression.

By the Ito formula (change of variables) the process $(Y(t), t \in [0,T])$ satisfies the following integral expression

$$\begin{aligned} Y(T) - Y(0) = \int_0^T &(2\langle PX, AX + BU\rangle dt + 2\langle PX, CdW\rangle \\ &+ tr(PCC^T)dt \\ &+ \langle X, (K_1 P + P K_1^T + K_2 P K_3 P K_2^T + K_4)X\rangle dt). \end{aligned} \tag{2.14}$$

Adding the integral terms for the cost functional to the previous equation it follows that

$$\begin{aligned} J_0(U) - \langle P(0)X(0), X(0)\rangle = \int_0^T &((\langle RU, U\rangle + \langle QX, X\rangle \\ &+ 2\langle PX, AX + BU\rangle)dt + 2\langle PX, CdW\rangle \\ &+ tr(PCC^T)dt \\ &+ \langle X, (K_1 P + P K_1^T + K_2 P K_3 P K_2^T + K_4)X\rangle dt). \end{aligned} \tag{2.15}$$

It is desired to choose linear transformations (K_1, K_2, K_3, K_4) so that the above equation can be expressed as

$$\begin{aligned} J_0(U) - \langle P(0)X(0), X(0)\rangle = \int_0^T &((|L_1(L_2 U - L_3 X)|^2 + L_4)dt \\ &+ 2\langle PX, CdW\rangle), \end{aligned} \tag{2.16}$$

where (L_1, L_2, L_3, L_4) are suitable linear transformations. Now equate the right hand sides of (2.15) and (2.16) canceling the common stochastic integral term to determine the unknown terms $(K_1, K_2, K_3, K_4, L_1, L_2, L_3, L_4)$.

$$\begin{aligned}
&\int_0^T (\langle RU, U\rangle + \langle QX, X\rangle + 2\langle PX, AX + BU\rangle)dt \\
&\quad + tr(PCC^T)dt \\
&\quad + \langle X, (K_1P + PK_1^T + K_2PK_3PK_2^T + K_4)X\rangle dt \\
&= (\langle L_1(L_2U + L_3X), L_1(L_2U + L_3X)\rangle + L_4)dt.
\end{aligned} \tag{2.17}$$

It follows immediately that $L_4 = tr(PCC^T)$. From the quadratic term in X it follows that $K_2PK_3PK_2^T = L_3^T L_1^T L_1 L_3$. From the squared term in U it follows that $L_2^T L_1^T L_1 L_2 = R$. From the cross term in U, X it follows that $L_3 L_1^T L_1 L_2 = PB$. Let $L_2 = R$ and $L_1 = R^{-\frac{1}{2}}$ so that $L_2^T L_1^T L_1 L_2 = R$ is satisfied. Thus $L_3 = PB$ and L_1, L_2, L_3, L_4 are determined. Choose $K_3 = R^{-1} = L_1^T L_1, K_2 = B^T P$, and from the above $K_1 = -A^T, K_4 = -Q$. Thus the Riccati equation is

$$\frac{dP}{dt} = -(A^TP + PA) + B^TPR^{-1}PB - Q, \tag{2.18}$$

$$P(T) = M. \tag{2.19}$$

It follows from (2.17) that an optimal control U^* is

$$U^*(t) = -R^{-1}(t)B^T(t)P(t)X(t). \tag{2.20}$$

The optimal cost is $J(U^*) = \mathbb{E}J_0(U^*)$ given by

$$J(U^*) = \int_0^T tr(PCC^T)dt + \mathbb{E}\langle P(0)X(0), X(0)\rangle. \tag{2.21}$$

□

A similar approach can be used to solve a linear-quadratic control problem for linear stochastic equations in an infinite dimensional separable Hilbert space e.g. [5] and for linear-quadratic stochastic differential games [3].

3. CONTROL PROBLEM FORMULATION WITH GENERAL NOISE PROCESSES

Now it is sketched how to extend the result for an optimal control to linear stochastic systems that have a fairly general noise process such as the family of fractional Brownian motions. In the controlled linear system (2.1) the Brownian motion is replaced by a stochastic process that is square integrable with continuous sample paths

$$dX(t) = A(t)X(t)dt + B(t)U(t)dt + C(t)dW(t), \tag{3.1}$$

$$X(0) = X_0, \tag{3.2}$$

where $X_0 \in \mathbb{R}^n$ is a random vector, $X(t) \in \mathbb{R}^n, A(t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B(t) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $C(t) \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^n)$, for each $t \in [0, T]$, and these deterministic functions are continuous and thus uniformly bounded, $U(t) \in \mathbb{R}^m$, for each $t \in [0, T]$, where $U \in \mathcal{U}$, is the control and $\mathcal{U}$ is the family of admissible controls, $(W(t), t \in [0, T])$ is an $\mathbb{R}^k$-valued square integrable process with continuous sample paths that is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0, T])$ is the filtration for the process W. The random elements X_0 and W are assumed to be independent and X_0 is defined on the same probability space as W. The positive integers (k, m, n) are arbitrary.

Theorem 3.1. *For the control problem with the linear system* (3.1) *and the quadratic cost functional* (2.3) *there is an optimal control, U^*, that is given by*

$$U^*(t) = -R^{-1}(t)B^T(t)P(t)X(t) + R^{-1}(t)B^T(t)C\mathbb{E}[W(t)|\mathcal{F}(t)]. \tag{3.3}$$

The proof is obtained by approximating the noise process W by a sequence of piecewise linear processes, replacing W by an element of this sequence and solving each of these control problems. The additional term on the RHS for an optimal control arises as the best estimate of the future noise. The proof can be verified using the results in [4].

4. CONCLUSIONS

A direct method is provided to obtain an optimal control for a linear-quadratic stochastic control problem that does not require solving a Hamilton-Jacobi-Bellman equation or a backward stochastic differential equation. The appropriate Riccati equation is derived from an algebraic requirement that the optimal control is obtained from a quadratic functional obtained from the system dynamics and the cost functional.

Acknowledgements

The authors thank the referees for their useful comments.
Department of Mathematics, University of Kansas, Lawrence, KS 66045 Research supported by NSF grant DMS 1411412, ARO grant W911NF-14-10390, AFOSR grant FA9550-17-1-0073 and a Simons Fellowship for the first author.

REFERENCES

[1] V.I. Arnold, *Characteristic class entering in quantization conditions*, Funktsional'nyi Analiz i Ego Prilozheniya (Funct. Anal. Appl.) **1** (1967), 1–14.

[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Grad. Texts in Math. 60, Springer, New York, 1978.

[3] T.E. Duncan, *Linear-quadratic stochastic differential games with general noise processes*, Models and Methods in Economics and Management Science: Essays in Honor of Charles S. Tapiero (eds. F. El Ouardighi and K. Kogan), Operations Research and Management Series, Springer Intern. Publishing, Switzerland, vol. 198, 2014, 17–26.

[4] T.E. Duncan, B. Pasik-Duncan, *Linear quadratic fractional Gaussian control*, SIAM J. Control. Optim. **51** (2013), 4604–4619.

[5] T.E. Duncan, B. Maslowski, B. Pasik-Duncan, *Linear-quadratic control for stochastic equations in a Hilbert space with fractional Brownian motions*, SIAM J. Control Optim. **50** (2012), 507–531.

[6] W.H. Fleming, R.W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, New York, 1975.

Tyrone E. Duncan
teduncan@ku.edu

University of Kansas
Department of Mathematics
503 Snow Hall, Lawrence, Kansas 66045, USA

Bozenna Pasik-Duncan
bozenna@ku.edu

University of Kansas
Department of Mathematics
503 Snow Hall, Lawrence, Kansas 66045, USA

Received: January 23, 2017.
Revised: April 3, 2017.
Accepted: April 8, 2017.