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EXISTENCE AND DECAY OF FINITE ENERGY SOLUTIONS FOR SEMILINEAR DISSIPATIVE WAVE EQUATIONS IN TIME-DEPENDENT DOMAINS

Mitsuhiro Nakao

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Abstract. We consider the initial-boundary value problem for semilinear dissipative wave equations in noncylindrical domain $\bigcup_{0\le t<\infty}\Omega(t)\times\{t\}\subset\mathbb{R}^N\times\mathbb{R}$. We are interested in finite energy solution. We derive an exponential decay of the energy in the case $\Omega(t)$ is bounded in $\mathbb{R}^N$ and the estimate

$$\int_0^\infty E(t)dt \le C(E(0),\|u(0)\|) < \infty$$

in the case $\Omega(t)$ is unbounded. Existence and uniqueness of finite energy solution are also proved.

Keywords: energy decay, global existence, semilinear wave equation, noncylindrical domains.

Mathematics Subject Classification: 35B35, 35L70.

1. INTRODUCTION

We consider the semilinear wave equation with a dissipative term

$$u_{tt} - \Delta u + u_t + f(u) = 0, \quad x\in\Omega(t), t>0, \tag{1.1}$$

with the initial-boundary conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{and} \quad u|_{\partial\Omega(t)} = 0, \tag{1.2}$$

where $\Omega(t)$, $t\ge 0$, is a domain in $\mathbb{R}^N$ for each $t\ge 0$ with the boundary $\partial\Omega(t)$. We set

$$Q(0,\infty) = \bigcup_{0\le t<\infty}\Omega(t)\times\{t\} \quad \text{and} \quad \Gamma(0,\infty) = \bigcup_{0\le t<\infty}\partial\Omega(t)\times\{t\}.$$

We assume that $\Gamma(0,\infty)$ is smooth, say, C^2 class in $\mathbb{R}^N \times \mathbb{R}^+$.

Concerning the function $f(u)$ we assume

Hyp. A: $f(u)$ is a C^1 function on $\mathbb{R}$, satisfying the conditions

$$f(0)=0, \quad 0 \le k_0 F(u) \le f(u)u \le k_1 F(u)$$

with some constants $k_0, k_1 > 0$, where we set $F(u) = \int_0^u f(s)ds$. Further,

$$|f(u)| \le k_3(|u| + |u|^{\alpha+1}) \quad \text{and} \quad |f'(u)| \le k_3(1+|u|^\alpha)$$

with an exponent α, $0 \le \alpha \le 2/(N-2)$ ($0 \le \alpha < \infty$ if $N=1,2$) and a constant $k_3 > 0$.

Concerning the movement of $\Omega(t)$ we assume that $\Omega(t)$ is time-like, that is, letting $\mathbf{n} = (n_x, n_t)$ be the outward unit normal at $(x,t) \in \Gamma(0,\infty)$ we assume

Hyp. B:

$$|n_t| < |n_x| \text{ at each } (x,t) \in \Gamma(0,\infty).$$

When $\Omega(t) = \Omega$, independent of t, the existence and uniqueness of the solution $u(t) \in C([0,\infty); H_0^1(\Omega)) \times C^1([0,\infty); L^2(\Omega))$ of the problem (1.1)–(1.2) with $(u(0), u_1(0)) \in H_0^1(\Omega) \times L^2(\Omega)$ is easy and we can prove the decay estimates:

$$E(t) \le CE(0)e^{-\lambda t}, \quad t > 0, \tag{1.3}$$

with some $\lambda > 0$ for the case Ω is bounded, and

$$\|u(t)\|^2 + \int_0^\infty E(t)dt \le C(E(0) + \|u_0\|^2) < \infty \tag{1.4}$$

for the case Ω is an exterior domain, where we set

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|^2 + \|\nabla u(t)\|^2\right) + \int_\Omega F(u(t))dx.$$

The latter estimate (1.4) easily implies

$$E(t) \le C(E(0) + \|u_0\|^2)(1+t)^{-1}$$

by use of the inequality $\frac{dE(t)}{dt} \le 0$.

The object of this paper is to derive such estimates (1.3) and (1.4) when $\Omega(t)$ is time-dependent. In fact, we shall make a further assumption on $\mathbf{n}$ requiring that the boundary $\partial\Omega(t)$ does not move so rapidly, where we set

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|^2_{\Omega(t)} + \|\nabla u(t)\|^2_{\Omega(t)}\right) + \int_{\Omega(t)} F(u(t))dx.$$

($\|\cdot\|_{\Omega(t)}$ denotes $L^2(\Omega(t))$ norm.)

When $\Omega(t)$ is time-dependent the existence and uniqueness of finite energy solution $u(t)$ such that

$$u(t) \in C([0,\infty); H_0^1(\Omega(t))) \quad \text{and} \quad u_t(t) \in C^1([0,\infty); L^2(\Omega(t)))$$

is never trivial (see the next section for the definitions of such function spaces). Inoue [3] considered the case $N = 3$ and $f(u) = u^3$ (without the dissipation u_t) and proved the existence and uniqueness of $C^\infty(Q(0,\infty))$ solution for each $(u_0, u_1) \in C^\infty(\Omega(0)) \times C^\infty(\Omega(0))$ which has compact support and satisfies the compatibility condition of infinite order at the boundary $\partial\Omega(0)$. (There, $\Gamma(0,\infty)$ is assumed to be C^∞ class.) His method is based on the linear theory for hyperbolic mixed problem due to Ikawa [2] and finite propagation property of the local solutions. Carefully checking the proof we see that the problem (1.1)–(1.2) admits a unique strong solution

$$u(t) \in C([0,\infty); H^2(\Omega(t)) \cap C^1([0,\infty); H_0^1(\Omega(t))) \cap C([0,\infty); L^2(\Omega(t)))$$

for each $(u_0, u_1) \in H_2(\Omega(0)) \times H_1(\Omega(0))$ satisfying the "0" order compatibility condition on $\partial\Omega(0)$ and having compact support. The compatibility condition is satisfied if $(u_0, u_1) \in C_0^2(\Omega(0)) \times C_0^1(\Omega(0))$. We first derive the estimates (1.3) and (1.4) for such strong solutions. Then by density argument we can show the existence of a finite energy solution for $(u_0, u_1) \in H_0^1(\Omega(0)) \times L^2(\Omega(0))$, satisfying the estimates (1.3) and (1.4). Further, combining these estimates with the uniqueness result for the linear equations due to Cooper [1] we see that our finite energy solution is unique for each $(u_0, u_1) \in H_0^1(\Omega(0)) \times L^2(\Omega(0))$. The estimate includes the boundedness of $\|u(t)\|_{\Omega(t)}$ and this is useful to treat a general nonlinear term $f(u)$ in Hyp. A for the case of exterior domain. To derive the estimates (1.3) and (1.4) we use an energy method as in our previous paper [6], where the linear equation in bounded domains in $\mathbb{R}^3$ is considered. The main idea using the multiplier $u_t + \beta \cdot \nabla u$ is taken from Lee [5], where the existence results of mixed problem for second order hyperbolic equations in time-dependent domains are proved by the method due to Ladyzhenskaya [4] and energy estimate for the solutions in $H_2(Q(0,T))$, $T > 0$, which is completely different from the methods in [2, 3].

Recently, linear and nonlinear wave equations with various dissipative terms have been investigated in details by many authors. However there seem to be very few papers which treat the equations in noncylindrical domains.

2. PRELIMINARIES AND STATEMENT OF THE RESULT

For the time-dependent domain $\Omega(t), 0 \le t < \infty$, and a, b with $0 \le a < b < \infty$ we set

$$Q(a,b) = \bigcup_{a \le t \le b} \Omega(t) \times \{t\} \quad \text{and} \quad \Gamma(a,b) = \bigcup_{a \le t \le b} \partial\Omega(t) \times \{t\}.$$

For a function $u(x,t)$ defined on $Q(a,b)$ we say that $u(t) \in C([a,b]; H_0^1(\Omega(t)))$ if $u(t) \in H_0^1(\Omega(t))$ for each $t, a \le t \le b$, and $u(x,t)$ has an extension $\tilde{u}(x,t)$ such that

$\tilde{u}(t) \in C([a,b]; H^1(\mathbb{R}^N))$ and $\tilde{u}(x,t) = u(x,t)$ if $(x,t) \in Q(a,b)$. Such a way of definition is applied to other spaces $C([a,b]; L^2(\Omega(t))), C([a,b]; H^2(\Omega(t)))$ and $C^1([a,b]; H_0^1(\Omega(t)))$ etc. We denote by $\|v\|_{\Omega(t)}$ and $(u,v)_{\Omega(t)}$ the L^2 norm and L^2 inner product on $\Omega(t)$, respectively.

The following existence and uniqueness result of strong solution is essentially included in [3].

Proposition 2.1. *Let* $(u_0, u_1) \in C_0^2(\Omega(0)) \times C_0^1(\Omega(0))$*. Then, under Hyp. A and Hyp. B, the problem* (1.1)–(1.2) *admits a unique solution* $u(t)$ *such that*

$$u(t) \in C([0,\infty); H^2(\Omega(t)) \cap H_0^1(\Omega(t))), \quad u_t \in C([0,\infty); H^1(\Omega(t)))$$

and

$$u_{tt}(t) \in C([0,\infty); L^2(\Omega(t))).$$

Let

$$\mathbf{n}(x,t) = (n_x, n_t)$$

be the outward unit normal of $\Gamma(0,\infty)$ at $(x,t) \in \partial\Omega(t) \times \{t\}$ and set

$$\beta = -n_t n_x / |n_x|^2 \in \mathbb{R}^N.$$

Since $\Gamma(0,\infty)$ is of C^2 class we can extend β to be a C^1 function on $\overline{Q(0,\infty)}$ which we denote by the same notation. We set

$$r^+ = \sup_{(x,t)\in Q(0,\infty)} (|\beta| + |\nabla\beta| + |\beta_t| + |\beta'|), \tag{2.1}$$

where β' denote the Jacobian of the mapping $\beta : x \mapsto \beta(x,t)$.

Definition 2.2. A function $u = u(x,t)$ on $Q(0,\infty)$ is called a finite energy solution of the problem (1.1)–(1.2) if

$$u(t) \in C([0,\infty); H_0^1(\Omega(t))), \quad u_t(t) \in C([0,\infty); L^2(\Omega(t)))$$

with $u(0) = u_0$, $u_t(0) = u_1$ and the equation (1.1) is satisfied in the sense of distribution on

$$Q^o(0,\infty) \equiv \bigcup_{0<t<\infty} \Omega(t) \times \{t\}.$$

Our results are read as follows:

Theorem 2.3. *Assume that* $\Omega(t)$ *is bounded for each* t *and*

$$\sup_{0\le t<\infty} \mathrm{diam}\Omega(t) \le d_0 < \infty$$

with a constant d_0*. Then there exists* $\delta_0 > 0$ *such that if* $r^+ < \delta_0$*, the problem* (1.1)–(1.2) *admits a unique finite energy solution* $u(t)$ *for each* $(u_0, u_1) \in H_0^1(\Omega(0)) \times L^2(\Omega(0))$*, satisfying the decay estimate*

$$E(t) \le CE(0)e^{-\lambda t}, \quad 0 \le t < \infty, \tag{2.2}$$

where C *and* λ *are positive constants independent of* (u_0, u_1)*.*

Theorem 2.4. *Let $\Omega(t)$ be an exterior domain of a bounded obstacle $V(t)$. Then there exists $\delta_0 > 0$ such that if $r^+ < \delta_0$, the problem (1.1)–(1.2) admits a unique finite energy solution $u(t)$ for each $(u_0, u_1) \in H_0^1(\Omega(0)) \times L^2(\Omega(0))$, satisfying the estimate*

$$E(t) + \|u(t)\|^2_{\Omega(t)} + \int_0^\infty E(s)ds \le C(E(0) + \|u_0\|^2_{\Omega(0)}) < \infty, \quad 0 \le t < \infty. \tag{2.3}$$

Remark 2.5. The existence and uniqueness part is valid under Hyp. A and Hyp. B only and this is applied also to the equation (1.1) without the dissipative term $\frac{\partial u}{\partial t}$ (see the last part of Section 4).

Remark 2.6. We do not know at this time whether the estimate (2.3) implies $E(t) \le C(E(0), \|u_0\|_{\Omega(0)})(1+t)^{-1}$ or not.

3. ENERGY ESTIMATES FOR STRONG SOLUTIONS

Let $u(t)$ be a strong solution in the sense of Proposition 2.1. We derive energy estimates including (2.1) and (2.2) for this solution by multiplier method.

First we see by integration by parts that

$$\begin{aligned}\int_{Q(t_1,t_2)} u_{tt}u_t dV &= \frac{1}{2}\int_{Q(t_1,t_2)} \frac{\partial}{\partial t}|u_t|^2 dV \\ &= \frac{1}{2}(\|u(t_2)\|^2_{\Omega(t_2)} - \|u(t_1)\|^2_{\Omega(t_1)}) + \frac{1}{2}\int_{\Gamma(t_1,t_2)} n_t|u_t|^2 dS.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{Q(t_1,t_2)} (-\Delta u + f(u))u_t dV = \frac{1}{2}\Big(\|\nabla u(t_2)\|^2_{\Omega(t_2)} - \|\nabla u(t_1)\|^2_{\Omega(t_1)}\Big) \\ + \int_{\Omega(t_2)} F(u(t_2))dx - \int_{\Omega(t_1)} F(u(t_1))dx \\ + \int_{\Gamma(t_1,t_2)} \Big(\frac{1}{2}|\nabla u|^2 n_t - n_x \cdot \nabla u u_t\Big)dS.\end{aligned}$$

Next, multiplying $\beta \cdot \nabla u$ we have

$$\begin{aligned}\int\limits_{Q(t_1,t_2)} u_{tt}\beta \cdot \nabla u dV &= \int\limits_{Q(t_1,t_2)} \Big\{(u_t\beta \cdot \nabla u)_t - u_t\beta_t \cdot \nabla u_t - \frac{1}{2}\beta \cdot \nabla u_t^2\Big\}dV \\ &= (u_t(t_2), \beta(t_2) \cdot \nabla u(t_2))_{\Omega(t_2)} - (u_t(t_2), \beta(t_2) \cdot \nabla u(t_2))_{\Omega(t_2)} \\ &\quad + \int\limits_{Q(t_1,t_2)} \Big\{ - \beta_t \cdot \nabla u u_t + \frac{1}{2}\nabla \cdot \beta |u_t|^2\Big\}dV \\ &\quad + \int\limits_{\Gamma(t_1,t_2)} \Big\{n_t u_t \beta \cdot \nabla u - \frac{1}{2}\beta \cdot n_x |u_t|^2\Big\}dS.\end{aligned}$$

Similarly,

$$\begin{aligned}-\int\limits_{Q(t_1,t_2)} \Delta u \beta \cdot \nabla u dV &= \int\limits_{Q(t_1,t_2)} \nabla u \nabla(\beta \cdot \nabla u) - \int\limits_{\Gamma(t_1,t_2)} \nabla u \cdot n_x \beta \cdot \nabla u dS \\ &= \int\limits_{Q(t_1,t_2)} \Big\{\nabla u \cdot \beta' \nabla u + \sum_{i,j} \frac{\partial u}{\partial x_i}\beta_j \frac{\partial^2 u}{\partial x_i \partial x_j}\Big\}dV \\ &\quad - \int\limits_{\Gamma(t_1,t_2)} \nabla u \cdot n_x \beta \cdot \nabla u dS.\end{aligned} \tag{3.1}$$

Here, integrating by parts,

$$\begin{aligned}A &\equiv \int\limits_{Q(t_1,t_2)} \sum_{i,j} \frac{\partial u}{\partial x_i}\beta_j \frac{\partial^2 u}{\partial x_i \partial x_j} dV \\ &= -A - \int\limits_{Q(t_1,t_2)} |\nabla u|^2 \nabla \cdot \beta dV + \int\limits_{\Gamma(t_1,t_2)} |\nabla u|^2 \beta \cdot n_x dS,\end{aligned}$$

and hence

$$A = -\frac{1}{2}\int\limits_{Q(t_1,t_2)} |\nabla u|^2 \nabla \cdot \beta dV + \frac{1}{2}\int\limits_{\Gamma(t_1,t_2)} |\nabla u|^2 \beta \cdot n_x dS.$$

Thus we have from (3.1)

$$\begin{aligned}-\int\limits_{Q(t_1,t_2)} \Delta u \beta \cdot \nabla u dV &= \int\limits_{Q(t_1,t_2)} \{\nabla u \cdot \beta' \nabla u - \frac{1}{2}|\nabla u|^2 \nabla \cdot \beta\}dV \\ &\quad - \frac{1}{2}\int\limits_{\Gamma(t_1,t_2)} \Big|\frac{\partial u}{\partial \mathbf{n}}\Big|^2 |n_x|^2 \beta \cdot n_x dS,\end{aligned}$$

where we have used the fact that $\nabla u = \frac{\partial u}{\partial n} n_x$ and $u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial n} n_t$ on $\Gamma(0, \infty)$ which follows from the boundary condition $u|_{\Gamma(0,\infty)} = 0$.

Further, we have

$$\begin{aligned}\int\limits_{Q(t_1,t_2)} f(u)\beta\cdot\nabla u dV &= \int\limits_{Q(t_1,t_2)} \beta\cdot\nabla(F(u))dV \\ &= -\int\limits_{Q(t_1,t_2)} \nabla\cdot\beta F(u)dV + \int\limits_{\Gamma(t_1,t_2)} \beta\cdot n_x F(u)dS \\ &= -\int\limits_{Q(t_1,t_2)} \nabla\cdot\beta F(u)dV.\end{aligned}$$

Thus multiplying the equation (1.1) by $u_t + \beta\cdot\nabla u$ and using the fact $n_t + \beta\cdot n_x = 0$ on $\Gamma(0,\infty)$ we obtain

$$\begin{aligned}&\tilde{E}(t_2) - \tilde{E}(t_1) + \int\limits_{Q(t_1,t_2)} |u_t(t)|^2 dV \\ &= \int\limits_{Q(t_1,t_2)} \Big\{ -\nabla u\cdot\beta'\nabla u + \frac{1}{2}(|\nabla u|^2 - |u_t|^2))\nabla\cdot\beta + \nabla\cdot\beta|u_t|^2 \\ &\qquad + \nabla\cdot\beta F(u) - u_t\beta\cdot\nabla u\Big\} dV \\ &\le Cr^+ \int\limits_{t_1}^{t_2} E(s)ds,\end{aligned} \tag{3.2}$$

where we set

$$\tilde{E}(t) = E(t) + (u_t(t), \beta\cdot\nabla u)_{\Omega(t)}.$$

We note that $\tilde{E}(t)$ is equivalent to $E(t)$ if $|\beta| = |n_t|/|n_x| \le \nu < 1$ with some constant ν.

Finally, multiplying the equation by u and integrating it we have

$$\begin{aligned}&\frac{1}{2}\Big(\|u(t_2)\|^2_{\Omega(t_2)} - \|u(t_1)\|^2_{\Omega(t_1)}\Big) + \int\limits_{Q(t_1,t_2)} (|\nabla u|^2 + f(u)u)dV \\ &= (u_t(t_2), u(t_2))_{\Omega(t_2)} - (u_t(t_1), u(t_1))_{\Omega(t_1)} + \int\limits_{Q(t_1,t_2)} |u_t|^2 dV.\end{aligned} \tag{3.3}$$

We take $t_2 = t+h, t_1 = t, h > 0$ and divide both sides of (3.2) and (3.3) by h. Taking the limits as $h \to 0$ we have

$$\frac{d}{dt}\tilde{E}(t) + \|u_t(t)\|^2_{\Omega(t)} \le Cr^+ E(t) \tag{3.4}$$

and

$$\frac{d}{dt}\left(\|u(t)\|^2_{\Omega(t)} - 2(u_t(t), u(t))_{\Omega(t)}\right) + 2\left(\|\nabla u(t)\|^2_{\Omega(t)} + \int\limits_{\Omega(t)} f(u)u dx\right) \le 2\|u_t(t)\|^2_{\Omega(t)}. \tag{3.5}$$

Combining (3.4) and (3.5) we obtain

$$\frac{d}{dt}\{3\tilde{E}(t) + \|u_t(t)\|^2_{\Omega(t)} + \|u(t)\|^2_{\Omega(t)} - 2(u_t(t), u(t))_{\Omega(t)}\} + (2\epsilon_0 - 3Cr^+)E(t) \le 0$$

with $\epsilon_0 = \min\{1, k_0\}$.

Case (1). $\Omega(t)$ is bounded.

In this case we assume further $\mathrm{diam}\Omega(t) \le d_0$ with a constant d_0 independent of t. Then, by Poincare's inequality we know $\|u(t)\|_{\Omega(t)} \le C\|\nabla u(t)\|_{\Omega(t)}$ with a constant C independent of t. Therefore we see that there exists $\delta_0 > 0$ such that if $r^+ < \delta_0$,

$$\frac{d}{dt}\{3\tilde{E}(t) + \|u_t(t)\|^2_{\Omega(t)} + \|u(t)\|^2_{\Omega(t)} - 2(u_t(t), u(t))_{\Omega(t)}\} + \epsilon_0 E(t) \le 0 \tag{3.6}$$

and further

$$3\tilde{E}(t) + \|u_t(t)\|^2_{\Omega(t)} + \|u(t)\|^2_{\Omega(t)} - 2(u_t(t), u(t))_{\Omega(t)}$$

is equivalent to $E(t)$. Thus we conclude that

$$E(t) \le CE(0)e^{-\lambda t}, \quad 0 \le t < \infty \tag{3.7}$$

with some $\lambda > 0$.

Case (2). $\Omega(t)$ is unbounded.

In this case we see that there exists $\delta_0 > 0$ such that if $r^+ < \delta_0$, (3.6) holds and

$$3\tilde{E}(t) + \|u_t(t)\|^2_{\Omega(t)} + \|u(t)\|^2_{\Omega(t)} - 2(u_t(t), u(t))_{\Omega(t)}$$

is equivalent to $E(t) + \|u(t)\|^2_{\Omega(t)}$. Thus integrating (3.6) we have

$$E(t) + \|u(t)\|^2_{\Omega(t)} + \int_0^t E(s)ds \le C(E(0) + \|u_0\|^2_{\Omega(0)}). \tag{3.8}$$

The decay estimations for strong solutions are finished.

For later use we consider the linear equation, instead of (1.1),

$$u_{tt} - \Delta u + u_t = g(x,t) \quad \text{in } Q(0,\infty),$$

where $g \in L^2_{loc}([0,\infty); L^2(\Omega(t)))$. Then, by the above arguments we easily see that

$$E(t) \le CE(0)e^{-\lambda t} + C\int_0^t e^{\lambda(s-t)}\|g(s)\|^2_{\Omega(t)}ds \tag{3.9}$$

and

$$
\begin{aligned}
&E(t)+\|u(t)\|^2_{\Omega(t)}+\int\limits_0^t E(s)ds\\
&\le C(E(0)+\|u_0\|^2_{\Omega(0)})+C\int\limits_0^t(\|g(s)\|^2_{\Omega(s)}+\|g(s)\|_{\Omega(s)}\|u(s)\|_{\Omega(s)})ds
\end{aligned}
\tag{3.10}
$$

for the cases (1) and (2), respectively, where

$$E(t)=\frac{1}{2}(\|u_t(t)\|^2_{\Omega(t)}+\|\nabla u(t)\|^2_{\Omega(t)})$$

in this situation.

4. PROOFS OF THEOREMS

We begin with the existence of finite energy solution. Let

$$(u_0,u_1)\in H^1_0(\Omega(0))\times L^2(\Omega(0)).$$

Then we can take a sequence

$$(u_{0,n},u_{1,n})\in C^2_0(\Omega(0))\times C^1_0(\Omega(0))$$

such that

$$(u_{0,n},u_{1,n})\to(u_0,u_1)\quad\text{in } H^1_0(\Omega(0))\times L^2(\Omega(0)).$$

Then, by Proposition 2.1, the problem admits a unique strong solution $u_n(t)$ of the problem (1.1)–(1.2) with $(u_n(0),u_{n,t}(0))=(u_{0,n},u_{1,n})$ for each n. These solutions satisfy the estimate (3.7) for the case (1) and (3.8) for the case (2), respectively, where we replace (u_0,u_1) by $(u_{0,n},u_{1,n})$. We denote $E(t)$ by $E_n(t)$ if $u(t)$ is replaced by $u_n(t)$. Since both cases (1) and (2) can be treated similarly based on the estimates (3.7), (3.8), (3.9) and (3.10) we consider the case (2) only. Setting

$$(w_{m,n})(t)=u_m(t)-u_n(t)$$

it satisfies

$$w_{m,n,tt}-\Delta w_{m,n}+w_{m,n,t}=g_{m,n}\equiv f(u_n)-f(u_m).$$

We set

$$E_{m,n}(t)=\frac{1}{2}(\|w_{m,n,t}(t)\|^2_{\Omega(t)}+\|\nabla w_{m,n}(t)\|^2_{\Omega(t)})$$

and

$$I^2_{0,m,n}=E_{m,n}(0)+\|w_{m,n}(0)\|^2_{\Omega(0)}\quad\text{and}\quad I^2_{0,n}=E_n(0)+\|u_n(0)\|^2_{\Omega(0)}.$$

Then, applying (3.10) to $w_{m,n}(t)$, we have in particular

$$\begin{aligned} &E_{m,n}(t) + \|w_{m,n}(t)\|^2_{\Omega(t)} \le CI^2_{0,m,n} \\ &+ C\int_0^t \left\{ \int_{\Omega(s)} (1+|u_n|^{2\alpha}+|u_m|^{2\alpha})|w_{m,n}|^2 dx + \|w_{m,n}(s)\|^2_{\Omega(s)} \right\} ds \\ &\le CI^2_{0,m,n} + C\left(1+I^{2\alpha}_{0,m}+I^{2\alpha}_{0,n}\right)\int_0^t \{E_{m,n}(s) + \|u_{m,n}(s)\|^2_{\Omega(s)}\} ds, \end{aligned} \tag{4.1}$$

where we have used the assumption $\alpha \le 2/(N-2)^+$. Applying Gronwall's lemma to (4.1) we obatain

$$E_{m,n}(t) + \|w_{m,n}(t)\|^2_{\Omega(t)} \le CI^2_{0,m,n} e^{\lambda_0 t}, \quad 0 \le t < \infty$$

with a constant $\lambda_0 > 0$ independent of sufficiently large m, n. This inequality means that there exists a function $u(t)$ such that

$$u \in C([0,\infty); H^1_0(\Omega(t))) \quad \text{and} \quad u_t(t) \in C([0,\infty); L^2(\Omega(t))$$

and $(u_n(t), u_{n,t}(t))$ converges to $(u(t), u_t(t))$ in $H^1_0(\Omega(t)) \times L^2(\Omega(t))$ uniformly on any interval $[0,T]$, $T > 0$. It is clear that $u(t)$ is a desired finite energy solution of the problem (1.1)–(1.2). (We extend $u(t)$ and $u_t(t)$ as functions on $\mathbb{R}^N$ for each $t \ge 0$ by setting $u(x,t) = 0$ and $u_t(x,t) = 0$ for $x \notin \Omega(t)$.) This solution, of course, satisfies the estimate (3.10).

It is left to show the uniqueness. We again consider the case (2) only. Let $u(t), v(t)$ be possible two finite energy solutions of the problem (1.1)–(1.2) with the same initial data. We set $w(t) = u(t) - v(t)$. Then $w(t)$ satisfies

$$w_{tt} - \Delta w + w_t = g \equiv f(v) - f(u).$$

We consider the equation

$$W_{tt} - \Delta W + W_t = g \tag{4.2}$$

with

$$W(0) = w(0) = 0, \quad W_t(0) = w_t(0) = 0.$$

Since $g \in C([0,\infty); L^2(\Omega(t))$ (see (4.1)) and (4.2) is a linear equation, this problem admits a unique finite energy solution W with $W \in C([0,\infty); H^1_0(\Omega(t)))$, $W_t \in C([0,\infty); L^2(\Omega(t))$ (cf. [1]), and we see $W(t) = w(t)$. We know that this solution is given by a limit of strong solutions. Indeed, for the existence we take $(W_{0,n}, W_{1,n}) = (0,0)$ and $g_n \in C^1_0((0,\infty); L^2(\Omega(t)))$ such that $g_n \to g \in C_{loc}([0,\infty); L^2(\Omega(t)))$ and consider the equation

$$W_{tt} - \Delta W + W_t = g_n.$$

This problem has a strong solution W_n (cf. Proposition 2.1), and using the estimate (4.1) we see that W_n converges to a finite energy solution $W(t)$.

Therefore we conclude that $W(t) = w(t)$ and the following estimate holds (see (4.1))

$$\begin{aligned} &E_w(t) + \|w(t)\|^2_{\Omega(t)} \\ &\le CI^2_{0,w} + C \sup_{0\le s\le t}\{1 + E_u(s) + \|u(s)\|^2_{\Omega(t)} + E_v(s) + \|v(s)\|^2_{\Omega(t)}\} \\ &\qquad \cdot \int_0^t \{E_w(s) + \|w(s)\|^2_{\Omega(s)}\} ds \qquad (4.3) \\ &\le CI^2_{0,w} + C(T)\int_0^t \{E_w(s) + \|w(s)\|^2_{\Omega(s)}\} ds, \quad 0 \le t \le T, \end{aligned}$$

for any $T > 0$ with some constant $C(T)$ depending on u, v and T, where we set

$$E_u(t) = \frac{1}{2}(\|u_t(t)\|^2_{\Omega(t)} + \|\nabla u(t)\|^2_{\Omega(t)}) \quad \text{and} \quad I_{0,u} = E(0) + \|u(0)\|^2_{\Omega(0)}.$$

Applying Gronwall's Lemma to (4.3) we have

$$E_w(t) + \|w(t)\|^2_{\Omega(t)} \le CI^2_{0,w} e^{C(T)t}, \quad 0 \le t < T.$$

Since $T > 0$ is arbitrary and $I_{0,w} = 0$, we conclude that $w(t) = 0$, that is, $u(t) = v(t)$ for any $t \ge 0$.

Finally, we mention the existence and uniqueness of finite energy solution for the equation (1.1) without the term $\partial u/\partial t$ under Hyp. A and Hyp. B. By Hyp. B, we can extend $\beta(x,t) = -n_t n_x/|n_x|^2$ as C^1 function on $\mathbb{R}^N \times \mathbb{R}^+$, satisfying

$$|\beta| \le \nu(T) < 1, \quad 0 \le t \le T,$$

for any $T > 0$. Therefore $\tilde{E}(t)$ is equivalent to $E(t)$ on each interval $[0, T]$. We may assume further that

$$r^+(t) \equiv \sup_x(|\beta_t| + |\nabla\beta| + |\beta'|) \le C(T) < \infty, \quad 0 \le t \le T,$$

for any $T > 0$. We have, instead of (3.4),

$$\frac{d}{dt}\tilde{E}(t) \le Cr^+(t)E(t),$$

which implies

$$E(t) \le C(T)E(0)e^{\lambda(T)t}, \quad 0 \le t \le T, \qquad (4.4)$$

with some $\lambda(T) > 0$ for any $T > 0$. Further, we see from the relation

$$\|u(t)\|^2_{\Omega(t)} = \|u(0)\|^2_{\Omega(0)} + 2\int_{Q(0,t)} uu_t dV$$

that

$$\|u(t)\|^2_{\Omega(t)} \leq 2\|u(0)\|^2_{\Omega(0)} + C(T)\int_0^T E(s)ds. \tag{4.5}$$

These estimates (4.4) and (4.5) are sufficient for the arguments giving the proofs of existence and uniqueness of finite energy solution given in the above.

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Mitsuhiro Nakao
nakao.mitsuhiro.678@m.kyushu-u.ac.jp

Faculty of Mathematics
Kyushu University
Moto-oka 744, Fukuoka 819-0395, Japan

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