DETERMINATION OF THE TEMPERATURE FIELD IN BURNED AND HEALTHY SKIN TISSUE USING THE BOUNDARY ELEMENT METHOD - PART I

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Abstract. In the paper the burned and healthy layers of skin tissue are considered. The temperature distribution in these layers is described by the system of two Pennes equations. The governing equations are supplemented by the boundary conditions. On the external surface the Robin condition is known. On the surface between burned and healthy skin the ideal contact is considered, while on the internal surface limiting the system the body temperature is taken into account. The problem is solved by means of the boundary element method.

Keywords: *bioheat transfer, Pennes equation, boundary element method*

Introduction

The boundary element method constitutes a very effective tool for numerical simulation of bioheat transfer processes. First of all, it assures the exact approximation of real shapes of the boundaries and also a very good exactness of boundary conditions approximation [1, 2]. These features of the BEM are very essential in the case of temperature determination in the burned and healthy tissue because small differences in temperature in these sub-domains require very accurate methods for predicting the temperature. In this paper two variants of the BEM are taken into account. For burned tissue, the classical boundary element algorithm for the Laplace equation is used, while for healthy tissue, the BEM algorithm for temperature-dependent source function is applied. These algorithms are coupled by the boundary condition on the contact surface between sub-domains. In numerical realization the linear boundary elements and the linear internal cells for burned tissue sub-domain are used. It should be pointed out that the burned sub-domain does not require the interior discretization. In the final part, the example of computations is presented and the conclusions are formulated.

1. Governing equations

The burned and healthy sub-domains, as shown in Figure 1, are considered.

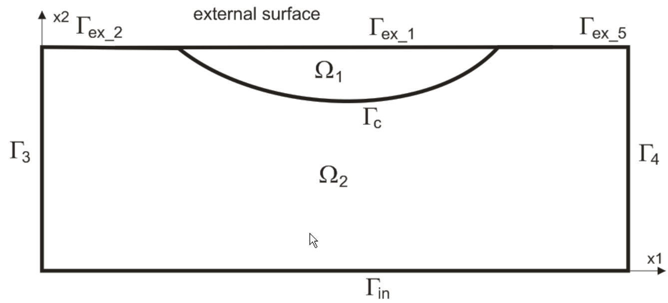

Fig. 1. Domain considered

The Laplace equation describes the steady temperature field in burned tissue

$$x \in \Omega_1 : \quad \lambda_1 \nabla^2 T_1(x) = 0 \tag{1}$$

where $x = (x_1, x_2)$ are the spatial coordinates, λ_1 is the thermal conductivity of burned tissue, $T_1(x)$ denotes the temperature.

The temperature field in healthy tissue is described by the Pennes equation [3]

$$x \in \Omega_2 : \quad \lambda_2 \nabla^2 T_2(x) + G_B c_B \left[T_B - T_2(x) \right] + Q_{met} = 0 \tag{2}$$

where λ_2 is the tissue thermal conductivity, $T_2(x)$ is the tissue temperature, G_B is the blood perfusion rate, c_B is the specific heat of blood, T_B is the arterial blood temperature, Q_{met} is the metabolic heat source.

On the external surface (cf. Fig. 1) the Robin condition is assumed

$$\begin{aligned} x \in \Gamma_{ex\_1} \quad &: \quad -\lambda_1 \frac{\partial T_1(x)}{\partial n} = \alpha \left[T_1(x) - T_a \right] \\ x \in \Gamma_{ex\_2} \cup \Gamma_{ex\_5} &: \quad -\lambda_2 \frac{\partial T_2(x)}{\partial n} = \alpha \left[T_2(x) - T_a \right] \end{aligned} \tag{3}$$

where T_a is the ambient temperature, α is the heat transfer coefficient, $\partial T_e / \partial n$ denotes the normal derivative (e = 1,2) and $n = [\cos\alpha_1, \cos\alpha_2]$ is the normal outward vector.

On the surface between sub-domains the continuity of heat flux and temperature field is taken into account

$$x \in \Gamma_c: \quad \begin{cases} -\lambda_1 \dfrac{\partial T_1(x)}{\partial n} = \lambda_2 \dfrac{\partial T_2(x)}{\partial n} \\ T_1(x) = T_2(x) \end{cases} \tag{4}$$

On the internal surface Γ_{in} the Dirichlet condition is known

$$x \in \Gamma_{in}: \quad T_2(x) = T_b \tag{5}$$

On the remaining boundaries the no-flux condition is accepted.

2. Boundary element method

The problem has been solved by means of the boundary element method [1, 2]. The boundary integral equation corresponding to the equation (1) is the following

$$B(\xi)T_1(\xi) + \int_{\Gamma_\mathrm{I}} q_1(\xi)T_1^*(\xi,x)\mathrm{d}\Gamma_\mathrm{I} = \int_{\Gamma_\mathrm{I}} T_1(\xi)\, q_1^*(\xi,x)\mathrm{d}\Gamma_\mathrm{I} \tag{6}$$

where $\Gamma_\mathrm{I} = \Gamma_{ex\_1} \cup \Gamma_\mathrm{c}$, ξ is the observation point, the coefficient $B(\xi)$ is dependent on the location of source point ξ, $q_1(x) = -\lambda_1 \partial T_1(x)/\partial n$ is the heat flux.

For the problem considered the fundamental solution $T_1^*(\xi, x)$ is the following

$$T_1^*(\xi,x) = \frac{1}{2\pi\lambda_1} \ln \frac{1}{r} \tag{7}$$

where r is the distance between the points ξ and x.

The heat flux resulting from the fundamental solution is defined as

$$q_1^*(\xi, x) = -\lambda_1 \frac{\partial T_1^*(\xi, x)}{\partial n} = \frac{d}{2\pi r^2} \tag{8}$$

where

$$d = (x_1 - \xi_1)\cos\alpha_1 + (x_2 - \xi_2)\cos\alpha_2 \tag{9}$$

The integral equation corresponding to the equation (2) is the following

$$B(\xi)T_2(\xi) + \int_{\Gamma_\mathrm{II}} q_2(x)T_2^*(\xi, x)\mathrm{d}\Gamma_\mathrm{II} = \int_{\Gamma_\mathrm{II}} T_2(x)q_2^*(\xi, x)\mathrm{d}\Gamma_\mathrm{II} + Q\int_{\Omega_2} T_2^*(\xi, x)\mathrm{d}\Omega_2 \tag{10}$$

where $\Gamma_\mathrm{II} = \Gamma_c \cup \Gamma_{ex\_2} \cup \Gamma_3 \cup \Gamma_{in} \cup \Gamma_4 \cup \Gamma_{ex\_5}$ (cf. Fig. 1), and

$$Q = G_B c_B T_B + Q_{met}, \quad q_2 = -\lambda_2 \frac{\partial T_2(x)}{\partial n}, \quad q_2^* = -\lambda_2 \frac{\partial T_2^*(\xi, x)}{\partial n} \tag{11}$$

The fundamental solution and heat flux resulting from the fundamental solution in the case discussed have a form

$$T_2^*(\xi,x)=\frac{1}{2\pi\lambda_2}\mathrm{K}_0\left(\sqrt{\frac{G_B c_B}{\lambda_2}}r\right)$$
$$q_2^*(\xi,x)=\frac{d}{2\pi r}\sqrt{\frac{G_B c_B}{\lambda_2}}\mathrm{K}_1\left(\sqrt{\frac{G_B c_B}{\lambda_2}}r\right) \tag{12}$$

where K$_0$ (·), K$_1$ (·) are the modified Bessel functions of second kind, the zero order and the first order, respectively [1, 2].

3. Numerical realization

To solve the equations (6) and (10) the boundary Γ is divided into N elements Γ_j, j = 1,2,..., N and the interior Ω_2 is divided into L internal cells as shown in Figure 2. Next, the integrals in the equations (6) and (10) are replaced by the sums of integrals over these elements, this means

$$B(\xi)T_1(\xi)+\sum_{j=1}^{N_1}\int_{\Gamma_j} q_1(x)T_1^*(\xi,x)\mathrm{d}\Gamma_j=\sum_{j=1}^{N_1}\int_{\Gamma_j} T_1(x)q_1^*(\xi,x)\mathrm{d}\Gamma_j \tag{13}$$

$$B(\xi)T_2(\xi)+\sum_{j=N_1+1}^{N}\int_{\Gamma_j} q_2(x)T_2^*(\xi,x)\mathrm{d}\Gamma_j=$$
$$\sum_{j=N_1+1}^{N}\int_{\Gamma_j} T_2(x)q_2^*(\xi,x)\mathrm{d}\Gamma_j+Q\sum_{l=1}^{L}\int_{\Omega_l} T_2^*(\xi,x)\mathrm{d}\Omega_l \tag{14}$$

where N_1 is the number of elements on the boundary limiting domain Ω_1. In the paper the linear boundary elements are applied. Finally one obtains the following systems of algebraic equations:

- for the burned region

$$\sum_{k=1}^{K_1} G_{ik}^1 q_k^1=\sum_{k=1}^{K_1} H_{ik}^1 T_k^1,\quad i=1,2,\ ...,K_1 \tag{15}$$

- for the healthy tissue domain

$$\sum_{k=K_1+1}^{K} G_{ik}^2 q_k^2=\sum_{k=K_1+1}^{K} H_{ik}^2 T_k^2+P_i,\quad i=K_1+1,K_1+2,\ ...,K \tag{16}$$

where K_1, K–K_1 are the number of boundary nodes located at the boundary limiting sub-domains Ω_1 and Ω_2, respectively.

The system of equations (15) and (16) can be written in the matrix convention

$$\mathbf{G}_1\mathbf{q}_1 = \mathbf{H}_1\mathbf{T}_1 \tag{17}$$

and

$$\mathbf{G}_2\mathbf{q}_2 = \mathbf{H}_2\mathbf{T}_2 + \mathbf{P} \tag{18}$$

Details of the calculation of matrix elements appearing in equations (17) and (18) are described in [3].

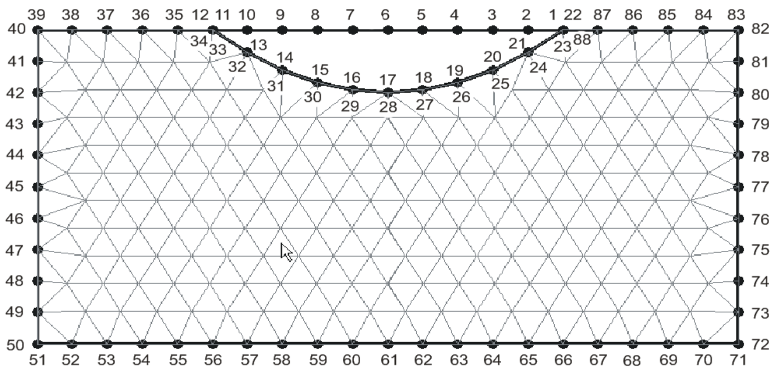

Fig. 2. Discretization of boundaries and interior Ω_2

The equations (17) and (18) can be written in the following form

- for the burned region

$$\begin{bmatrix} \mathbf{G}_1^{ex\_1} & \mathbf{G}_{c1} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^{ex\_1} \\ \mathbf{q}_{c1} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1^{ex\_1} & \mathbf{H}_{c1} \end{bmatrix} \begin{bmatrix} \mathbf{T}_1^{ex\_1} \\ \mathbf{T}_{c1} \end{bmatrix} \tag{19}$$

- for the healthy tissue domain

$$\begin{bmatrix} \mathbf{G}_{c2} & \mathbf{G}_2^{ex\_2} & \mathbf{G}_2^3 & \mathbf{G}_2^{in} & \mathbf{G}_2^4 & \mathbf{G}_2^{ex\_5} \end{bmatrix} \begin{bmatrix} \mathbf{q}_{c2} \\ \mathbf{q}_2^{ex\_2} \\ \mathbf{q}_2^3 \\ \mathbf{q}_2^{in} \\ \mathbf{q}_2^4 \\ \mathbf{q}_2^{ex\_5} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{c2} & \mathbf{H}_2^{ex\_2} & \mathbf{H}_2^3 & \mathbf{H}_2^{in} & \mathbf{H}_2^4 & \mathbf{H}_2^{ex\_5} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{c2} \\ \mathbf{T}_2^{ex\_2} \\ \mathbf{T}_2^3 \\ \mathbf{T}_2^{in} \\ \mathbf{T}_2^4 \\ \mathbf{T}_2^{ex\_5} \end{bmatrix} + \mathbf{P} \tag{20}$$

where:

- $\mathbf{T}_1^{ex\_1}$, $\mathbf{q}_1^{ex\_1}$ are the vectors of functions T and q at the boundary Γ_{ex_1} of domain Ω_1,
- $\mathbf{T}_{c1}, \mathbf{T}_{c2}, \mathbf{q}_{c1}, \mathbf{q}_{c2}$ are the vectors of functions T and q on the contact surface Γ_c between sub-domains Ω_1 and Ω_2,

- $\mathbf{T}_2^{ex\_2}, \mathbf{T}_2^3, \mathbf{T}_2^{in}, \mathbf{T}_2^4, \mathbf{T}_2^{ex\_5}, \mathbf{q}_2^{ex\_2}, \mathbf{q}_2^3, \mathbf{q}_2^{in}, \mathbf{q}_2^4, \mathbf{q}_2^{ex\_5}$ are the vectors of functions T and q at the boundary $\Gamma_{ex\_2} \cup \Gamma_3 \cup \Gamma_{in} \cup \Gamma_4 \cup \Gamma_{ex\_5}$ of domain Ω_2.

The condition (4) written in the form

$$\begin{cases} \mathbf{q}_{c1} = -\mathbf{q}_{c2} = \mathbf{q} \\ \mathbf{T}_{c1} = \mathbf{T}_{c2} = \mathbf{T} \end{cases} \tag{21}$$

should be introduced to equations (19), (20).
Next, coupling these systems of equations and taking into account the remaining boundary conditions, one has

$$\begin{bmatrix} \alpha\mathbf{G}_1^{ex\_1} - \mathbf{H}_1^{ex\_1} & -\mathbf{H}_{c1} & \mathbf{G}_{c1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{H}_{c2} & -\mathbf{G}_{c1} & \alpha\mathbf{G}_2^{ex\_2} - \mathbf{H}_2^{ex\_2} & -\mathbf{H}_2^3 & \mathbf{G}_2^{in} & -\mathbf{H}_2^4 & \alpha\mathbf{G}_2^{ex\_5} - \mathbf{H}_2^{ex\_5} \end{bmatrix} \begin{bmatrix} \mathbf{T}_1^{ex\_1} \\ \mathbf{T} \\ \mathbf{q} \\ \mathbf{T}_2^{ex\_2} \\ \mathbf{T}_2^3 \\ \mathbf{q}_2^{in} \\ \mathbf{T}_2^4 \\ \mathbf{T}_2^{ex\_5} \end{bmatrix} =$$

$$= \begin{bmatrix} \alpha\mathbf{G}_1^{ex\_1} T_a \\ \alpha\mathbf{G}_2^{ex\_2} T_a + \alpha\mathbf{G}_2^{ex\_5} T_a + \mathbf{H}_2^{in} T_b + \mathbf{P} \end{bmatrix} \tag{22}$$

Finally, the system of equations (22) can be written in the form

$$\mathbf{AX} = \mathbf{B} \tag{23}$$

where **A** is the main matrix, **X** is the unknown vector and **B** is the free terms vector. The system of equations (23) enables the determination of the missing boundary values. Knowledge of nodal boundary temperatures and heat fluxes allows to calculate the internal temperatures at the optional points selected from the burned and healthy tissue sub-domains [1, 2].

4. Results of computations

The rectangular domain of dimensions $2L \times L$ ($L = 0.02$ m) shown in Figure 1 has been considered. The shape of internal surface Γ_c is defined by the parabola with vertex (0.02, 0.016). The following input data have been assumed: thermal conductivity of burned tissue $\lambda_1 = 0.1$ W/(mK), thermal conductivity of healthy tissue $\lambda_2 = 0.2$ W/(mK) [4], blood perfusion rate $G_B = 0.5$ kg/(m^{3}s), specific heat of blood $c_B = 4200$ J/(kgK), arterial blood temperature $T_B = 37$°C, metabolic heat source Q_{met} = 200 W/m^3, heat transfer coefficient α = 10 W/(m^{2}K), ambient temperature T_a = 20°C, boundary temperature T_b = 37°C (cf. condition (5)).

In Figure 3 the temperature distribution in the domain considered is presented, while Figure 4 illustrates the course of temperature on the external surface of skin tissue.

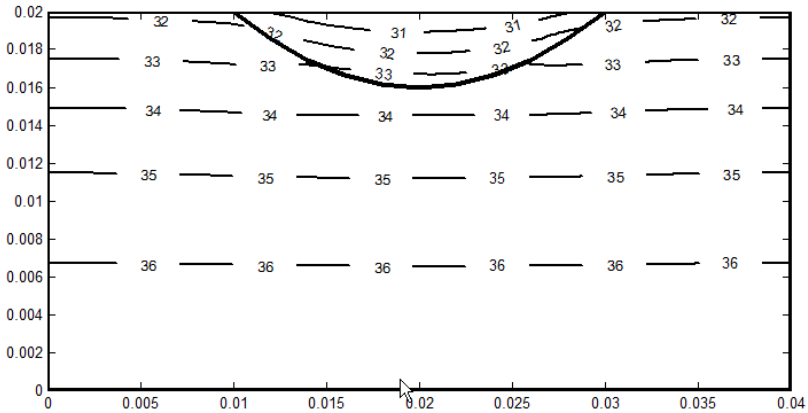

Fig. 3. Temperature distribution

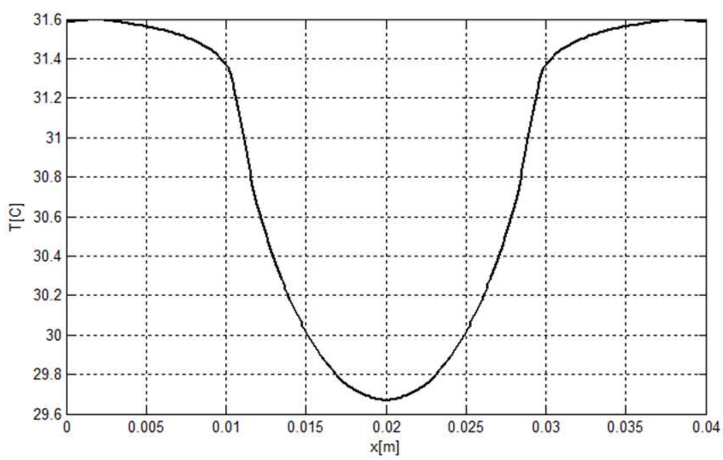

Fig. 4. Temperature distribution on the external surface

Conclusions

The heterogeneous domain being the composition of burned and healthy layers of skin tissue has been considered. The temperature distribution has been described by the system of two Pennes equations with different thermophysical parameters. The problem has been solved by means of the boundary element method. The algorithm proposed allows one to determine the temperature field in burned and healthy tissue and can be used as part of a computer system that enables one to predict the shape of the burn wound [5].

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