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Stability Tests of Positive Fractional Continuous-time Linear Systems with Delays

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ABSTRACT: Necessary and sufficient conditions for the asymptotic stability of positive fractional continuous-time linear systems with many delays are established. It is shown that:
1) the asymptotic stability of the positive fractional system is independent of their delays,
2) the checking of the asymptotic stability of the positive fractional systems with delays can be reduced to checking of the asymptotic stability of positive standard linear systems without delays.

1 INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial states remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in monographs [10, 16]. The problems of stability and control of system with delays have been considered in [3, 11, 12, 13, 25]. The stability and the robust stability of positive discrete-time linear systems without delays and with delays have been investigated in [1-10, 14-26].

The stability of positive continuous-time linear systems with delays have been addressed in [17].and the stability of positive fractional systems with one delay in [22].

In this paper new necessary and sufficient conditions for asymptotic stability of positive fractional continuous-time linear systems with delays will be presented. It will be shown that the asymptotic stability of positive fractional continuous-time linear systems is independent of their delays and checking of asymptotic stability of the system with delays can be reduced to checking of the stability of positive systems without delays.

The paper is organized as follows. In section 2 the fractional continuous-time linear systems and their solutions are recalled. Necessary and sufficient conditions for the positivity of this class of fractional systems with delays are given in section 3. The main result of the paper is presented in section 4, where the necessary and sufficient conditions for the asymptotic stability of the positive fractional linear systems with delays are established. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, Z_+ - the set of nonnegative integers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n. -the $n\times n$ identity matrix. The strictly positive vector x with all positive components will be denoted by $x > 0$.

2 FRACTIONAL CONTINUOUS-TIME LINEAR SYSTEMS AND THEIR SOLUTIONS

In this paper the Caputo definition will be used

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{2.1}$$
$$n-1 < \alpha \le n \in N = \{1,2,...\}$$

where $\alpha \in \Re$ is the order of fractional derivative, $f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the gamma function.

Consider the continuous-time fractional linear system with delays

$$\frac{d^\alpha}{dt^\alpha} x(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t-d_k) + Bu(t), \tag{2.2}$$
$$0 < \alpha \le 1$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input vector and $A_k \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, d_k is a delay $k = 1,...,q$.

The initial conditions for (2.2) have the form

$$x(t) = x_0(t) \text{ for } t \in [-d,0], \ d = \max_k d_k \tag{2.3}$$

The solution to the equation (2.2) with (2.3) can be found by the use of the step method [17].

For $q = 1$ and $0 \le t \le d$ the solution has the form [17]

$$x(t) = \Phi_0(t) x_0(0) + \int_0^t \Phi(t-\tau)[A_1 x_0(\tau - d) + Bu(\tau)] d\tau \tag{2.4}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A_0^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad \Phi(t) = \sum_{k=0}^{\infty} \frac{A_0^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{2.5}$$

Knowing the state vector $x(t)$ for $0 \le t \le d$ in a similar way we can find the state vector for $d \le t \le 2d$ and next for $2d \le t \le 3d$, …

3 POSITIVE FRACTIONAL CONTINUOUS-TIME SYSTEMS WITH DELAYS

Definition 3.1. The fractional continuous-time linear systems with delays (2.2) is called positive if $x(t) \in \Re_+^n$, $t \ge 0$ for any initial conditions

$$x_0(t) \in \Re_+^n \text{ for } t \in [-d,0] \tag{3.1}$$

and all input vectors $u(t) \in \Re_+^n$, $t \ge 0$.

A real matrix $A \in \Re^{n \times n}$ is called the Metzler matrix if its off-diagonal entries are nonnegative.

Let M_n be the set of $n \times n$ Metzler matrices.

Theorem 3.1. The fractional continuous-time linear systems (2.2) for $0 < \alpha < 1$ is positive if and only if

$$A_0 \in M_n, \ A_k \in \Re_+^{n \times n}, \ k = 1,...,q; \ B \in \Re_+^{n \times m}. \tag{3.2}$$

Proof. It is well-known [17] that $\Phi_0(\tau) \in \Re_+^{n \times n}$ and $\Phi(\tau) \in \Re_+^{n \times n}$ if and only if $A_0 \in M_n$. From (2.4) it follows that $x(t) \in \Re_+^n$, $t \ge 0$ if $A_k \in \Re_+^{n \times n}$, $k = 1,...,q$; $B \in \Re_+^{n \times m}$ and $u(t) \in \Re_+^n$ for $t \ge 0$. The necessity can be shown in similar way as in [17].□

4 ASYMPTOTIC STABILITY OF THE POSITIVE FRACTIONAL SYSTEMS

Consider the autonomous fractional positive linear system with delays

$$\frac{d^\alpha}{dt^\alpha} x(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t-d_k), \quad 0 < \alpha \le 1 \tag{4.1}$$

where $A_0 \in M_n$, $A_k \in \Re_+^{n \times n}$ and $d_k > 0$, $k = 1,...,q$.

Definition 4.1. The positive system (4.1) is called asymptotically stable if

$$\lim_{t \to \infty} x(t) = 0 \text{ for any initial conditions (3.1)} \tag{4.2}$$

Definition 4.2. A vector $x_e \in \Re_+^n$ is called the equilibrium point of the positive asymptotically stable system (2.2) for $Bu(t) = 1_n = [1 \ \ldots \ 1]^T \in \Re_+^n$ if the following condition is satisfied

$$0 = Ax_e + 1_n \text{ and } A = \sum_{k=0}^{q} A_k \tag{4.3}$$

From (4.3) we have

$$x_e = (-A)^{-1} 1_n \tag{4.4}$$

since for asymptotically stable system (2.2) the matrix A is invertible and the inverse matrix $(-A)^{-1} \in \Re_+^{n \times n}$ [16].

Theorem 4.1. The positive fractional system with delay (4.1) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \Re_+^n$ such that

$$A\lambda < 0 \tag{4.5}$$

Proof. First we shall show that if the positive system (4.1) is asymptotically stable then there exists a strictly positive vector $\lambda > 0$ satisfying (4.5). If the positive system (4.1) is a asymptotically stable then the equilibrium point (4.4) is a strictly positive vector and we can choose $\lambda = x_e = (-A)^{-1}1_n$. This vector satisfies the condition (4.5) since

$$A\lambda = -AA^{-1}1_n = -1_n \tag{4.6}$$

Now we shall show that the positive system (4.1) is asymptotically stable if there exists strictly positive vector λ satisfying (4.5). It is well-known that the positive system (4.1) is asymptotically stable if and only if the corresponding transpose system

$$\frac{d^\alpha}{dt^\alpha}x(t) = A_0^T x(t) + \sum_{k=1}^{q} A_k^T x(t-d_k)$$

(T denotes the transpose) (4.7)

is asymptotically stable. As candidate for a Lyapunov function for the positive system (4.7) we chose the function

$$V[x(t)] = x^T(t)\lambda + \sum_{k=1}^{q} \int_{t-d_k}^{t} x^T(\tau)d\tau^\alpha \ A_k\lambda \tag{4.8}$$

which is positive for any nonzero $x(t) \in \Re_+^n,\ t \ge 0$. Using (4.8) and (4.7) we obtain

$$\begin{aligned}\frac{d^\alpha V[x(t)]}{dt^\alpha} &= \frac{d^\alpha x^T(t)}{dt^\alpha}\lambda + \frac{d^\alpha}{dt^\alpha}\left[\int_{t-d}^{t} x^T(\tau)d\tau^\alpha\right]A_1\lambda \\ &= [A_0^T x(t) + \sum_{k=1}^{q} A_k^T x(t-d_k)]^T\lambda + \\ &+ \sum_{k=1}^{q}[x^T(t)A_k\lambda - x^T(t-d_k)A_k\lambda] = x^T(t)A\lambda\end{aligned} \tag{4.9}$$

If (4.5) holds then from (4.9) we have $\frac{d^\alpha V[x(t)]}{dt^\alpha} < 0$ and the system (4.1) is asymptotically stable. □

Theorem 4.2. The positive fractional system with delay (4.1) is asymptotically stable if and only one of the following equivalent conditions is satisfied:

- The positive system without delay

  $$\dot{x}(t) = Ax(t), \ _1 \tag{4.10}$$

  is asymptotically stable,
- The matrix A is asymptotically stable Metzler matrix.

Proof. In [20] it was shown that the positive system (4.10) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \Re_+^n$ such that (4.5) holds. Hence by Theorem 2 the positive system (4.1) is asymptotically stable if and only if the positive system (4.10) is asymptotically stable. It is well-known [16] that the positive system (4.10) (and (4.1)) is asymptotically stable if and only if the matrix A is asymptotically stable Metzler matrix. □

To check the asymptotic stability of the Metzler matrix A the following theorem is recommended [23, 24].

Theorem 4.3. The matrix $A \in \Re^{n\times n}$ is a asymptotically stable Metzler matrix if and only if one of the following equivalent conditions is satisfied:

- all coefficients $a_0,\dots,a_{n-1}$ of the characteristic polynomial

  $$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 \tag{4.11}$$

  are positive, i.e. $a_i \ge 0$, $i = 0,1,\dots,n-1$,
- the diagonal entries of the matrices

  $$A_{n-k}^{(k)} \text{ for } k = 1,\dots,n-1 \tag{4.12}$$

  are negative, where

$$A_n^{(0)} = A = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n,1}^{(0)} & \dots & a_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} A_{n-1}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & a_{n,n}^{(0)} \end{bmatrix},$$

$$A_{n-1}^{(0)} = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n-1}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n-1,1}^{(0)} & \dots & a_{n-1,n-1}^{(0)} \end{bmatrix}$$

$$b_{n-1}^{(0)} = \begin{bmatrix} a_{1,n}^{(0)} \\ \vdots \\ a_{n-1,n}^{(0)} \end{bmatrix},\quad c_{n-1}^{(0)} = [a_{n,1}^{(0)} \ \dots \ a_{n,n-1}^{(0)}]$$

$$A_{n-k}^{(k)} = A_{n-k}^{(n-1)} - \frac{b_{n-k}^{(k-1)}c_{n-k}^{(k-1)}}{a_{n-k+1,n-k+1}^{(k-1)}} = \begin{bmatrix} a_{11}^{(k)} & \dots & a_{1,n-k}^{(k)} \\ \vdots & \dots & \vdots \\ a_{n-k,1}^{(k)} & \dots & a_{n-k,n-k}^{(k)} \end{bmatrix} = \begin{bmatrix} A_{n-k-1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & a_{n-k,n-k}^{(k)} \end{bmatrix},$$

$$b_{n-k-1}^{(k)} = \begin{bmatrix} a_{1,n-k}^{(k)} \\ \vdots \\ a_{n-k-1,n-k}^{(k)} \end{bmatrix},$$

$$c_{n-k-1}^{(k)} = [a_{n-k,1}^{(k)} \ \dots \ a_{n-k,n-k-1}^{(k)}] \tag{4.13}$$

for $k = 0,1,\dots,n-1$.

– the diagonal entries $\tilde{a}_{kk}$ of the lower triangular matrix

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & 0 & \dots & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \tilde{a}_{n,2} & \dots & \tilde{a}_{n,n} \end{bmatrix}$$

are negative, i.e. $\tilde{a}_{kk} < 0$ for $k = 1,...,n$; where $\tilde{A}$ is obtained from A by elementary column operations [23].

From Theorem 4.2 we have the following important corollary.

Corollary 4.1. The asymptotic stability of the positive fractional linear systems (4.1) is independent of its delays.

Theorem 4.4. The positive fractional linear system (4.1) is unstable if at least one diagonal entry of the Metzler matrix A is nonnegative.

The proof follows immediately from Theorem 4.2 and Theorem in [16].

Example 4.1. Consider the positive fractional system (4.1) with the matrices

$$A_0 = \begin{bmatrix} a & 1 \\ 0.5 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.05 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.05 & 0.2 \end{bmatrix} \tag{4.14}$$

and arbitrary delay $d_k > 0$, $k = 1,2$. Find the value of the parameter a for which the system is asymptotically stable.

By Theorem 4.4 the positive fractional system (4.1) with (4.14) is unstable if the diagonal entry (1,1) of the Metzler matrix

$$A = A_0 + A_1 + A_2 = \begin{bmatrix} a+0.4 & 1.2 \\ 0.6 & -1.5 \end{bmatrix} \tag{4.15}$$

is nonnegative i.e. $a + 0.4 \geq 0$. Using the condition i) of Theorem 4.2 we obtain

$$\det[Is - A] = \begin{vmatrix} s-a-0.4 & -1.2 \\ -0.6 & s+1.5 \end{vmatrix} = s^2 + (1.1-a)s - (1.5a + 1.32) \tag{4.16}$$

and the positive fractional system is asymptotically stable if and only if the coefficients of the polynomial (4.16) are positive

$$1.1 - a > 0 \quad \text{and} \quad 1.5a + 1.32 < 0 \tag{4.17}$$

Therefore, the positive fractional system (4.1) with (4.14) is asymptotically stable for arbitrary delay $d > 0$ if $a < -\frac{1.32}{1.5} = -0.88$.

The same result can be obtained by the use of the condition ii) and iii) of Theorem 4.3.

5 CONCLUDING REMARKS

Necessary and sufficient conditions for the asymptotic stability of continuous-time linear systems with delays have been established (Theorem 4.1 and 4.2). It has been shown that: 1) The asymptotic stability of the positive fractional system is independent of their delays, 2) The checking of the asymptotic stability of the positive fractional systems with delays can be reduced to checking of the asymptotic stability of positive standard linear systems without delays. The considerations can be also extended for fractional positive 2D continuous-discrete linear systems with delays.

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REFERENCES

[1] Busłowicz M., Robust stability of positive discrete-time linear systems with multiple delays with unity rank uncertainty structure or non-negative perturbation matrices, Bull. Pol. Acad. Techn. Sci. Vol. 55, No. 1, 2007, 347-350.

[2] Busłowicz M., Simple stability conditions for linear positive discrete-time systems with delays, Bull. Pol. Acad. Techn. Sci. Vol. 50, No. 4, 2008.

[3] Busłowicz M., Robust stability of dynamical linear stationary systems with delays, Publishing Department of Technical University of Białystok, Warszawa-Białystok 2000, (in Polish).

[4] Busłowicz M., Robust stability of scalar positive discrete-time linear systems with delays, Proc. Int. Conf. on Power Electronics and Intelligent Control, Warszawa 2005, Paper 163 (on CD-ROM).

[5] Busłowicz M., Stability of positive singular discrete-time system with unit delay with canonical forms of state matrices, Proc. 12th IEEE int. Conf. on Methods and Models in Automation and Robotics, Międzyzdroje 2006, pp. 215-218.

[6] Bysłowicz M., Robust stability of positive discrete-time linear systems with multiple delays with linear unit rank uncertainty structure or non-negative perturbation matrices, Bull. Of Pol. Acad. of Sci., Tech. Sci., Vol. 52, No. 2, 2004, pp. 99-102.

[7] Busłowicz M. and Kaczorek T., Robust stability of positive discrete-time interval systems with time-delays, Bull. of Pol. Acad. of Sci., Tech. Sci., Vol. 55, No. 1, 2007, pp. 1-5.

[8] Busłowicz M. and Kaczorek T., Stability and robust stability of positive discrete-time systems with pure delays, Proc. of the 10th IEEE Int. Conf. on Methods and

Models in Automation and Robotics, Międzyzdroje 2004, Vol. 1, pp.105-108.
[9] Busłowicz M. and Kaczorek T., Robust stability of positive discrete-time systems with pure delays with linear unit rank uncertainty structure, Proc. of the 11th IEEE Int. Conf. on Methods and Models in Automation and Robotics, Międzyzdroje 2005, Paper 0169 (on CD-ROM).
[10] Farina L., Rinaldi S., Positive linear systems; Theory and applications, J. Wiley, New York, 2000.
[11] Górecki H., Analysis and synthesis of control systems with delays, WNT, Warszawa (1971), (in Polish).
[12] Górecki H., Fuksa S., Grabowski P. and Korytowski A., Analysis and synthesis of time delay systems, PWN-J. Willey, Warszawa – Chichester (1989).
[13] Górecki H., Korytowski A., Advances in optimizations and stability analysis of dynamical systems, Publishing Department of University Mining and Metallurgy, Kraków (1993).
[14] Hinrichsen D., Hgoc P. H. A. and Son N.K., Stability radii of positive higher order difference systems, System & Control Letters, Vol. 49, 2003, pp.377-388.
[15] Hmamed A., Benzaouia A., Ait Rami M. and Tadeo F. Positive stabilization of discrete-time systems with unknown delays and bounded control, Proc. European Control Conference, Kos, Greece, Jul 2007, 5616-5622 (paper ThD07.3).
[16] Kaczorek T., Positive 1D and 2D Systems, Springer-Verlag, London 2002.
[17] Kaczorek T., "Stability of positive continuous-time linear systems with delays", Bul. Pol. Acad. Sci. Techn. vol. 57. no. 4, 2009, 395-398.
[18] Kaczorek T., Stability of positive discrete-time systems with time-delays, Proc. 8th World Multiconference on Systemics, Cybernetics and Informatics, Orlando Florida USA, July 2004, pp.321-324.
[19] Kaczorek T., Choice of the forms of Lyapunov functions for positive 2D Roesser model, Int. J. Applied Math. And Comp. Sci. Vol. 17 (2007), No. 4, pp.471-475.
[20] Kaczorek T., Asymptotic stability of positive 1D and 2D linear systems, Recent Advances in Control and Automation, Academ. Publ. House EXIT 2008, pp.41-52.
[21] Kaczorek T., Practical stability of positive fractional discrete-time linear systems, Bull. Pol. Acad. Techn. Sci. Vol. 56, No. 4.
[22] Kaczorek T., Stability tests of positive fractional continuous-time linear systems with delays, Proc. 10th Inter. Conf. ICANNGA 2011, Springer, Ljubljana, Slovenia, April 2011, 305-311.
[23] Kaczorek T., New stability tests of positive standard and fractional systems, Circuits and Systems 2011 (in press).
[24] Narendra K.S. and Shorten R., "Hurwitz stability of Metzler matrices", IEEE Trans. Autom. Contr. vol. 55, no. 6 June 2010, 1484-1487.
[25] Niculescu S.-I., Delay effects on stability. A robust control approach, Springer-Verlag, London 2001.
[26] Twardy M., On the alternative stability criteria for positive systems, Bull. Pol. Acad. Techn. Sci. Vol. 55, 2007, No. 4, pp.385-303.