Vol. 32, No. 4 (2022) | DOI: 10.37190/ord220406

OPEN ACCESS

Operations Research and Decisions

www.ord.pwr.edu.pl

OPERATIONS RESEARCH AND DECISIONS QUARTERLY

Optimality conditions for preinvex functions using symmetric derivative

Sachin Rastogi*[1] Akhlad Iqbal[2] Sanjeev Rajan[1]

[1]*Department of Mathematics, Hindu College, M.J.P. Rohilkhand University, Bareilly-243003, UP, India*

[2]*Department of Mathematics, Aligarh Muslim University, Aligarh-202002, UP, India*

**Corresponding author: sachin.sachras@gmail.com*

Abstract

As a generalization of convex functions and derivatives, in this paper, the authors study the concept of a symmetric derivative for preinvex functions. Using symmetrical differentiation, they discuss an important characterization for preinvex functions and define symmetrically pseudo-invex and symmetrically quasi-invex functions. They also generalize the first derivative theorem for symmetrically differentiable functions and establish some relationships between symmetrically pseudo-invex and symmetrically quasi-invex functions. They also discuss the Fritz John type optimality conditions for preinvex, symmetrically pseudo-invex and symmetrically quasi-invex functions using symmetrical differentiability.

Keywords: *invex sets, preinvex functions, symmetric derivative, Fritz John optimality conditions*

1. Introduction

Convex programming problem is one of the major parts of non-linear analysis. But in many problems, applying only convexity is not sufficient to provide a satisfactory solution to a problem. Hence it is necessary to generalize the concept of convexity notion. Weir and Mond [22] and Weir and Jeyakumar [21] introduced an important generalization of convex functions named preinvex functions. An important generalization of convexity named invexity was introduced by Hanson [6] and Yang et al. [23]. A remarkable work on invexity was done by Mishra. et al. [11] and others [12–16, 18, 19]. However, Quyon [7] studied necessary and sufficient Karush–Kuhn–Tucker (KKT) optimality conditions for non-convex optimization. The second-order KKT optimality conditions in non-smooth settings have been discussed by Ivanov [8].

In 1971, Minch [10] introduced the notion of symmetric derivative for convex programming and derived the KKT type optimality conditions for convex functions. But after that, not much literature

Received 19 April 2022, accepted 12 December 2022, published online 8 February 2023
ISSN 2391-6060 (Online)/

The costs of publishing this issue have been cofinansed by the program *Development of Academic Journals* of the Polish Ministry of Education and Science under agreement No. RCN/SP/0241/2021/1

could be found in that direction. In 2019, Guo et al. [4] generalized the idea of Minch [10] to gH-symmetrical derivative for interval-valued optimization problems. For more work on gH-symmetrical differentiation; see [5] [3]. Motivated by the above work, we extend the idea of symmetric derivatives to find the optimality conditions for symmetrically differentiable preinvex functions. For more details on symmetric differentiation, cf. Aull [1] and Larson [9].

This paper is divided into five sections as follows. In Section 2, some preliminaries about symmetrical differentiation and invexity have been described. In Section 3, we discuss a characterization theorem for symmetrically differentiable preinvex functions and introduce symmetrically pseudoinvex (s-pseudoinvex) and symmetrically quasiinvex (s-quasiinvex) functions. We present also a condition under which s-quasiinvex function is s-pseudoinvex function. Section 4 is devoted to the derivation of the Fritz John type optimality conditions for preinvex and s-pseudoinvex functions using symmetric differentiation. We conclude this paper in Section 5.

2. Preliminaries

The concept of convexity has been generalized in different directions. Among them, a significant generalization is invex sets and preinvex functions, introduced by Hanson [6], Weir and Mond [22] and Weir and Jeyakumar [21].

Definition 1 ([6]). A set $M \subseteq \mathbb{R}^m$ is said to be invex if there exists a vector function $\zeta : M \times M \to \mathbb{R}^m$ such that

$$\forall v, w \in M \text{ and } t \in [0,1] \implies w + t\zeta(v,w) \in M$$

Definition 2 ([21]). A function ϕ: $M \subseteq \mathbb{R}^m \to \mathbb{R}$ is preinvex with respect to ζ: $M \times M \to \mathbb{R}^m$, if

$$\phi(w + t\zeta(v,w)) \leq t\phi(v) + (1-t)\phi(w)$$

for all $v, w \in M$ and $t \in [0,\ 1]$.

If $\zeta(v,\ w) = v - w$, then the invex set reduces to the convex set and preinvex function reduces to the convex function. For every $v, w \in M$, the ζ-path P_{wy} joining w and $y = w + \zeta(v,w)$ is defined as follows

$$P_{wy} := \{z : z = w + t\zeta(v,w) : t \in [0,1]\}$$

Now we recall the definition and some properties of symmetrically differentiable functions defined by Thomson [20]. Throughout the paper, M is considered an open subset of $\mathbb{R}^m$ unless otherwise it is mentioned.

Definition 3 ([20]). Let $\phi : S \subseteq \mathbb{R} \to \mathbb{R}$ be a real-valued function. ϕ is said to be symmetrically differentiable (SD) at $v_0 \in M$, if there exists a real number A such that

$$\lim_{h \to 0} \frac{\phi(v_0 + h) - \phi(v_0 - h)}{2h} = A = \phi^s(v_0)$$

This concept was extended for the functions for several variables by Minch [10] as follows:

Definition 4 ([10]). Let $v \in M$. If there exists a linear operator $\phi^s : \mathbb{R}^m \to \mathbb{R}$ such that for sufficiently small $h \in \mathbb{R}^m$,

$$\phi(v+h) - \phi(v-h) = 2\phi^s(v)h + \alpha(v,h)\|h\|$$

where $\alpha(v,h) \in \mathbb{R}$ and $\alpha(v,h) \to 0$ as $\|h\| \to 0$, then ϕ is said to be SD at v. $\phi^s(v)$ is called the symmetric derivative of ϕ at v.

Definition 5 ([10]). Let ϕ be SD at $v \in M$. The symmetric gradient of ϕ at v is that vector $v \in \mathbb{R}^m$, such that

$$v^T h = \phi^s(v)h \quad \forall h \in \mathbb{R}^m$$

Henceforth, $\phi^s(v)$ will denote the symmetric gradient of ϕ at v.

Next, Minch [10] proved that every differentiable function is SD but the converse is not true.

Theorem 1 ([10]). If ϕ is differentiable at v_0, then it is also SD at v_0 and their values are equal.

It is an example that defines differentiability the class of convex and preinvex functions as well as ordinary differentiability and SD, respectively.

Example 1. Let $\phi(v) = -|v|$, then ϕ is preinvex with respect to ζ given by

$$\zeta(v,w) = \begin{cases} v - w & \text{if } v \leq 0, \quad w \leq 0 \\ v - w & \text{if } v \geq 0, \quad w \geq 0 \\ v - w & \text{otherwise} \end{cases}$$

and it is also SD at v_0 but it is neither convex nor ordinary differentiable at $v_0 = 0$.

Theorem 2 ([10]). Suppose $v, w \in M$ be fixed such that $v \neq w$. Let $N = \{t \in \mathbb{R} : w + t(v-w) \in$M$\}$. If ϕ is SD at $v_0 = w + t_0(v-w) \in M$, then the function ψ defined on N, where

$$\psi(t) = \phi(w + t(v-w))$$

is SD at t_0 and

$$\psi^s(t_0) = \phi^s(v_0)(v-w)$$

Another important characterization theorem of SD convex function proved by Minch [10] is:

Theorem 3 ([10]). Suppose ϕ is SD on M. Then ϕ is convex on M if and only if

$$\phi(v) - \phi(w) \geq \phi^s(w)^T(v-w) \forall v, \quad w \in M$$

The following Condition C was given by Mohan and Neogy [17].

Condition C. Let $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, we say that the function ζ satisfies Condition C if for any $v, w \in M$ and $t \in [0,1]$.

$$\zeta(w, w + t\zeta(v,w)) = -t\zeta(v,w), \quad \zeta(v, w + t\zeta(v,w)) = (1-t)\zeta(v,w)$$

Another important consequence of Condition C is

$$\zeta(w+t_2\zeta(v,w), w+t_1\zeta(v,w)) = (t_2-t_1)\zeta(v,w), \forall v, \quad w \in M$$

and

$$t_1, t_2 \in [0,1]$$

An important characterization for preinvex functions was given by Barani et al. [2]:

Proposition 1 ([2]). Let $M \subseteq \mathbb{R}^m$ be an invex set with respect of $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and suppose that ζ satisfies Condition C on M, then for every $v, w \in M$ the function $\phi : M \to \mathbb{R}$ is preinvex with respect to ζ on ζ-path P_{wy}, if and only if the function $\psi : [0,1] \to \mathbb{R}$ defined by $\psi(t) = \phi(w + t\zeta(v,w))$ is convex on $[0, 1]$.

3. Symmetrically differentiable preinvex functions

In this section, we discuss an important characterization for SD preinvex functions. For that, firstly we extend Theorem 2 proved by Minch [10] for the invex sets.

Theorem 4. Let $v, w \in$M$\subseteq \mathbb{R}^m$ and M be an open invex set with respect to $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ such that $w + t\zeta(v,w) \in M$. If ϕ is SD at $v_0 = w + t_0\zeta(v,w) \in M$, then the function ψ defined on M, where $\psi(t) = \phi(w + t\zeta(v,w))$ is SD at t_0 and $\psi^s(t_0) = \phi^s(v_0)^T\zeta(v,w)$.

Proof. By Theorem 1 we can say that a symmetric derivative is a generalization of an ordinary derivative. Hence, the above theorem is a special chain rule of symmetric differentiation. □

Theorem 5. Suppose that ϕ is an SD function on an open invex set M with respect to ζ, where $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ satisfying Condition C. Then, ϕ is preinvex if and only if

$$\phi(v) - \phi(w) \geq \phi^s(w)^T\zeta(v,w), \quad \forall v, w \in M$$

Proof. Let $p, q \in M$ and $t \in [0,1]$, and suppose that

$$\phi(v) - \phi(w) \geq \phi^s(w)^T\zeta(v,w), \ \forall\, v, w \in M$$

Since M is an invex set, $p + t\zeta(q,p) \in M$. So,

$$\phi(p) - \phi(p + t\zeta(q,p)) \geq \phi^s(p + t\zeta(q,p))^T\zeta(p, p + tn(q,p))$$

and

$$\phi(q) - \phi(p + t\zeta(q,p)) \geq \phi^s(p + t\zeta(q,p))^T\zeta(q, p + tn(q,p))$$

By Condition C, we get

$$\zeta(p, p + t\zeta(q,p)) = -tn(q,p)$$

and

$$\zeta(q, p + t\zeta(q,p)) = (1-t)n(q,p)$$

Applying these conditions in the above equation, we get

$$\phi(p) - \phi(p + t\zeta(q,p)) \geq \phi^s(p + t\zeta(q,p))^T(-t\zeta(q,p)) \tag{1}$$

$$\phi(q) - \phi(p + t\zeta(q,p)) \geq \phi^s(p + t\zeta(q,p))^T(1-t)\zeta(q,p) \tag{2}$$

On multiplying equation (1) by $(1-t)$ and equation (2) by t and then adding, we get

$$(1-t)\phi(p) + t\phi(q) - \phi(p + t\zeta(q,p)) \geq 0$$

or

$$\phi(p + t\zeta(q,p)) \leq t\phi(q) + (1-t)\phi(p)$$

Thus ϕ is preinvex function on M.

Conversely, let ϕ be preinvex on an invex set M with respect of ζ, now we define the function $\psi : [0,1] \to \mathbb{R}$ such that

$$\psi(t) = \phi(w + t\zeta(v,w))$$

Since ϕ is SD, by Theorem 4, $\psi(t)$ is also SD and

$$\psi^s(t) = \phi^s(w + t\zeta(v,w))^T\zeta(v,w)$$

By Proposition 1, $\psi(t)$ is convex on [0, 1]. On applying Theorem 3 on $\psi(t)$, we get

$$\psi(1) - \psi(0) \geq \psi^s(0)(1-0),$$
$$\text{or}\phi(w + \zeta(v,w)) - \phi(w) \geq \phi^s(w)^T\zeta(v,w).$$

And we know that

$$\phi(w + \zeta(v,w)) \leq \phi(v).$$

Hence,

$$\phi(v) - \phi(w) \geq \phi^s(w)^T\zeta(v,w)$$

□

Corollary 1. Let ϕ be SD on an open invex set M with respect to $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$. If ϕ is preinvex on M, then

$$\phi^s(w)^T\zeta(v,w) \geq 0 \implies \phi(v) \geq \phi(w) \quad \forall v,\ w \in M$$

Using the above corollary, we introduce s-pseudoinvex and s-quasiinvex functions as follows:

Definition 6. The SD function $\phi : M \to \mathbb{R}$ is s-pseudoinvex at $v_0 \in M$, if there exists $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ such that for all $v \in M$

$$\phi^s(v_0)^T\zeta(v,v_0) \geq 0 \implies \phi(v) \geq \phi(v_0)$$

The function ϕ is s-pseudoinvex on M if it is s-pseudoinvex at each point of M. Also the function ϕ is s-pseudoincave at v_0 if $-\phi$ is s-pseudoinvex at v_0 and ϕ is s-pseudoincave on M if it is s-pseudoincave at each point of M.

Analogously to the differentiable quasiinvexity, we introduce the following definition.

Definition 7. The SD function $\phi : M \to \mathbb{R}$ is said to be s-quasiinvex at $v_0 \in M$, if there exists $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ such that for all $v \in M$

$$\phi(v) - \phi(v_0) \leq 0 \implies \phi^s(v_0)^T \zeta(v, v_0) \leq 0$$

The function ϕ is s-quasiinvex on M if it is s-quasiinvex at each point of M. Also the function ϕ is s-quasiincave at v_0 if $-\phi$ is s-quasiinvex at v_0 and ϕ is s-quasiincave on M if it is s-quasiincave at each point of M.

Theorem 6. Every s-pseudoinvex function is s-quasiinvex.

Proof. The proof is obvious as every pseudoconvex function is quasiconvex function. □

We know that, for a one-dimensional function f, defined and differentiable on (a, b), if a point $x_0 \in (a, b)$ is local extremum, then $f'(x_0) = 0$. For a multi-dimensional function, the gradient is zero at local optimum points. But this result is not true for symmetric derivatives. However, for continuous functions we can prove it as follows:

Lemma 1. Let the function $\phi^s(v)$ be continuous at a point v_0 and let $\phi(v)$ be continuous on some neighborhood of v_0, then $\phi'(v_0)$ exists and $\phi'(v_0) = \phi^s(v_0)$. Where, ϕ' represents ordinary differentiation. Moreover, if v_0 is a stationary point of ϕ, then $\phi^s(v_0) = 0$.

Proof. The first part of the theorem was proved by Aull [1] in Theorem 3, i.e., $\phi'(v_0) = \phi^s(v_0)$. Now, if v_0 is a stationary point of ϕ, then by first derivative test, $\phi'(v_0) = 0$. Hence, $\phi^s(v_0) = 0$. □

For continuous functions, we can extend the result for multi-dimensional SD functions as follows:

Theorem 7. Suppose $\phi : M \to \mathbb{R}$ is a continuous and SD function such that all the symmetric partial derivatives of ϕ exist at $v_0 \in Int(M)$. Let $\phi^s(v)$ be also continuous at v_0. If v_0 is a local optimum (stationary point) of ϕ, then $\phi^s(v_0) = 0$.

Proof. Consider the one-dimensional function $g(t) = \phi(v_0 + te_i)$, where e_i is the standard basis of $\mathbb{R}^m$, $i \in \{1, 2, \ldots, m\}$. Since, ϕ is SD at v_0, by Theorem 2, $g(t)$ is also SD at $t = 0$. Hence,

$$g^s(t) = [\phi^s(v_0 + te_i)e_i]^T \implies g^s(0) = \phi^s(v_0).$$

Since, v_0 is local optimum of ϕ, it follows that $t = 0$ is a local optimum of g, by Lemma 1, $g^s(0) = 0$. Therefore, $\phi^s(v_0) = 0$. □

Theorem 8. Assume that $\phi(v)$ and $\phi^s(v)$ are continuous at the stationary points. Let $M \subseteq \mathbb{R}^m$ be an open invex set with respect to $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$. The function $\phi : M \to \mathbb{R}$ is s-pseudoinvex if and only if every stationary point of ϕ is a global minimum of ϕ over M.

Proof. Let v be a stationary point of ϕ, then $\phi^s(v) = 0$. Since ϕ is s-pseudoinvex function, for every $w \in M$ we have

$$\phi^s(v)^T \zeta(w, v) \geq 0 \implies \phi(w) \geq \phi(v)$$

which by condition $\phi^s(v) = 0$ yields $\phi(v) \leq \phi(w) \forall w \in M$. Hence $v \in M$ is a global minimum of ϕ over M.

Conversely, let $\phi^s(v) = 0 \implies \phi(w) \geq \phi(v) \forall w \in$ M.$Wedefine \zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ as follows:

$$\zeta(w, v) = \begin{cases} 0 \quad \text{if} \quad \phi^s(v) = 0 \\ \\ \phi^s(v) \dfrac{(\phi(w) - \phi(v))\phi(w, v)}{\phi^s(v)^T \phi^s(v)} \quad \text{if} \quad \phi^s(v) \neq 0 \end{cases}$$

Now, if $\phi^s(v) = 0 \implies \phi(w) \geq \phi(v)$, which is equivalent to

$$\phi^s(v)^T \zeta(w, v) = 0 \implies \phi(w) \geq \phi(v)$$

If $\phi^s(v) \neq 0$, then

$$\phi^s(v)^T \zeta(w, v) \geq 0 \implies \phi^s(v)^T \phi^s(v) \frac{(\phi(w) - \phi(v))\phi(w, v)}{\phi^s(v)^T \phi^s(v)} \geq 0$$

$$\implies \phi(w) - \phi(v) \geq 0 \implies \phi(w) \geq \phi(v)$$

Hence, in both cases, ϕ is s-pseudoinvex concerning ζ. □

Next, we present a characterization of the s-quasiinvex function.

Theorem 9. Assume that $\phi(u)$ and $\phi^s(u)$ are continuous at the stationary points. Let $M \subseteq \mathbb{R}^m$ be an invex set with respect to $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ and $\phi : M \to \mathbb{R}$ be a s-quasiinvex function. Then ϕ is s-pseudoinvex if and only if every stationary point is a global minimum of ϕ over M.

Proof. Let ϕ be s-pseudoinvex, then by Theorem 8, every stationary point is a global minimum of ϕ over M.

For the converse part, on the contrary, suppose that every stationary point of ϕ is a global minimum but ϕ is not s-pseudoinvex. Then there exists $v, w \in M$, such that

$$\phi^s(v)^T \zeta(w, v) \geq 0 \implies \phi(w) < \phi(v) \tag{3}$$

Now, we define $n : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ as follows:

$$\zeta(w, v) = \begin{cases} 0 \quad \text{if} \quad \phi^s(v) = 0 \\ \\ \dfrac{(\phi(w) - \phi(v))\eta(w, v)}{\phi^s(v)^T \phi^s(v)} \phi^s(v) \text{ if } \phi^s(v) \neq 0 \end{cases}$$

where $\eta(w, v) : \mathbb{R}^m \times \mathbb{R}^m \to (0, 1) \cup (1, \infty)$. Here it is clear that $\phi^s(v) \neq 0$, otherwise by the assumption v must be global minimum, which is not so.

Now, since ϕ is s-quasiinvex by Definition 7

$$\phi(w) \leq \phi(v) \implies \phi^s(v)^T\zeta(w,v) \leq 0 \tag{4}$$

from (3) and (4), we can say that

$$\phi^s(v)^T\zeta(w,v) = 0$$

or

$$\phi^s(v)^T\zeta(w,v) = \phi^s(v)^T\frac{(\phi(w)-\phi(v))\eta(w,v)}{\phi^s(v)^T\phi^s(v)}\phi^s(v) = 0$$

which implies

$$(\phi(w)-\phi(v))\eta(w,v) = 0$$

By definition, $\eta(w,v)$ is a positive scalar function, so

$$\phi(w)-\phi(v) = 0 \implies \phi(w) = \phi(v)$$

which is a contradiction by (3). Hence, ϕ is s-pseudoinvex with respect to $\zeta(w,v)$. ☐

Remark 3.1. For $\eta(w,v) \neq 1$, the function ϕ is not invex with respect to ζ because of the equality

$$\phi^s(v)^T\zeta(w,v) = \eta(w,v)(\phi(w)-\phi(v))$$

Example 2. The following example shows that s-pseudoinvexity is more general than pseudoinvexity. Define $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(v) = \min\{|v|, v^2\},$$

then ϕ is not differentiable at -1 and 1. But it is SD at -1 and 1

$$\phi^s(v) = \begin{cases} -1, & v \in (-\infty,-1] \\ 2v, & v \in (-1,1] \\ 1, & v \in (1,\infty) \end{cases}$$

Thus due to the symmetricity of the function and definition 3.1, ϕ is an s-pseudoinvex function concerning $\zeta(v,w) = \{(v-w) if vw \geq 0\}$ but it is not pseudoinvex for same $\zeta(v,w)$.

4. Optimality conditions for symmetrically differentiable preinvex functions

In this section, we obtain sufficient optimality conditions for SD preinvex functions. Let ϕ and ψ_i, $i = 1, 2, 3, \ldots, m$, be real-valued SD functions defined on M. Consider the following problem:

$$\text{(P)} \quad \begin{array}{ll} \min & \phi(v) \\ \text{s.t.} & v \in X \end{array}$$

where $X = \{v \in M : \psi_i(v) \leq 0,\ i = 1, 2, 3, \dots, m\}$. X is called the constraint set. If there exists $v_0 \in X$ such that $\phi(v_0) \leq \phi(v)\,\forall v \in X$, then v_0 is called an optimal solution of problem P.

Theorem 10. Let M be an open invex set with respect to $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ satisfying Condition C and let ϕ and $\psi_i,\ i = 1, 2, 3, \dots, m$ be preinvex functions on M. If there exists real-valued multipliers $\alpha_i \geq 0, i = 1, 2, 3, \dots, m$, such that the following conditions hold:

(i) $\phi^s(v_0) + \sum_{i=1}^m \alpha_i \psi_i^s(v_0) = 0$

(ii) $\sum_{i=1}^m \alpha_i \psi_i(v_0) = 0$

then v_0 is an optimal solution of the problem P.

Proof. Since ϕ is preinvex and SD, by Theorem 5

$$\phi(v) - \phi(v_0) \geq \phi^s(v_0)^T \zeta(v, v_0)\ \forall v \in X$$

Now by the assumption (i) we have

$$\phi^s(v_0) = -\sum_{i=1}^{m} \alpha_i \psi_i(v_0)$$

By applying it to the above inequality, we get

$$\phi(v) - \phi(v_0) \geq -\sum_{i=m}^{m} \alpha_i \psi_i(v_0)^T \zeta(v, v_0) \tag{5}$$

Now since each ψ_i is preinvex and SD, applying again Theorem 5 for $i = 1, 2, 3, \dots, m$, we get

$$\psi_i(v) - \psi_i(v_0) \geq \psi_i^s(v_0)^T \zeta(v, v_0)\ \forall v \in X$$

or

$$\sum_{i=1}^{m} \alpha_i(\psi_i(v) - \psi_i(v_0) \geq \sum_{i=1}^{m} \alpha_i \psi_i^s(v_0)^T \zeta(v, v_0) \tag{6}$$

On adding (5) and (6), we get

$$\phi(v) - \phi(v_0) + \sum_{i=1}^{m} \alpha_i(\psi_i(u) - \psi_i(u_0) \geq 0 \forall u \in X$$

By assumption (ii),

$$\phi(v) - \phi(v_0) \geq -\sum_{i=1}^{m} \alpha_i \psi_i(v), \forall v \in X$$

but for each $i,\ \alpha_i \geq 0$ and $\psi_i(v) \leq 0$, which gives

$$\phi(v) - \phi(v_0) \geq 0 \forall v \in X \quad \text{or} \quad \phi(v) \geq \phi(v_0)\ \forall v \in X$$

This shows that v_0 is the optimal solution to the problem P. □

Now using s-pseudoinvexity and s-quasiinvexity concepts, we deduce the following generalized optimality theorem for invex programming.

Theorem 11. Let $v_0 \in X$, $S = \{i : \psi_i(v_0) = 0\}$, ϕ be s-pseudoinvex at v_0 with respect to $\zeta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, and ψ_i be s-quasiinvex at v_0 with respect to same ζ for $i \in S$. If there exists real-valued multipliers $\alpha_i \geq 0,\ i = 1, 2, 3,\ \ldots,\ m$, such that following conditions hold:

(i) $\phi^s(v_0) + \sum_{i=1}^m \alpha_i \psi_i^s(v_0) \ = \ 0,$

(ii) $\sum_{i=1}^m \alpha_i \psi_i(v_0) \ = \ 0, i = 1, 2, 3,\ \ldots,\ m.$

Then v_0 is a solution to problem P.

Proof. Since $\alpha_i \geq 0$, each $\psi_i(v_0) \leq 0$, by assumption (ii), $\sum_{i=1}^m \alpha_i \psi_i(v_0) = 0$, it follows that for $i \notin S$, $\alpha_i = 0$. Now, $\forall v \in X$, $\psi_i(v_0)$ is s-quasiinvex for $i \in S$, we have

$$\psi_i(v) \leq \psi_i(v_0) \implies \psi_i^s(v_0)^T \zeta(v, v_0) \leq 0 \implies \alpha_i \psi_i^s(v_0)^T \zeta(v, v_0) \leq 0$$
$$\implies \sum_{i=1}^m \alpha_i \psi_i^s(v_0)^T \zeta(v, v_0) \leq 0\ \forall v \in X$$

By assumption (i),

$$-\phi^s(v_0)^T \zeta(v, v_0) \leq 0,\ \forall v \in X \implies \phi^s(v_0)^T \zeta(v, v_0) \geq 0, \forall\ v \in X$$

s-pseudoinvexity of F at v_0 implies that

$$\phi^s(v_0)^T \zeta(v, v_0) \geq 0 \implies \phi(v) \geq \phi(v_0) \forall v \in X$$

This shows that v_0 is a solution to problem P. □

5. Conclusions

In this paper, we have introduced the concept of symmetric derivative for preinvex functions which is the generalization of ordinary derivatives. Using condition C, a necessary and sufficient condition for SD preinvex functions has been derived. Symmetrically pseudoinvex (s-pseudoinvex) and symmetrically quasiinvex (s-quasiinvex) functions have also been defined and some essential properties of these functions are discussed. The first derivative theorem has been generalized for SD functions and some relationships between s-pseudoinvex and s-quasiinvex functions have been established. Suitable examples have been constructed and sufficient optimality conditions for SD preinvex functions have also been obtained. In future, one can extend this work for the Hadamard manifolds but for that firstly they have to define symmetric differentiability on the manifolds.

Acknowledgement

The authors are grateful to the anonymous reviewers for their valuable comments and suggestions made on the previous draft of this manuscript.

References

[1] AULL, C. E. The first symmetric derivative. *The American Mathematical Monthly 74*, 6 (1967), 708–711.

[2] BARANI, A., GHAZANFARI A, G., AND DRAGOMIR S, S. Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex. *Journal of Inequalities and Applications 2012* (2012), 247.

[3] GUO, Y., YE, G., LIU, W., ZHAO, D., AND TREANŢĂ, S. On symmetric gH-derivative: Applications to dual interval-valued optimization problems. *Chaos, Solitons and Fractals 158* (2022), 112068.

[4] GUO, Y., YE, G., ZHAO, D., AND LIU, W. gH-symmetrically derivative of interval-valued functions and application in interval-valued optimization. *Symmetry 11* (2019), 1203.

[5] GUO, Y., YE, G., LIU, W., ZHAO, D., AND TREANŢĂ, S. Optimality conditions and duality for a class of generalized convex interval-valued optimization problems. *Mathematics 9* (2021), 2979.

[6] HANSON, M. A. On sufficiency of the Kuhn-Tucker conditions. *Journal of Mathematical Analysis and Applications 80*, 2 (1981), 545–550.

[7] HO, Q. Necessary and sufficient KKT optimality conditions in non-convex optimization. *Optimization Letters 11* (2017), 41–46.

[8] IVANOV, V. I. Second-order optimality conditions for inequality constrained problems with locally Lipschitz data. *Optimization Letters 4* (2010), 597–608.

[9] LARSON, L. The symmetric derivative. *Transactions of the American Mathematical Society 277* (1983), 589–599.

[10] MINCH, R. A. Applications of symmetric derivatives in mathematical programming. *Mathematical Programming 1*, 1 (1971), 307–320.

[11] MISHRA, S. K. Generalized fractional programming problems containing locally subdifferentiable and ρ-Univex functions. *Optimization 41*, 2 (1997), 135–158.

[12] MISHRA, S. K., AND GIORGI, G. *Invexity and optimization*. Springer Science and Business Media, Berlin, 2008.

[13] MISHRA, S. K., WANG, S. Y., AND LAI, K. K. Multiple objective fractional programming involving semi locally type I-preinvex and related functions. *Journal of Mathematical Analysis and Applications 310*, 2 (2005), 626–640.

[14] MISHRA, S. K., WANG, S. Y., AND LAI, K. K. Optimality and duality for a multi-objective programming problem involving generalized d-type-I and related n-set functions. *European Journal of Operational Research 173*, 2 (2006), 405–418.

[15] MISHRA, S. K., WANG, S. Y., AND LAI, K. K. On non-smooth α-invex functions and vector variational-like inequality. *Optimization Letters 2*, 1 (2008), 91–98.

[16] MISHRA, S. K., WANG, S. Y., AND LAI, K. K., AND SHI, J. M. Nondifferentiable minimax fractional programming under generalized univexity. *Journal of Computational and Applied Mathematics 158*, 2 (2003), 379–395.

[17] MOHAN, S. R., AND NEOGY, S. K. On invex sets and preinvex functions. *Journal of Mathematical Analysis and Applications 189*, 3 (1995), 901–908.

[18] SHARMA, N., MISHRA, S. K., AND HAMDI, A. A weighted version of Hermite-Hadamard type inequalities for strongly GA-convex functions. *International Journal of Advanced and Applied Sciences 7*, 3 (2020), 113–118.

[19] SHARMA, N., MISHRA, S. K., AND HAMDI, A. Hermite–Hadamard-type inequalities for interval-valued preinvex functions via Riemann–Liouville fractional integrals. *Journal of Inequalities and Applications 1* (2021), 98.

[20] THOMSON, B. S. *Symmetric properties of real functions*. Dekker, New York, NY, USA, 1994.

[21] WEIR, T., AND JEYAKUMAR, V. A class of nonconvex functions and mathematical programming. *Bulletin of the Australian Mathematical Society 38*, 2 (1988), 177–189.

[22] WEIR, T., AND MOND, B. Pre-invex functions in multiobjective optimization. *Journal of Mathematical Analysis and Applications 136*, 1 (1988), 29–38.

[23] YANG, X. Q., AND CHEN, G.-Y. A class of nonconvex functions and pre-variational inequalities. *Journal of Mathematical Analysis and Applications 169*, 2 (1992), 359–373.