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MONOTONIC PROPERTIES OF KNESER SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH DELAYED ARGUMENT

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Abstract. In this paper new monotonic properties of nonoscillatory solutions for second order linear functional differential equations with delayed argument

$$y''(t) = p(t)y(\tau(t))$$

have been established. New properties are used to introduce criteria for elimination of bounded nonoscillatory solutions for studied equations.

Keywords: second order, differential equations, delayed argument, monotonic properties, oscillation.

Mathematics Subject Classification: 34K11, 34C10.

1. INTRODUCTION

We consider the linear functional differential equation with delayed argument

$$y''(t) = p(t)y(\tau(t)). \tag{E}$$

In this paper it is assumed that

(H_1) $p(t) \in C([t_0,\infty))$, $p(t) > 0$,
(H_2) $\tau(t) \in C^1([t_0,\infty))$, $\tau(t) < t$, $\tau'(t) > 0$, $\lim\limits_{t\to\infty} \tau(t) = \infty$.

By a proper solution of Eq. (E) we mean a function $y : [t_0,\infty) \to (-\infty,\infty)$ which satisfies (E) for all sufficiently large t and $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq t_0$. We make the standing hypothesis that (E) does possess proper solutions. The oscillatory character of the solutions is understood in the standard way, that is, a proper solution is termed to be oscillatory or nonoscillatory according to whether it does or does not have infinitely many zeros.

If $y(t)$ is a nonoscillatory solution of (E), then there exists a number $\ell \in \{0,2\}$ such that

$$\begin{aligned} y(t)y^{(i)}(t) > 0 \quad &\text{for} \quad 0 \le i \le \ell, \\ (-1)^i y(t)y^{(i)}(t) > 0 \quad &\text{for} \quad \ell \le i \le 2. \end{aligned} \tag{1.1}$$

Such $y(t)$ is said to be a (nonoscillatory) solution of degree ℓ and the totality of solutions of degree ℓ is denoted by $\mathcal{N}_\ell$. If we denote the set of all nonoscillatory solutions of (E) by $\mathcal{N}$, then we have

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2.$$

It is obvious that if $y(t)$ is a solution of (E), then also $-y(t)$ is the solution of (E) and hence as usually we can restrict our attention only to positive nonoscillatory solutions of considered equation. Solutions from the class $\mathcal{N}_0$ are called Kneser solutions.

There are numerous papers devoted to monotonic properties and oscillation of second order differential equations, see e.g. [1–22] and the references included. We mention some of them in detail.

Koplatadze and Chanturia [13] (written in Russian, and so we refer also to [15]) formulated the following result:

Theorem 1.1. If $\tau(t) \le t$ and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} (s - \tau(t))p(s)\,\mathrm{d}s > 1, \tag{1.2}$$

then $\mathcal{N}_0 = \emptyset$ for (E).

This result has been extended to more general differential equations. For example Džurina in [9] generalized it to trinomial differential equations with retarded and advanced arguments

$$y''(t) = p(t)y(\tau(t))) + g(t)y(\sigma(t)).$$

On the other hand, Kusano and Lali in [17] have shown that equation

$$\left(|y'(t)|^{\alpha-1}y'(t)\right)' = p(t)|y(\tau(t))|^{\alpha-1}y(\tau(t)), \quad \alpha > 0$$

does not allow solutions of degree 0, i.e. $\mathcal{N}_0 = \emptyset$ if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} (\tau(t) - \tau(s))^{\alpha} p(s)\,\mathrm{d}s > 1.$$

A number of the authors focused on the development of similar techniques based on the transformation of the nonlinear, neutral delay differential equations to the first-order Riccati type equation or inequality, see, for instance [2, 19, 20, 22]. In [7] authors present a comparison result in which the oscillation of second-order neutral differential equations is deduced from that a first-order delay differential equation.

The above mentioned results concern sublinear, superlinear, nonlinear, neutral functional second-order differential equations, which is the current trend to study more and more complicated differential equations. In this paper, we turn our attention to simple linear differential equations and provide new criteria for such types of equations.

Our goal is to improve essentially the above mentioned criterion (1.2). The main idea of the Koplatadze and Chanturia's proof consists of the following estimates:

$$\begin{aligned} y(\tau(t)) &= \int_{\tau(t)}^{\infty} (s-\tau(t))p(s)y(\tau(s))\,\mathrm{d}s \\ &\geq \int_{\tau(t)}^{t} (s-\tau(t))p(s)y(\tau(s))\,\mathrm{d}s \\ &\geq y(\tau(t)) \int_{\tau(t)}^{t} (s-\tau(t))p(s)\,\mathrm{d}s. \end{aligned} \tag{1.3}$$

So, there are two ways how to improve their result. The first one, Koplatadze and Chanturia used the fact that $y(t)$ is decreasing (see the third line of (1.3)). Our progress in the present paper is in establishing a new monotonicity for $y(t)$ in the form that $\alpha(t)y(t)$ is decreasing for a suitable function $\alpha(t)$ such that $\alpha(t) \to \infty$ as $t \to \infty$.

The second one, the information about function $(s-\tau(t))p(s)y(\tau(s))$ is lost on the interval (t, ∞) (see the second line of (1.3)). We eliminate also this insufficiency by establishing "the opposite" monotonicity of $y(t)$ in the sense that $\beta(t)y(t)$ is increasing for certain function $\beta(t)$.

Some new methods for asymptotic properties of (E) have been presented in latest papers [6] and [9]. The comparison with obtained our criteria will be provided in Example 2.9.

2. MAIN RESULTS

We are about to offer new monotonic properties of nonoscillatory solutions of degree 0 for equation (E).

To simplify our notation we employ the following functions:

$$\begin{aligned} P(t) &= \int_{t}^{\tau^{-1}(t)} p(s)\,\mathrm{d}s, \\ \alpha(t) &= e^{\int_{t_1}^{t} P(s)\,\mathrm{d}s}, \end{aligned} \tag{2.1}$$

where $t_1 \geq t_0$ is arbitrary but fixed constant.

Lemma 2.1. *Let $y(t)$ be a positive solution of degree 0 for equation (E). Then*

$$\alpha(t)y(t) \text{ is decreasing function} \tag{2.2}$$

for $t \geq t_1$.

Proof. It is easy to see that for considered solution $y(t)$ we have $y'(t) < 0$ and $y''(t) > 0$. Then, in view of (2.1), an integration of (E) from t to ∞ yields

$$-y'(t) \geq \int_t^\infty p(s)y(\tau(s))\,\mathrm{d}s \geq \int_t^{\tau^{-1}(t)} p(s)y(\tau(s))\,\mathrm{d}s \geq y(t)P(t)$$

which is equivalent to

$$y'(t) + P(t)y(t) \leq 0.$$

Using standard methods of calculus it is easy to verify that for any $t_1 \geq t_0$

$$\left[y(t)e^{\int_{t_1}^t P(s)\,\mathrm{d}s}\right]' = [y(t)\alpha(t)]' \leq 0,$$

and we conclude that $y(t)\alpha(t)$ is decreasing function. □

Our first improvement of Theorem 1.1 is based on the established monotonicity.

Theorem 2.2. *If*

$$\limsup_{t\to\infty}\left[\alpha(\tau(t))\int_{\tau(t)}^t \frac{p(s)}{\alpha(\tau(s))}(s-\tau(t))\,\mathrm{d}s\right] > 1, \tag{2.3}$$

then $\mathcal{N}_0 = \emptyset$ for (E).

Proof. Assume on the contrary that $y(t)$ is a positive solution of degree 0 of equation (E). Integrating (E) twice from u to t and changing the order of integration, we obtain

$$y(u) \geq \int_u^t p(s)y(\tau(s))(s-u)\,\mathrm{d}s.$$

Employing the fact that $y(t)\alpha(t)$ is decreasing function, we get

$$y(u) \geq y(\tau(t))\alpha(\tau(t))\int_u^t \frac{p(s)}{\alpha(\tau(s))}(s-u)\,\mathrm{d}s$$

which for $u = \tau(t)$ provides

$$1 \geq \alpha(\tau(t))\int_{\tau(t)}^t \frac{p(s)}{\alpha(\tau(s))}(s-\tau(t))\,\mathrm{d}s.$$

This contradicts (2.3) and the proof is complete. □

It is easy to see that Theorem 2.2 upgrades classical Theorem 1.1. We illustrate this fact with the Euler differential equation.

Example 2.3. Consider the second order delay differential equation

$$y''(t) = \frac{p_0}{t^2} y(\lambda t), \tag{E_1}$$

where $p_0 > 0$ and $\lambda \in (0,1)$. By Theorem 1.1 of [13], equation (E_1) for $\lambda = 0.5$ has not nonoscillatory solutions from the class $\mathcal{N}_0$ provided that $p_0 > 5.177$. On the other hand, it is easy to verify that

$$P(t) = \frac{p_0}{t}(1-\lambda)$$

and if we choose $t_1 = 1$, then

$$\alpha(t) = t^{\alpha_0}, \text{ where } \alpha_0 = p_0(1-\lambda).$$

Therefore, condition (2.3) reduces to

$$p_0 \left[\frac{\lambda^{-\alpha_0} - 1}{\alpha_0} + \frac{\lambda - \lambda^{-\alpha_0}}{1+\alpha_0} \right] > 1$$

which for $\lambda = 0.5$ takes the form $p_0 > 3.32$, and this by Theorem 2.2 guarantees that $\mathcal{N}_0 = \emptyset$ for (E_1). The progress is obvious.

For the next improvement we need to establish the opposite monotonicity for $y(t)$.

It is easy to see that (H_2) guaranties the existence of the inverse function $\tau^{-1}(t)$ and therefore the auxiliary function $\xi(t) \in C^1([t_0,\infty))$

$$\xi(\xi(t)) = \tau^{-1}(t) \tag{2.4}$$

is well-defined. To simplify our notation we employ the following auxiliary functions:

$$\begin{aligned}
P_1(t) &= \alpha(\tau(\xi(t))) \int_t^{\xi(t)} \frac{s-t}{\alpha(\tau(s))} p(s) \, \mathrm{d}s, \\
P_2(t) &= \alpha(t) \int_{\xi(t)}^{\tau^{-1}(t)} \frac{s-t}{\alpha(\tau(s))} p(s) \, \mathrm{d}s, \\
P_3(t) &= \alpha(\xi(t)) \int_{\tau^{-1}(t)}^{\tau^{-1}(\xi(t))} \frac{s-t}{\alpha(\tau(s))} p(s) \, \mathrm{d}s, \\
P_1^*(t) &= \frac{P_1(t)}{1-P_2(t)}, \\
P_3^*(t) &= \frac{P_3(t)}{1-P_2(t)}.
\end{aligned} \tag{2.5}$$

Lemma 2.4. *Assume that there exists a function* $\xi(t) \in C^1([t_0,\infty))$ *satisfying* (2.4) *and* $y(t)$ *is a positive solution of degree* 0 *of* (E), *then*

$$y(\tau(t)) \le \frac{1 - P_3^*(\xi^{-1}(t))P_1^*(t) - P_1^*(\xi(t))P_3^*(t)}{P_1^*(t)P_1^*(\xi^{-1}(t))}y(t). \tag{2.6}$$

Proof. Assume that $y(t) \in \mathcal{N}_0$ is a positive solution of equation (E). Integrating (E) twice from t to ∞ and changing the order of integration, we are led to

$$y(t) \ge \int_t^\infty p(s)y(\tau(s))(s-t)\,\mathrm{d}s.$$

Employing auxiliary function $\xi(t)$, we have

$$\begin{aligned} y(t) \ge &\int_t^{\xi(t)} p(s)y(\tau(s))(s-t)\,\mathrm{d}s + \int_{\xi(t)}^{\tau^{-1}(t)} p(s)y(\tau(s))(s-t)\,\mathrm{d}s \\ &+ \int_{\tau^{-1}(t)}^{\tau^{-1}(\xi(t))} p(s)y(\tau(s))(s-t)\,\mathrm{d}s \end{aligned} \tag{2.7}$$

which in view of (2.2) and (2.5) implies that

$$y(t) \ge y(\tau(\xi(t)))P_1(t) + y(t)P_2(t) + y(\xi(t))P_3(t).$$

Thus, for $P_2(t) < 1$, we obtain

$$y(t) \ge y(\xi^{-1}(t))P_1^*(t) + y(\xi(t))P_3^*(t). \tag{2.8}$$

Therefore,

$$y(\xi^{-1}(t)) \ge y(\tau(t))P_1^*(\xi^{-1}(t)) + y(t)P_3^*(\xi^{-1}(t)) \tag{2.9}$$

and

$$y(\xi(t)) \ge y(t)P_1^*(\xi(t)) + y(\tau^{-1}(t))P_3^*(\xi(t)). \tag{2.10}$$

Setting (2.9) and (2.10) into (2.8), one gets

$$\begin{aligned} y(t) \ge &\left[P_1^*(t)P_1^*(\xi^{-1}(t))\right] y(\tau(t)) \\ &+ \left[P_3^*(\xi^{-1}(t))P_1^*(t) + P_1^*(\xi(t))P_3^*(t)\right] y(t). \end{aligned} \tag{2.11}$$

Finally, we have

$$y(\tau(t)) \le \frac{1 - P_3^*(\xi^{-1}(t))P_1^*(t) - P_1^*(\xi(t))P_3^*(t)}{P_1^*(t)P_1^*(\xi^{-1}(t))}y(t).$$

The proof is complete. □

In what follows, we shall assume that there exist positive constants P_i^*, $i = 1, 3$, such that

$$P_i^*(t) \geq P_i^*. \tag{2.12}$$

The following criteria immediately result from the proof of Lemma 2.4.

Corollary 2.5. *If* $\limsup_{t\to\infty} P_2(t) > 1$, *then* $\mathcal{N}_0 = \emptyset$.

Corollary 2.6. *Let* (2.12) *hold and there exist a function* $\xi(t) \in C^1([t_0, \infty))$ *satisfying* (2.4). *Assume that* $y(t) \in \mathcal{N}_0$ *is a positive solution of* (E). *Then*

$$y(\tau(t)) \leq \frac{1 - 2P_3^* P_1^*}{(P_1^*)^2} y(t). \tag{2.13}$$

Now, we are prepared to provide the opposite monotonicity for $y(t) \in \mathcal{N}_0$. We set a couple of auxiliary functions:

$$\begin{aligned} Q(t) &= \frac{1 - 2P_3^* P_1^*}{(P_1^*)^2} \alpha(t) \int_t^\infty \frac{p(s)}{\alpha(s)} \, \mathrm{d}s, \\ \beta(t) &= e^{\int_{t_1}^t Q(s) \, \mathrm{d}s}, \end{aligned} \tag{2.14}$$

where $t_1 \geq t_0$ is arbitrary constant and the improper integral is assumed to be convergent.

Lemma 2.7. *Let* (2.12) *hold and there exist a function* $\xi(t) \in C^1([t_0, \infty))$ *satisfying* (2.4) *and* $y(t)$ *be a positive solution of degree* 0 *of* (E). *Then*

$$\beta(t)y(t) \text{ is an increasing function} \tag{2.15}$$

for $t \geq t_1$.

Proof. Assume that $y(t) \in \mathcal{N}_0$ is a positive solution of (E). Integrating (E) from t to ∞, in view of (2.13) and (2.2) we get

$$-y'(t) = \int_t^\infty p(s) y(\tau(s)) \, \mathrm{d}s \leq \int_t^\infty \frac{1 - 2P_3^* P_1^*}{(P_1^*)^2} p(s) y(s) \, \mathrm{d}s \leq y(t) Q(t).$$

But the inequality $y'(t) + y(t)Q(t) \geq 0$ implies that $y(t)\beta(t)$ is an increasing function. The proof is complete. □

Now, we present the final improvement of Theorems 1.1 and 2.2.

Theorem 2.8. *Let* (2.12) *hold and there exist a function* $\xi(t) \in C^1([t_0,\infty))$ *satisfying* (2.4)*. If*

$$\limsup_{t\to\infty}\left[\alpha(\tau(t))\int_{\tau(t)}^{t}\frac{p(s)(s-\tau(t))}{\alpha(\tau(s))}\,\mathrm{d}s + \beta(\tau(t))\int_{t}^{\infty}\frac{p(s)(s-\tau(t))}{\beta(\tau(s))}\,\mathrm{d}s\right] > 1, \tag{2.16}$$

then $\mathcal{N}_0 = \emptyset$ *for* (E)*.*

Proof. Assume on the contrary that $y(t) \in \mathcal{N}_0$ is a positive solution of (E). Integrating (E) twice from t to ∞ and changing the order of integration, we get

$$y(t) \geq \int_t^\infty p(s)y(\tau(s))(s-t)\,\mathrm{d}s.$$

Using the fact that $y(t)\alpha(t)$ is a decreasing function and $y(t)\beta(t)$ is increasing, we see that

$$\begin{aligned} y(\tau(t)) &\geq \int_{\tau(t)}^{t} p(s)y(\tau(s))(s-\tau(t))\,\mathrm{d}s + \int_t^\infty p(s)y(\tau(s))(s-\tau(t))\,\mathrm{d}s \\ &\geq y(\tau(t))\alpha(\tau(t))\int_{\tau(t)}^{t}\frac{p(s)(s-\tau(t))}{\alpha(\tau(s))}\,\mathrm{d}s + y(\tau(t))\beta(\tau(t))\int_t^\infty\frac{p(s)(s-\tau(t))}{\beta(\tau(s))}\,\mathrm{d}s \end{aligned}$$

which contradicts (2.16) and the proof is finished. □

By comparing (2.3) and (2.16) we see that main goal of the paper has been achieved since the information about the function $(s-\tau(t))p(s)y(\tau(s))$ is saved also over the interval (t,∞). We support our progress with help of the several examples.

Example 2.9. Consider once more the differential equation

$$y''(t) = \frac{p_0}{t^2}y(\lambda t), \quad p_0 > 0, \lambda \in (0,1) \tag{E_1}$$

It follows from Example 2.3 that $\alpha(t) = t^{\alpha_0}$, where $\alpha_0 = p_0(1-\lambda)$. It is easy to verify that

$$P_1(t) = p_0\lambda^{-\frac{\alpha_0}{2}}\left(\frac{1-\lambda^{\frac{\alpha_0}{2}}}{\alpha_0} + \frac{\lambda^{\frac{1+\alpha_0}{2}}-1}{1+\alpha_0}\right),$$
$$P_2(t) = p_0\lambda^{-\frac{\alpha_0}{2}}\left(\frac{1-\lambda^{\frac{\alpha_0}{2}}}{\alpha_0} + \frac{\lambda^{\frac{2+\alpha_0}{2}}-\lambda^{\frac{1}{2}}}{1+\alpha_0}\right),$$
$$P_3(t) = p_0\lambda^{-\frac{\alpha_0}{2}}\left(\frac{1-\lambda^{\frac{\alpha_0}{2}}}{\alpha_0} + \frac{\lambda^{\frac{3+\alpha_0}{2}}-\lambda}{1+\alpha_0}\right).$$

Then

$$P_1^* = \frac{P_1(t)}{1-P_2(t)}, \quad P_3^* = \frac{P_3(t)}{1-P_2(t)}$$

and

$$\beta(t) = t^\gamma, \quad \text{where} \quad \gamma = \frac{p_0}{1+\alpha_0}\frac{1-2P_3^*P_1^*}{(P_1^*)^2}.$$

For (E_1) condition (2.16) gives

$$p_0\left[\frac{\lambda^{-\alpha_0}-1}{\alpha_0} + \frac{\lambda-\lambda^{-\alpha_0}}{1+\alpha_0}\right] + p_0\left[\frac{1}{\gamma} - \frac{\lambda}{\gamma+1}\right] > 1. \tag{2.17}$$

Consequently, for $\lambda = 0.5$ condition (2.17) simplifies to $p_0 > 2.56$ which by Theorem 2.8 yields $\mathcal{N}_0 = \emptyset$ for (E_1). On the other hand, Theorem 2.1 in [6] requires

$$\frac{Q_1^2 + 2Q_1Q_3}{(1-Q_2)^2} > 1$$

with

$$Q_1 = p_0\left(\ln\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda} - 1\right),$$
$$Q_2 = p_0\left(\ln\frac{1}{\sqrt{\lambda}} + \lambda - \sqrt{\lambda}\right),$$
$$Q_3 = p_0\left(\ln\frac{1}{\sqrt{\lambda}} + \lambda\sqrt{\lambda} - \lambda\right)$$

which is satisfies for $p_0 > 3.39$. The progress is obvious.

Example 2.10. Consider the equation

$$y''(t) = p_0 y(t-\tau_0), \quad p_0 > 0, \tau_0 > 0. \tag{E_2}$$

By Theorem 1.1, Eq. (E_2) for $\tau_0 = 0.5$ has no nonoscillatory solution from the class $\mathcal{N}_0$ provided that $p_0 > 8$. On the other hand, condition (2.3) from Theorem 2.2 takes the form

$$-1 + \frac{e^{p_0\tau_0^2}}{p_0\tau_0^2} - \frac{1}{p_0\tau_0^2} > 1.$$

Now for $\tau_0 = 0.5$, we see that if $p_0 > 5.026$, then $\mathcal{N}_0 = \emptyset$. Finally, to apply Theorem 2.8 we find out that

$$P_1 = -\frac{1}{2} - \frac{1}{p_0\tau_0^2} + \frac{e^{\frac{p_0\tau_0^2}{2}}}{p_0\tau_0^2},$$
$$P_2 = -1 + \frac{e^{\frac{p_0\tau_0^2}{2}}}{2} + \frac{1}{p_0\tau_0^2}\left(e^{\frac{p_0\tau_0^2}{2}} - 1\right),$$
$$P_3 = -\frac{3}{2} + e^{\frac{p_0\tau_0^2}{2}} + \frac{1}{p_0\tau_0^2}\left(e^{\frac{p_0\tau_0^2}{2}} - 1\right).$$

Moreover,

$$P_1^* = \frac{P_1(t)}{1 - P_2(t)}, \quad P_3^* = \frac{P_3(t)}{1 - P_2(t)}$$

and

$$\beta(t) = e^{\frac{\delta t}{\tau_0}}, \quad \text{where} \quad \delta = \frac{1 - 2P_3^*P_1^*}{(P_1^*)^2}.$$

Criterion (2.16) for (E_2) takes the form

$$-1 + \frac{e^{p_0\tau_0^2}}{p_0\tau_0^2} - \frac{1}{p_0\tau_0^2} + \frac{p_0\tau_0^2}{\delta}\left(1 + \frac{1}{\delta}\right) > 1,$$

which for $\tau_0 = 0.5$ gives that if $p_0 > 3.872$, then (E_2) does not have nonoscillatory solutions from the class $\mathcal{N}_0$. Again the progress is outstanding.

In this paper we presented a new technique for investigation of the second order linear differential equation. We demonstrated the progress of our criteria on standardly used equations (E_1) and (E_2). It remains an open problem to extend our technique to higher order equations or eventually nonlinear equations and what to do if an improper integral in (2.14) is divergent.

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