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Extremal problems for parabolic systems with time-varying lags

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Extremal problems for parabolic systems with time-varying lags are presented. An optimal boundary control problem for parabolic systems in which time-varying lags appear in the state equations and in the boundary conditions simultaneously is solved. The time horizon is fixed. Making use of Dubovicki-Milutin scheme, necessary and sufficient conditions of optimality for the Neumann problem with the quadratic performance functionals and constrained control are derived.

Key words: boundary control, parabolic systems, time-varying lags

1. Introduction

Extremal problems are now playing an ever-increasing role in applications of mathematical control theory. It has been discovered that notwithstanding the great diversity of these problems, then can be approached by a unified functional-analytic approach, first suggested by Dubovicki and Milutin. The general theory of extremal problems has been developed so intensely recently that its basic concepts may now be considered complete.

Extremal problems were the object of mathematical research at the very earliest stages of the development of mathematics.

The first results were then systematized and brought together under the heading of the calculus of variations with its innumerable applications to physics, automatic control, and mechanics.

In 1962 Dubovicki and Milutin found a necessary condition for an extremum in the form of an equation set down in the language of functional analysis. They were able to derive, as special cases of this condition, almost all previously known necessary extremum conditions and thus to recover the lost theoretical unity of the calculus of variations.

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For example, in the papers [3, 4] the Dubovicki-Milutin method was applied for solving boundary optimal control problems for the case of time lag parabolic equations [3] and for the case of parabolic equations involving time-varying lags in the Neumann boundary conditions [4].

Such equations with deviating arguments are a well-known mathematical tool for representing many physical problems.

Parabolic equations with deviating arguments are widely applied in optimal control problems of distributed parameter systems with time lags.

Consequently, in the papers [3,4] linear quadratic problems of optimal control for the case of parabolic systems with time lags given in various forms (constant time lags [3], time-varying lags [4], etc.) were solved.

Extremal problems for time-varying lag parabolic systems are investigated. The purpose of this paper is to show the use of Dubovicki-Milutin method in solving optimal control problems for parabolic systems in which time-varying lags appear both in the state equations and in the Neumann boundary conditions.

As an example, an optimal boundary control problem for a system described by a linear time-varying lag partial differential equation of parabolic type with the Neumann boundary condition involving a time-varying lag is considered. Such equation constitutes in a linear approximation universal mathematical model for many diffusion processes. The right-hand side of this equation and the initial condition are not continuous functions usually, but they are measurable functions belonging to L^2 or L^∞ spaces. Therefore, the solution of this equation is given in a certain Sobolev space.

The performance functionals have the quadratic form. The time horizon is fixed. Finally, we impose same constraints on the boundary control. Making use of the Dubovicki-Milutin theorem, necessary and sufficient conditions of optimality with the quadratic performance functionals and constrained control are derived for the Neumann problem.

2. The Dubovicki-Milutin method [3]

The Dubovicki-Milutin theorem arises from the geometric form of the Hahn-Banach theorem (a theorem about the separation of convex sets). We shall show it on the example.

Let us assume that:

E – a linear topological space, locally convex,

$I(x)$ – a functional defined on E,

$A_i, i = 1, 2, \ldots, n$ – sets in E with inner points which represent inequality constraints,

B – a set in E without inner points representing equality constraint.

We must find some conditions necessary for a local minimum of the functional $I(x)$ on the set $Q = \bigcap_{i=1}^{n} A_i \cap B$, or find a point $x_0 \in E$, so that $I(x_0) = \min\limits_{Q \cap U} I(x)$, where U means a certain environment of the point x_0.

We define the set $A_0 = \{x : I(x) < I(x_0)\}$.

Then we formulate the necessary condition of optimality as follows: in the environment of the local minimum point, the intersection of system of sets (the set on which the functional attains smaller values than $I(x_0)$ and the sets representing constraints) is empty or $\bigcap_{i=0}^{n} A_i \cap B = \emptyset$.

The condition $\bigcap_{i=0}^{n} A_i \cap B = \emptyset$ is also equivalent to the one in which there are approximations of the sets $A_i, i = 1, 2, \ldots, n$ and B instead of A_i or B ones. These approximations are cones with the vertex in a point $\{0\}$.

We shall approximate the inequality constraints by the regular admissible cones $RAC(A_i, x_0)(i = 1, 2, \ldots, n)$, the equality constraint by the regular tangent cone $RTC(B, x_0)$ and for the performance functional we shall construct the regular improvement cone $RFC(I, x_0)$.

Then we have the necessary condition of the optimality $I(x)$ on the set $Q = \bigcap_{i=1}^{n} A_i \cap B$ in the form of Euler-Langrange's equation $\sum_{i=1}^{n+1} f_i = 0$, where f_i $(i = 1, 2, \ldots, n+1)$ – are linear, continuous functionals, all of them are not equal to zero at the same time and they belong to the adjoint cones

$$f_i \in [RAC(A_i, x_0)]^*, \quad i = 1, 2, \ldots, n,$$

$$f_{n+1} \in [RTC(B, x_0)]^*, \quad f_0 \in [RFC(I, x_0)]^*,$$

$$\{f_i \in [RAC(A_i, x_0)]^* \Leftrightarrow f_i(x) \geqslant 0 \ \ \forall x \in RAC(A_i, x_0)\}.$$

For convex problems, i.e. problems in which the constraints are convex sets and the performance functional is convex, the Euler-Lagrange equation is the necessary and sufficient condition of optimality if only certain additional assumptions are fulfilled (the so-called Slater's condition).

Using the Dubovicki-Milutin theorem we shall derive the necessary and sufficient conditions of optimality for time lag parabolic systems with the quadratic performance functionals and the constrained control.

3. Preliminaries

Consider now the distributed parameter system described by the following parabolic equation

$$\frac{\partial y}{\partial t} + A(t)y + y(x, t - h(t)) = u \qquad x \in \Omega,\ t \in (0, T), \tag{1}$$

$$y(x, t') = \Phi_0(x, t') \qquad x \in \Omega,\ t' \in [-h(0), 0), \tag{2}$$

$$y(x,0) = y_0(x) \qquad x \in \Omega, \tag{3}$$

$$\frac{\partial y}{\partial \eta_A} = y(x,t-h(t)) + v \qquad x \in \Gamma,\ t \in (0,T), \tag{4}$$

$$y(x,t') = \Psi_0(x,t') \qquad x \in \Gamma,\ t' \in [-h(0),0), \tag{5}$$

where: $\Omega \subset \mathrm{R}^n$ – a bounded, open set with boundary Γ which is a C^∞-manifold of dimension $(n-1)$. Locally, Ω is totally on one side of Γ.

$$y \equiv y(x,t;v), \qquad u \equiv u(x,t), \qquad v \equiv v(x,t),$$

$$Q = \Omega \times (0,T), \qquad \bar{Q} = \Omega \times [0,T], \qquad Q_0 = \Omega \times [-h(0),0),$$

$$\Sigma = \Gamma \times (0,T), \qquad \Sigma_0 = \Gamma \times [-h(0),0).$$

$h(t)$ is a function representing a time-varying lag, Φ_0 is an initial function defined on Q_0, Ψ_0 is an initial function defined on Σ_0.

The parabolic operator $\frac{\partial}{\partial t} + A(t)$ in the state equation (1) satisfies the hypothesis of Section 1, Chapter 4 ([10], vol. 2, p. 2), $A(t)$ is given by

$$A(t)y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right), \tag{6}$$

and the functions $a_{ij}(x,t)$ satisfy the ellipticity condition

$$\begin{gathered} \sum_{i,j=1}^{n} a_{ij}(x,t)\varphi_i\varphi_j \geqslant \alpha \sum_{i=1}^{n} \varphi_i^2, \\ \alpha > 0, \quad \forall (x,t) \in \overline{Q}, \quad \forall \varphi_i \in \mathscr{R}, \end{gathered} \tag{7}$$

where $a_{ij}(x,t)$ are real C^∞ functions defined on $\overline{Q}$ (closure of Q).

The equations (1)–(3) constitute a Neumann problem. Then the left-hand side of (4) is written in the form

$$\frac{\partial y}{\partial \eta_A} = \sum_{i,j=1}^{n} a_{ij}(x,t)\cos(n,x_i)\frac{\partial y(x,t)}{\partial x_j} = q(x,t), \tag{8}$$

where $\frac{\partial}{\partial \eta_A}$ is a normal derivative at Γ, directed towards the exterior of Ω, $\cos(n,x_i)$ is an i-th direction cosine of n, n-being the normal at Γ exterior to Ω and

$$q(x,t) = y(x,t-h(t)) + v(x,t). \tag{9}$$

Let $t \to t - h(t)$ be a strictly increasing function on $[0,T]$, $h(t)$ being non-negative in $[0,T]$ and also being a C^1 function. Then, there exists the inverse function of $t \to t - h(t)$.

Let us denote $r(t) \hat{=} t - h(t)$, then the inverse function of $r(t)$ has the form $t = f(r) = r + s(r)$, where $s(r)$ is a time-varying prediction.

Let $f(t)$ be the inverse function of $t \to t - h(t)$. Thus we define the following iteration $f_j(t) \hat{=} \underbrace{f...[f[f(t)]]}_{j}$ such that $f_0(t) = t$, where $f_j(t)$ is a j-th iteration of the operation $f(t)$ for $j = 0, 1 \ldots .$

For simplicity, we introduce the notation

$$E_j \hat{=} (f_{j-1}(0), f_j(0)), \qquad Q_j = \Omega \times \Sigma_j, \qquad Q_0 = \Omega \times [-h(0), 0),$$

$$\Sigma_j = \Gamma \times E_j, \quad \Sigma_0 = \Gamma \times [-h(0), 0) \text{ for } j = 1, \ldots .$$

Then the following result is fulfilled [5]:

Theorem 1 *Let y_0, Φ_0, Ψ_0, v and u be given with $y_0 \in H^{1/2}(\Omega)$, $\Phi_0 \in H^{3/2,3/4}(Q_0)$, $\Psi_0 \in L^2(\Sigma_0)$, $v \in L^2(\Sigma)$ and $u \in H^{-1/2,-1/4}(Q)$. Then, there exists a unique solution $y \in H^{3/2,3/4}(Q)$ for the mixed initial-boundary value problem (1)–(3). Moreover, $y(\cdot, f_j(0)) \in H^{1/2}(\Omega)$ for $j = 1, \ldots .$*

We refer to Lions and Magenes ([10], vol. 2) for the definition and properties of $H^{r,s}$ and $(H^{r,s})'$ respectively.

In the sequal, we shall fix $u \in H^{-1/2,-1/4}(Q)$.

4. Problem formulation. Optimization theorems

We shall restrict our considerations to the case of the boundary control. Therefore we shall formulate the optimal control problem in the context of Theorem 1 [5].

Let us denote by $Y = H^{3/2,3/4}(Q)$ the space of states and by $U = L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem.

The performance functional is given by

$$I(v) = \lambda_1 \int_Q | y(x,t;v) - z_d |^2 \, dx dt + \lambda_2 \int_0^T \int_\Gamma (Nv) v \, d\Gamma \, dt, \tag{10}$$

where: $\lambda_i \geqslant 0$ and $\lambda_1 + \lambda_2 > 0$; z_d is a given element in $L^2(Q)$ and N is a strictly positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$. We note from Theorem 1 [5] that for any $v \in U_{ad}$ the cost function (10) is well-defined since $y(v) \in H^{3/2,3/4}(Q) \subset L^2(Q)$.

We assume the following constraints on controls: $v \in U_{ad}$ is a closed, convex subset of U with non-empty interior,

$$\text{a subset of } U. \tag{11}$$

The optimal control problem (1)–(3), (10), (11) will be solved as the optimization one in which the function v is the unknown function.

Making use of the Dubovicki-Milutin theorem [3] we shall derive the necessary and sufficient conditions of optimality for the optimization problem (1)–(3), (10), (11).

The solution of the stated optimal control problem is equivalent to seeking a pair $(y^0, v^0) \in E = H^{3/2,3/4}(Q) \times L^2(\Sigma)$ that satisfies the equation (1)–(3) and minimizing performance functional (10) with the constraints on control (11).

We formulate the necessary and sufficient conditions of the optimality in the form of Theorem 2.

Theorem 2 *The solution of the optimization problem (1)–(3), (10), (11) exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities:*

$$\frac{\partial y^0}{\partial t} + A(t)y^0 + y^0(x, t-h(t)) = u \qquad (x,t) \in \Omega \times (0,T), \tag{12}$$

$$y^0(x,t') = \Phi_0(x,t') \qquad (x,t') \in \Omega \times [-h(0),0), \tag{13}$$

$$y^0(x,0) = y_0(x) \qquad x \in \Omega, \tag{14}$$

$$\frac{\partial y^0}{\partial \eta_A} = y^0(x, t-h(t)) + v^0 \qquad (x,t) \in \Gamma \times (0,T), \tag{15}$$

$$y^0(x,t') = \Psi_0(x,t') \qquad (x,t') \in \Gamma \times [-h(0),0). \tag{16}$$

Adjoint equations

$$-\frac{\partial p}{\partial t} + A^*(t)p + p(x, t+s(t))(1+s'(t)) = \lambda_1(y^0 - z_d)$$
$$(x,t) \in \Omega \times (0, T-h(T)), \tag{17}$$

$$-\frac{\partial p}{\partial t} + A^*(t)p = \lambda_1(y^0 - z_d) \qquad (x,t) \in \Omega \times (T-h(T), T), \tag{18}$$

$$\frac{\partial p}{\partial \eta_{A^*}} = p(x, t+s(t))(1+s'(t)) \qquad (x,t) \in \Gamma \times (0, T-h(T)), \tag{19}$$

$$\frac{\partial p}{\partial \eta_{A^*}} = 0 \qquad (x,t) \in \Gamma \times (T - h(T), T), \tag{20}$$

$$p(x,T) = 0 \qquad x \in \Omega. \tag{21}$$

Maximum condition

$$\int_0^T \int_\Gamma (p + \lambda_2 N v^0)(v - v^0)\, d\Gamma dt \geqslant 0 \qquad \forall v \in U_{ad}\,. \tag{22}$$

We can also notice that

$$\begin{aligned} \frac{\partial p}{\partial \eta_{A^*}} &= \sum_{i,j=1}^{n} a_{ji}(x,t)\cos(n,x_i)\frac{\partial p}{\partial x_j}, \\ A^*(t)p &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial p}{\partial x_i}\right). \end{aligned} \tag{23}$$

Outline of the proof:
According to the Dubovicki-Milutin theorem, we approximate the set representing the inequality constraints by the regular admissible cone, the equality constraint by the regular tangent cone and the performance functional by the regular improvement cone.

a) Analysis of the equality constraint

The set Q_1 representing the equality constraint has the form

$$Q_1 = \left\{ \begin{array}{ll} \frac{\partial y}{\partial t} + A(t)y + y(x,t-h(t)) = u & (x,t) \in \Omega \times (0,T) \\ y(x,t') = \Phi_0(x,t') & (x,t') \in \Omega \times [-h(0),0) \\ y(x,0) = y_0(x) & x \in \Omega \\ \frac{\partial y}{\partial \eta_A} = y(x,t-h(t)) + v & (x,t) \in \Gamma \times (0,T) \\ y(x,t') = \Psi_0(x,t') & (x,t') \in \Gamma \times [-h(0),0) \end{array} \right\}. \tag{24}$$

We construct the regular tangent cone of the set Q_1 using the Lusternik theorem (Theorem 9.1 [2]). For this purpose, we define the operator P in the form

$$\begin{aligned} P(y,v) = \Bigg(& \frac{\partial y}{\partial t} + Ay + y(x,t-h(t)) - u,\;\; y(x,t') - \Phi_0(x,t'),\;\; y(x,0) - y_0(x), \\ & \frac{\partial y}{\partial \eta_A} - y(x,t-h(t)) - v,\;\; y(x,t') - \Psi_0(x,t')\Bigg). \end{aligned} \tag{25}$$

The operator P is the mapping from the space $H^{3/2,3/4}(Q) \times L^2(\Sigma)$ into the space $H^{-1/2,-1/4}(Q) \times H^{3/2,3/4}(Q_0) \times H^{1/2}(\Omega) \times L^2(\Sigma) \times L^2(\Sigma_0)$.

The Fréchet differential of the operator P can be written in the following form:

$$P'(y^0,v^0)(\bar{y},\bar{v}) = \left(\frac{\partial \bar{y}}{\partial t} + A\bar{y} + \bar{y}(x,t-h(t)),\ \bar{y}\big|_{Q_0}(x,t'),\ \bar{y}(x,0),\right.$$
$$\left.\frac{\partial \bar{y}}{\partial \eta_A} - \bar{y}(x,t-h(t)) - \bar{v},\ \bar{y}\big|_{\Sigma_0}(x,t')\right). \tag{26}$$

Really, $\frac{\partial}{\partial t}$ (Theorem 2.8 [11]), $A(t)$ (Theorem 2.1 [9]) and $\frac{\partial}{\partial \eta_A}$ (Theorem 2.3 [10]) are linear and bounded mappings.

Using Theorem 1 [5], we can prove that P' is the operator "one to one" from the space $H^{3/2,3/4}(Q) \times L^2(\Sigma)$ onto the space $H^{-1/2,-1/4}(Q) \times H^{3/2,3/4}(Q_0) \times H^{1/2}(\Omega) \times L^2(\Sigma) \times L^2(\Sigma_0)$.

Considering that the assumptions of the Lusternik's theorem are fulfilled, we can write down the regular tangent cone for the set Q_1 in a point (y^0,v^0) in the form

$$RTC(Q_1,(y^0,v^0)) = \{(\bar{y},\bar{v}) \in E, P'(y^0,v^0)(\bar{y},\bar{v}) = 0\}. \tag{27}$$

It is easy to notice that it is a subspace. Therefore, using Theorem 10.1 [2] we know the form of the functional belonging to the adjoint cone

$$f_1(\bar{y},\bar{v}) = 0 \qquad \forall(\bar{y},\bar{v}) \in RTC(Q_1,(y^0,v^0)). \tag{28}$$

b) Analysis of the constraint on controls

The set $Q_2 = Y \times U_{ad}$ representing the inequality constraints is a closed and convex one with non-empty interior in the space E.

Using Theorem 10.5 [2] we find the functional belonging to the adjoint regular admissible cone, i.e.

$$f_2(\bar{y},\bar{v}) \in [RAC(Q_2,(y^0,v^0))]^*.$$

We can note if E_1, E_2 are two linear topological spaces, then the adjoint space to $E = E_1 \times E_2$ has the form

$$E^* = \{f = (f_1,f_2);\ f_1 \in E_1^*, f_2 \in E_2^*\}$$

and

$$f(x) = f_1(x_1) + f_2(x_2).$$

So we note the functional $f_2(\overline{y},\overline{v})$ as follows

$$f_2(\overline{y},\overline{v}) = f_1'(\overline{y}) + f_2'(\overline{v}) \tag{29}$$

where:
$f_1'(\overline{y}) = 0\ \forall y \in Y$ (Theorem 10.1 [2])
$f_2'(\overline{v})$ is a support functional to the set U_{ad} in a point v_0 (Theorem 10.5 [2]).

c) Analysis of the performance functional

Using Theorem 7.5 [2] we find the regular improvement cone of the performance functional (10)

$$RFC(I,(y^0,v^0)) = \{(\overline{y},\overline{v}) \in E, I'(y^0,v^0)(\overline{y},\overline{v}) < 0\}, \tag{30}$$

where: $I'(y^0,v^0)(\overline{y},\overline{v})$ is the Fréchet differential of the performance functional (10) and it can be written as

$$\begin{aligned} I'(y^0,v^0)(\overline{y},\overline{v}) &= 2\lambda_0\lambda_1 \int_Q (y^0 - z_d)\overline{y}\, dx\, dt \\ &\quad + 2\lambda_0\lambda_2 \int_0^T \int_\Gamma (Nv^0)\overline{v}\, d\Gamma\, dt\,. \end{aligned}$$

On the basis of Theorem 10.2 [2] we find the functional belonging to the adjoint regular improvement cone, which has the form

$$\begin{aligned} f_3(\overline{y},\overline{v}) = &- \lambda_0\lambda_1 \int_Q (y^0 - z_d)\overline{y}\, dx\, dt \\ &- \lambda_0\lambda_2 \int_0^T \int_\Gamma (Nv^0)\overline{v}\, d\Gamma\, dt, \end{aligned} \tag{31}$$

where: $\lambda_0 > 0$.

d) Analysis of Euler-Lagrange's equation

The Euler-Lagrange's equation for our optimization problem has the form

$$\sum_{i=1}^{3} f_i = 0. \tag{32}$$

Let $p(x,t)$ be the solution of (17)–(21) for (y^0, v^0) and denote by $\overline{y}$ the solution of $P'(\overline{y}, \overline{v}) = 0$ for any fixed $\overline{v}$. Then taking into account (28), (29) and (31) we can express (32) in the form

$$f_2'(\overline{v}) = \lambda_0\lambda_1 \int_Q (y^0 - z_d)\overline{y}\, dx\, dt + \lambda_0\lambda_2 \int_0^T \int_\Gamma (Nv^0)\overline{v}\, d\Gamma\, dt \qquad \forall(\overline{y}, \overline{v}) \in RTC(Q_1, (\overline{y}, \overline{v})). \tag{33}$$

We transform the first component of the right-hand side of (33) introducing the adjoint variable by adjoint equations (17)–(21).

For this purpose, multiplying both sides of (17)–(18) by $\overline{y}$, then integrating over $\Omega \times (0, T - h(T))$ and $\Omega \times (T - h(T), T)$ respectively, and then adding both sides of (17)–(18), we get

$$\begin{aligned} \lambda_0\lambda_1 &\int_Q (y^0 - z_d)\overline{y}\, dx\, dt \\ &= \lambda_0 \int_Q \left(-\frac{\partial p}{\partial t} + A^* p \right) \overline{y}\, dx\, dt + \lambda_0 \int_0^{T-h(T)} \int_\Omega p(x, t + s(t))(1 + s'(t))\overline{y}\, dx\, dt \\ &= \lambda_0 \int_Q p \frac{\partial \overline{y}}{\partial t} dx\, dt + \lambda_0 \int_Q A^* p \overline{y} dx\, dt \\ &\quad + \lambda_0 \int_0^{T-h(T)} \int_\Omega p(x, t + s(t))(1 + s'(t))\overline{y}\, dx\, dt. \end{aligned} \tag{34}$$

Using the equation (1), the first integral on the right-hand side of (34) can be written as

$$\begin{aligned} \lambda_0 \int_Q p \frac{\partial \overline{y}}{\partial t}\, dx\, dt &= -\lambda_0 \int_Q pA\overline{y}\, dx\, dt - \lambda_0 \int_0^T \int_\Omega p(x,t)\overline{y}(x, t - h(t)) dx\, dt \\ &= -\lambda_0 \int_Q pA\overline{y}\, dx\, dt \\ &\quad - \lambda_0 \int_{-h(0)}^{T-h(T)} \int_\Omega p(x, t' + s(t'))(1 + s'(t'))\overline{y}(x, t')\, dx\, dt. \end{aligned} \tag{35}$$

The second integral on the right-hand side of (34) in view of Green's formula can be expressed as

$$\lambda_0 \int_Q A^* p\, \bar{y}\, dx\, dt = \lambda_0 \int_Q pA\bar{y}\, dx\, dt + \lambda_0 \int_0^T \int_\Gamma p \frac{\partial \bar{y}}{\partial \eta_A} d\Gamma\, dt - \lambda_0 \int_0^T \int_\Gamma \frac{\partial p}{\partial \eta_{A^*}} \bar{y} d\Gamma\, dt. \tag{36}$$

Using the boundary condition (4), the second term on the right-hand side of (36) can be written as

$$\begin{aligned}
\lambda_0 \int_0^T \int_\Gamma p \frac{\partial \bar{y}}{\partial \eta_A}\, d\Gamma\, dt &= \lambda_0 \int_0^T \int_\Gamma p(x,t)[\bar{y}(x,t-h(t)) + \bar{v}]\, d\Gamma\, dt \\
&= \lambda_0 \int_{-h(0)}^{T-h(T)} \int_\Gamma p(x,t'+s(t'))(1+s'(t'))\bar{y}(x,t')\, d\Gamma dt' + \lambda_0 \int_0^T \int_\Gamma p\bar{v}\, d\Gamma\, dt \\
&= \lambda_0 \int_{-h(0)}^{0} \int_\Gamma p(x,t'+s(t'))(1+s'(t'))\bar{y}(x,t')\, d\Gamma\, dt' \\
&\quad + \lambda_0 \int_0^{T-h(T)} \int_\Gamma p(x,t'+s(t'))(1+s'(t'))\bar{y}(x,t')\, d\Gamma\, dt' \\
&\quad + \lambda_0 \int_0^T \int_\Gamma p\bar{v}\, d\Gamma\, dt.
\end{aligned} \tag{37}$$

The last term in (36) can be rewritten as

$$\lambda_0 \int_0^T \int_\Gamma \frac{\partial p}{\partial \eta_{A^*}} \bar{y} d\Gamma\, dt = \lambda_0 \int_0^{T-h(T)} \int_\Gamma \frac{\partial p}{\partial \eta_{A^*}} \bar{y} d\Gamma\, dt + \lambda_0 \int_{T-h(T)}^{T} \int_\Gamma \frac{\partial p}{\partial \eta_{A^*}} \bar{y} d\Gamma\, dt. \tag{38}$$

Substituting (37), (38) into (36) and then (35), (36) into (34) we obtain

$$\begin{aligned}
&\lambda_0\lambda_1\int\limits_Q (y^0 - z_d)\bar{y}\,dxdt = -\lambda_0\int\limits_Q pA\bar{y}\,dxdt \\
&-\lambda_0\int\limits_{-h(0)}^{0}\int\limits_{\Omega} p(x,t+s(t))(1+s'(t))\bar{y}\,dxdt - \lambda_0\int\limits_{0}^{T-h(T)}\int\limits_{\Omega} p(x,t+s(t))(1+s'(t))\bar{y}\,dxdt \\
&+\lambda_0\int\limits_Q pA\bar{y}\,dxdt + \lambda_0\int\limits_{-h(0)}^{0}\int\limits_{\Gamma} p(x,t+s(t))(1+s'(t))\bar{y}\,d\Gamma dt \\
&+\lambda_0\int\limits_{0}^{T-h(T)}\int\limits_{\Gamma} p(x,t+s(t))(1+s'(t))\bar{y}\,d\Gamma dt \\
&+\lambda_0\int\limits_0^T\int\limits_\Gamma p\bar{v}\,d\Gamma dt - \lambda_0\int\limits_{0}^{T-h(T)}\int\limits_{\Gamma}\frac{\partial p}{\partial\eta_{A^*}}\bar{y}d\Gamma dt - \lambda_0\int\limits_{t-h(T)}^{T}\int\limits_{\Gamma}\frac{\partial p}{\partial\eta_{A^*}}\bar{y}d\Gamma dt \\
&+\lambda_0\int\limits_{0}^{T-h(T)}\int\limits_{\Omega} p(x,t+s(t))(1+s'(t))\bar{y}\,dxdt = \lambda_0\int\limits_0^T\int\limits_\Gamma p\bar{v}\,d\Gamma dt. \qquad (39)
\end{aligned}$$

Substituting (39) into (33) gives

$$f_2'(\bar{v}) = \lambda_0\int\limits_0^T\int\limits_\Gamma (p + \lambda_2 N v^0)\bar{v}\,d\Gamma dt. \qquad (40)$$

Using the definition of the support functional [2] and dividing both members of the obtained inequality by λ_0, we finally get

$$\int\limits_0^T\int\limits_\Gamma (p + \lambda_2 N v^0)(v - v^0)\,d\Gamma dt \geqslant 0 \quad \forall v \in U_{ad}. \qquad (41)$$

The last inequality is equivalent to the maximum condition (22).

In order to prove the sufficiency of the derived conditions of the optimality, we use the fact that constraints and the performance functional are convex and that the Slater's condition is satisfied (Theorem 15.3 [2]). Then, there exists a point $(\tilde{y},\tilde{v}) \in \operatorname{int} Q_2$ such that $(\tilde{y},\tilde{v}) \in Q_1$.

This fact follows immediately from the existence of non-empty interior of the set Q_2 and from the existence of the solution of the equation (1)–(3) as well.

This last remark finishes the proof of Theorem 2.

One may also consider analogous optimal control problem with the performance functional

$$\widehat{I}(y,v) = \lambda_1 \int_{\Sigma} |\, y(v) \,|_{\Sigma} - z_{\Sigma d} \,|^2 \, d\Gamma dt + \lambda_2 \int_0^T \int_{\Gamma} (Nv)v \, d\Gamma \, dt, \tag{42}$$

where: $z_{\Sigma d}$ is a given element in $L^2(\Sigma)$.

From Theorem 1 [5] and the trace theorem ([10], vol.2, p.9) for such $v \in L^2(\Sigma)$, there exists a unique solution $H^{3/2,3/4}(Q)$ with $y\big|_{\Sigma} \in L^2(\Sigma)$. Thus $\widehat{I}(y,v)$ is well-defined. Then the solution of the formulated optimal control problem is equivalent to seeking a pair $(y^0, v^0) \in E = H^{3/2,3/4}(Q) \times L^2(\Sigma)$ that satisfies the equation (1)-(3) and minimizing the cost function (42) with the constraints on controls (11).

We can prove the following theorem:

Theorem 3 *The solution of the optimization problems (1)-(3), (42), (11) exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities:*
State equations (12)–(16).
Adjoint equations

$$-\frac{\partial p}{\partial t} + A^*(t)p + p(x,t+s(t))(1+s'(t)) = 0 \quad (x,t) \in \Omega \times (0, T-h(T)), \tag{43}$$

$$-\frac{\partial p}{\partial t} + A^*(t)p = 0 \quad (x,t) \in \Omega \times (T-h(T), T), \tag{44}$$

$$\frac{\partial p}{\partial \eta_{A^*}} = p(x,t+s(t))(1+s'(t)) + \lambda_1(y^0 - z_{\Sigma d}) \quad (x,t) \in \Gamma \times (0, T-h(T)), \tag{45}$$

$$\frac{\partial p}{\partial \eta_{A^*}} = \lambda_1(y^0 - z_{\Sigma d}) \quad (x,t) \in \Gamma \times (T-h(T), T), \tag{46}$$

$$p(x,T) = 0 \quad x \in \Omega. \tag{47}$$

Maximum condition

$$\int_0^T \int_{\Gamma} (p + \lambda_2 N v^0)(v - v^0) \, d\Gamma dt \geqslant 0 \qquad \forall v \in U_{ad}\,. \tag{48}$$

The idea of the proof of the Theorem 3 is the same as in the case of the Theorem 2.

Remark 1 *The coupled system (12)–(16) with (43)–(47) corresponds to the case of the observation on the boundary for the optimal control problem (1)–(3) with (11) and (42).*

Remark 2 *The existence of a unique solution for the adjoint problem (43)–(47) on the cylinder Q can be proved using a constructive method. It is easy to notice that for given z_{Σ_d} and v, the problem (43)–(47) can be solved backwards in time starting from $t = T$, i.e. first, solving (43)–(47) on subcylinder Q_K and in turn on Q_{K-1}, etc. until the procedure covers the whole cylinder Q. For this purpose, we may apply Theorem 1 (with an obvious change of variables) to the adjoint problem (43)–(47) (with reversed sense of time, i.e. $t' = T - t$). Then, for given $z_{\Sigma_d} \in L^2(\Sigma)$ and any $v \in L^2(\Sigma)$, there exists a unique solution $p(v) \in H^{3/2,3/4}(Q)$ for the adjoint problem (43)–(47).*

We must notice that the conditions of optimality derived above (Theorems 2 and 3) allow us to obtain an analytical formula for the optimal control in particular cases only (e.g. there are no constraints on boundary control). It results from the following: the determining of the function $p(x,t)$ in the maximum condition from the adjoint equation is possible if and only if we know that $y^0(x,t)$ will suit the control $v^0(x,t)$. These mutual connections make the practical use of the derived optimization formulas difficult. Thus we resign from the exact determining of the optimal control and we use approximation methods.

In the case of performance functionals (10) and (42) with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one [6–8] which can be solved by the use of the well-known algorithms, e.g. Gilbert's [1, 6–8] ones.

The practical application of Gilbert's algorithm to optimal control problem for a parabolic system with the boundary condition involving a time lag is presented in [8]. Using of the Gilbert's algorithm a one dimensional numerical example of the plasma control process is solved.

5. Conclusions and perspectives

The derived conditions of the optimality (Theorems 2 and 3) are original from the point of view of application of the Dubovicki-Milutin theorem for solving optimal control problems for parabolic systems in which time-varying lags appear both in the state equations and in the Neumann boundary conditions.

The proved optimization results (Theorems 2 and 3) constitute a novelty of the paper with respect to references [5–8] concerning application of the Lions scheme [9] for solving linear quadratic problems of optimal control for the case of the Neumann problem.

The results presented in the paper can be treated as a generalization of the results obtained in [4] to the case of additional time-varying lags appearing in the state equations.

Moreover, the optimization problems presented here constitute a generalization of optimal control problems considered in [3] to the case of parabolic systems with time-varying lags appearing in the state equations and in the Neumann boundary conditions simultaneously.

The obtained optimization theorems (Theorems 2 and 3) demand the assumption dealing with the non-empty interior of the set Q_2 representing the inequality constraints.

Therefore, we approximate the set Q_2 by the regular admissible cone (if $\text{int}Q_2 = \emptyset$, then this cone does not exist).

It is worth mentioning that the obtained results can be reinforced by omitting the assumption concerning the non-empty interior of the set Q_2 and utilizing the fact that the equality constraints in the form of the parabolic equations are "decoupling". The optimal control problem reduces to seeking $v^0 \in Q_2'$ and minimizing the performance index $I(v)$. Then we approximate the set Q_2' representing the inequality constraints by the regular tangent cone and for the performance index $I(v)$ we construct the regular improvement cone.

Making use of the Dubovicki-Milutin method the similar conditions of the optimality may be derived for a parabolic system with the Dirichlet boundary condition involving a time-varying lag.

One may also derive the necessary and sufficient conditions of optimality for parabolic system with more complex boundary conditions involving integral time lags.

Finally, one may consider more complex optimization problems with non-differentiable and non-continuous performance functionals.

According to the author similar optimal control problems can be solved for hyperbolic systems.

The ideas mentioned above will be developed in forthcoming papers.

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