SOME DIFFERENTIAL EQUATIONS OF ELASTICITY AND THEIR LIE POINT SYMMETRY GENERATORS

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Abstract: The formal models of physical systems are typically written in terms of differential equations. A transformation of the variables in a differential equation forms a symmetry group if it leaves the differential equation invariant. Symmetries of differential equations are very important for understanding of their properties. It can be said that the theory of Lie group symmetries of differential equations is general systematic method for finding solutions of differential equations. Despite of this fact, the Lie group theory is relatively unknown in engineering community. The paper is devoted to some important questions concerning this theory and for several equations resulting from the theory of elasticity their Lie group infinitesimal generators are given.

Key words: Lie Groups, Symmetry, Differential Equations, Elasticity, Group Generator

1. INTRODUCTION

The group theory was discovered by Évariste Galois, who applied it to study of polynomial equations. The so-called finite groups were used as permutation groups and later the symmetry groups were applied in geometry. The idea of continuous groups was first used by Norwegian mathematician Sophus Lie for description of properties of differential equations. The application of continuous groups started systematically in his works and the terms Lie group and Lie algebra are used in honor of this great mathematician. The theory of Lie groups is applied in many different areas of mathematics, physics and engineering (Azad et al., 2010; Drew and Kloster, 1989; Olver, 1986; Sansour and Bednarczyk, 1995; Schwarz, 1982, 1984, 1988; Simo and Fox, 1989). In the field of physics work of German mathematician Emmy Noether, who found connection between symmetries of differential equations and the conservation laws, is very known. The paper describes Lie theory of symmetries of differential equations and for some equations resulting from the elasticity theory infinitesimal generators of their Lie groups are given. The generators have been found by program for symbolic manipulation. More details concerning Lie symmetries of differential equations can be found in Drew and Kloster (1989); Euler and Steeb (1992), Head (1993, 1996), Sansour and Bufler (1992).

2. DIFFERENTIAL EQUATIONS AND LIE POINT GROUPS

Starting point is the description of the system of partial differential equations:

$$W_v(x_i, u^a, u_i^a, u_{ij}^a, ..., u_{i_1,...,i_k}^a) = 0 \tag{1}$$

$v = 1, ..., N$, where $\mathbf{u} = (u^1, ..., u^m)$ are the functions depending on independent variables $\mathbf{x} = (x_1, ..., x_n)$ and:

$$u_{i_1,...,i_n}^\alpha = \frac{\partial^{i_1 + ... + i_n} u^\alpha}{\partial x_1^{i_1} ... \partial x_n^{i_n}} \tag{2}$$

are the partial derivatives.

Shortly, a Lie group is a group and a manifold at the same time. For any two points a and b in the manifold, there exists multiplication operation giving ab and this group operation has a continuous structure of the manifold.

Change of independent and dependent variables can be represented by the finite transformations:

$$\begin{aligned} \overline{x}_i &= \phi_i(\mathbf{x}, \mathbf{u}, \varepsilon) \\ \overline{u}^\alpha &= \psi^\alpha(\mathbf{x}, \mathbf{u}, \varepsilon) \end{aligned} \tag{3}$$

where $\varepsilon \in \mathbb{R}$ is a parameter of a group. Expanding (3) by Taylor series at $\varepsilon = 0$ gives relations:

$$\begin{aligned} \overline{x}_i &= x_i + \varepsilon \xi_i(\mathbf{x}, \mathbf{u}) + o(\varepsilon^2) \\ \overline{u}^\alpha &= u^\alpha + \varepsilon \eta^\alpha(\mathbf{x}, \mathbf{u}) + o(\varepsilon^2) \end{aligned} \tag{4}$$

with the substitutions:

$$\begin{aligned} \xi_i &= \left. \frac{\partial \phi_i(\mathbf{x}, \mathbf{u}, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} \\ \eta^\alpha &= \left. \frac{\partial \psi^\alpha(\mathbf{x}, \mathbf{u}, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} \end{aligned} \tag{5}$$

The value $\varepsilon = 0$ is the identity element of the group. Equations (5) allow us to write infinitesimal generator, or Lie point symmetry vector field by relation:

$$\mathbf{U} = \xi_i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \tag{6}$$

The transformation of partial derivatives influences the so-called k-th prolongation of a vector field U:

$$\mathbf{U}^{(k)} = \mathbf{U} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + ... + \zeta_{i_1...i_k}^\alpha \frac{\partial}{\partial u_{i_1...i_k}^\alpha} \tag{7}$$

where the functions $\xi^{\alpha}_{i_1\ldots i_k}$ describe the transformations of partial derivatives of order k. The functions $\xi^{\alpha}_{i_1\ldots i_k}$ are determined according to relations:

$$\zeta^{\alpha}_i = D_i\left(\eta^{\alpha}\right) - u^{\alpha}_s D_i\left(\xi_s\right) \tag{8}$$

and:

$$\zeta^{\alpha}_{i_1\ldots i_k} = D_{i_k}\left(\zeta^{\alpha}_{i_1\ldots i_{k-1}}\right) - u^{\alpha}_{i_1\ldots i_{k-1}},{}_s D_{i_k}\left(\xi_s\right) \tag{9}$$

where:

$$D_i = \frac{\partial}{\partial x_i} + u^{\alpha}_i \frac{\partial}{\partial u^{\alpha}} + u^{\alpha}_{ki} \frac{\partial}{\partial u^{\alpha}_k} + u^{\alpha}_{kli} \frac{\partial}{\partial u^{\alpha}_{kl}} \ldots \tag{10}$$

is the operator of total differentiation with respect to the independent variable x_i. The group is symmetry of equation system (2) if and only if the invariant surface condition (symmetry condition):

$$\mathbf{U}^{(k)}\Omega_{v} = 0 \tag{11}$$

is satisfied, where all $\Omega_v = 0$, $v = 1,\ldots,N$. The components ξ_i and η^{α} of the infinitesimal generator U are determined from equations (11).

The infinitesimal generator and Lie group are connected by relation:

$$\left(\bar{\mathbf{x}},\bar{\mathbf{u}}\right) = g_{\varepsilon}.\left(\mathbf{x},\mathbf{u}\right) = \left(\boldsymbol{\phi}_{\varepsilon}\left(\mathbf{x},\mathbf{u}\right),\boldsymbol{\psi}_{\varepsilon}\left(\mathbf{x},\mathbf{u}\right)\right) = e^{\varepsilon\mathbf{U}}\left(\mathbf{x},\mathbf{u}\right) \tag{12}$$

Here, the Greek letter ε is reserved for the parameter of the group and accordingly it does not represent partial differentiation. The function $\mathbf{f}$ undergoes the transformation by the group element g_{ε} in accordance with relation:

$$\bar{\mathbf{u}} = \bar{\mathbf{f}}_{\varepsilon}\left(\bar{\mathbf{x}}\right) = \left(g_{\varepsilon}\mathbf{f}\right)\left(\bar{\mathbf{x}}\right) = \left[\boldsymbol{\psi}_{\varepsilon}\circ\left(\mathbf{1}\times\mathbf{f}\right)\right]\circ\left[\boldsymbol{\phi}_{\varepsilon}\circ\left(\mathbf{1}\times\mathbf{f}\right)\right]^{-1}\left(\bar{\mathbf{x}}\right) \tag{13}$$

where $\mathbf{1}$ represents the identity function $\mathbf{1}(\mathbf{x}) = \mathbf{x}$.

Calculation of the Lie vector fields from equations (11) is tedious work. It involves a large amount of symbolic calculations that is better done by computer. Fortunately, different packages in computer algebra systems exist implementing Lie symmetry computations (Champagne et al., 1991; Lie, 1891, 1896; Vu et al., 2012).

3. THICK-WALLED PIPE

Thick-walled pressure vessels and pipes have many applications in engineering practice. Differential equation describing radial stress in a pipe is:

$$\frac{d^2\sigma\left(r\right)}{\partial r^2} + \frac{3}{r}\frac{d\sigma\left(r\right)}{dr} = 0 \tag{14}$$

where $\sigma(r)$ is the radial stress and r is the independent variable representing radius. As was mentioned above, there are number programs for solving determining equations of the vector fields resulting from equation (11). They work under different systems for symbolic manipulations, e.g. Reduce, Mathematica, Maple, and so on (Champagne et al., 1991; Lie, 1891, 1986; Vu et al., 2012). In our case we have used DESOLVII [19] working under

system Maple. For equation (14) the program DESOLVII gives us the following Lie symmetry vector fields:

$$\mathbf{U}_1 = -2\sigma^2 \frac{\partial}{\partial\sigma} + \sigma\,r\frac{\partial}{\partial r}$$

$$\mathbf{U}_2 = \frac{1}{r^2}\frac{\partial}{\partial\sigma}$$

$$\mathbf{U}_3 = r\frac{\partial}{\partial r}$$

$$\mathbf{U}_4 = r^3\frac{\partial}{\partial r}$$

$$\mathbf{U}_5 = -\frac{2\sigma}{r^2}\frac{\partial}{\partial\sigma} + \frac{1}{r}\frac{\partial}{\partial r} \tag{15}$$

$$\mathbf{U}_6 = \sigma\frac{\partial}{\partial\sigma}$$

$$\mathbf{U}_7 = \sigma\,r^3\frac{\partial}{\partial r}$$

$$\mathbf{U}_8 = \frac{\partial}{\partial\sigma}$$

These vector fields are infinitesimal generators of Lie groups of symmetries of differential equation describing radial stress in thick-walled pipe. The corresponding Lie groups can be established from vector fields according to equation (12).

4. AXISYMMETRIC PLATE

Differential equation for deformation of axisymmetric plate loaded by uniformly distributed load can be written as:

$$\frac{d^3w\left(r\right)}{dr^3} + \frac{1}{r}\frac{d^2w\left(r\right)}{dr^2} - \frac{1}{r^2}\frac{dw\left(r\right)}{dr} = Q \tag{16}$$

where $w(r)$ is the deflection of the plate at the radius r, the constant $Q = \frac{T}{D}$ depends on the constant uniformly distributed load T and the constant plate stiffness D. The plate stiffness is

$$D = \frac{E\,h^3}{12(1-v^2)} \tag{17}$$

where E and v is Young modulus and Poisson ratio of plate material, respectively; h is the plate thickness.

Here again the program DESOLVII has been used for solution of determining equations that correspond to the equation (16). Resulting infinitesimal generators are:

$$\mathbf{U}_1 = Q\,r^3\frac{\partial}{\partial w} + 3\,r\frac{\partial}{\partial r}$$

$$\mathbf{U}_2 = r^2\frac{\partial}{\partial w}$$

$$\mathbf{U}_3 = \left(Q\,r^3 - 9w\right)\frac{\partial}{\partial w} \tag{18}$$

$$\mathbf{U}_4 = \ln\left(r\right)\frac{\partial}{\partial w}$$

$$\mathbf{U}_5 = \frac{\partial}{\partial w}$$

5. VIBRATING BEAM WITH DAMPING

Differential equation for vibration of a beam with damping is:

$$\frac{\partial^4 u(x,t)}{\partial x^4} + \frac{k^4}{\alpha^2}\frac{\partial^2 u(x,t)}{\partial t^2} + \mu\frac{\partial^5 u(x,t)}{\partial x^4 \partial t} = 0 \qquad (19)$$

where $u(x,t)$ is the deflection of beam at the position x and the time instant t. Constant $\frac{k^4}{\alpha^2} = \frac{\rho A}{EJ}$ depends on the material density ρ, Young modulus E, the cross-section area A and moment of inertia of the beam's cross-section J. μ is the coefficient of internal damping of material.

Determining equations (11) for differential equation (19) have been solved by program DESOLVII. Resulting infinitesimal generators are:

$$\mathbf{U}_1 = u\frac{\partial}{\partial u}$$

$$\mathbf{U}_2 = \frac{\partial}{\partial t}$$

$$\mathbf{U}_3 = \frac{\partial}{\partial x} \qquad (20)$$

$$\mathbf{U}_4 = f(x,t)\frac{\partial}{\partial u}$$

Here, function $F(x,t)$ is any solution of equation (19). The vector field U_2 represents simple shifting in time, the vector U_3 shifting in the direction x.

6. MEMBRANE

Differential equation of a stretched membrane in the plane x, y which is loaded by the constant pressure p can be written in the form:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{p}{S} \qquad (21)$$

where $w(x,y)$ is the deflection of a membrane perpendicular to the membrane plane and S is the tension force per unit length of the membrane. Here, the infinitesimal generators of equation (21) are:

$$\mathbf{U}_1 = w\frac{\partial}{\partial w}$$

$$\mathbf{U}_2 = g_1(x,y)\frac{\partial}{\partial x}$$

$$\mathbf{U}_3 = g_2(x,y)\frac{\partial}{\partial y} \qquad (22)$$

$$\mathbf{U}_4 = g_3(x,y)\frac{\partial}{\partial w}$$

The functions $g_1(x,y)$, $g_2(x,y)$, $g_3(x,y)$ represent any solution of differential equation (21).

7. PLATE ON ELASTIC FUSS-WINKLER FOUNDATION

Differential equation describing deformation of a plate on elastic Fuss-Winkler foundation is:

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} - \frac{\partial^4 w}{\partial y^4}\right) + K\,w(x,y) = p(x,y) \qquad (23)$$

where $w(x,y)$ is the deflection of the plate at the point with the coordinates x, y, constant D is the plate stiffness given by equation (17), K is the coefficient of subgrade reaction and $p(x,y)$ is the pressure acting on the plate. The Lie group generators of given differential equation are:

$$\mathbf{U}_1 = \frac{\partial}{\partial x}$$

$$\mathbf{U}_2 = \frac{\partial}{\partial y}$$

$$\mathbf{U}_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad (24)$$

$$\mathbf{U}_4 = w\frac{\partial}{\partial w}$$

$$\mathbf{U}_5 = h(x,y)\frac{\partial}{\partial w}$$

Here, $h(x,y)$ is any solution of differential equation (23). The vector fields U_1, U_2 represent simple shifting along the coordinate x and y respectively.

Let us now compute less trivial example of transformation that belong to the Lie vector:

$$\mathbf{U}_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad (25)$$

For the transformation of the independent variable x we have:

$$\mathbf{U}_3(x) = -y$$
$$\mathbf{U}_3^2(x) = -\mathbf{U}_3(y) = -x$$
$$\mathbf{U}_3^3(x) = -\mathbf{U}_3(x) = y$$
$$\mathbf{U}_3^4(x) = \mathbf{U}_3(y) = x \qquad (26)$$
$$\vdots$$

$$\bar{x} = x + \varepsilon(-y) + \frac{\varepsilon^2}{2!}(-x) + \frac{\varepsilon^3}{3!}(y) + \frac{\varepsilon^4}{4!}(x) + \ldots =$$
$$x\cos\varepsilon - y\sin\varepsilon$$

For the independent variable y we have similar relation:

$$\mathbf{U}_3(y) = x$$
$$\mathbf{U}_3^2(y) = -y$$
$$\mathbf{U}_3^3(y) = -x \qquad (27)$$
$$\mathbf{U}_3^4(y) = y$$
$$\vdots$$

$$\bar{y} = y + \varepsilon(x) + \frac{\varepsilon^2}{2!}(-y) + \frac{\varepsilon^3}{3!}(-x) + \frac{\varepsilon^4}{4!}(y) + \ldots =$$
$$y\cos\varepsilon + x\sin\varepsilon$$

We see that vector U_3 is connected with the rotation of variables in the plain xy.

All transformation groups that belong to vectors (24) are:

$$G_1 : (x, y, w) \mapsto (x + \varepsilon, y, w)$$

$$G_2 : (x, y, w) \mapsto (x, y + \varepsilon, w)$$

$$G_3 : (x, y, w) \mapsto (x \cos \varepsilon - y \sin \varepsilon, x \sin \varepsilon + y \cos \varepsilon, w) \qquad (28)$$

$$G_4 : (x, y, w) \mapsto (x, y, e^{\varepsilon} w)$$

$$G_5 : (x, y, w) \mapsto (x, y, w + \varepsilon h(x, y))$$

The groups (28) represent transformations that convert solutions of differential equation (23) into new solutions of the same equation.

8. CONCLUSIONS

The notion of Lie group is very important in the current mathematics and physics. The paper analyzes differential equations resulting from different branches of elasticity theory from the point of view of their symmetries. Lie vectors of corresponding Lie groups symmetries of differential equations have been computed by computer program DESOLVII. Infinitesimal generators of Lie group symmetries give us additional information that is not visible during classical solutions of differential equations.

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