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NUMBER THEORY

Standard Models of Abstract Intersection Theory for Operators in Hilbert Space

by

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Summary. For an operator in a possibly infinite-dimensional Hilbert space of a certain class, we set down axioms of an abstract intersection theory, from which the Riemann hypothesis regarding the spectrum of that operator follows. In our previous paper (2011) we constructed a GNS (Gelfand–Naimark–Segal) model of abstract intersection theory. In this paper we propose another model, which we call a standard model of abstract intersection theory. We show that there is a standard model of abstract intersection theory for a given operator if and only if the Riemann hypothesis and semisimplicity hold for that operator. (For the definition of semisimplicity of an operator in Hilbert space, see the Introduction.) We show this result under a condition for a given operator which is much weaker than the condition in the previous paper. An operator satisfying this condition can be constructed by using the method of automorphic scattering of Uetake (2009).

Combining this with a result from Uetake (2009), we can show that a Dirichlet L-function, including the Riemann zeta-function, satisfies the Riemann hypothesis and its all nontrivial zeros are simple if and only if there is a corresponding standard model of abstract intersection theory. Similar results can be proven for GNS models since the same technique of proof for standard models can be applied.

1. Introduction. In the 1940s Weil [W1] developed an intersection theory on surfaces over finite fields to apply it to the proof of the Riemann hypothesis for curves over finite fields.

In this paper we introduce axioms ((AIT1)–(AIT3) in §3) of abstract intersection theory for an operator in a Hilbert space. These axioms are analogous to the properties of Weil's intersection theory on surfaces. We consider a collection $\mathbb{AIT}$ that consists of a vector space, distinguished vectors and corresponding maps, satisfying the axioms (AIT1)–(AIT3). From

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this collection we can derive the Riemann hypothesis regarding the spectrum of that operator. Therefore we call $\mathbb{AIT}$ an abstract intersection theory.

Let H be a $\mathbb{C}$-Hilbert space. Let $A\colon H \supset \mathrm{dom}(A) \to H$ be a $\mathbb{C}$-linear operator (possibly unbounded) where $\mathrm{dom}(A)$ is the domain of A. We assume that the spectrum $\sigma(A)$ of A consists only of the point spectrum $\sigma_p(A)$.

Definition 1.1. We say that the operator A *satisfies the Riemann hypothesis* (briefly, RH) if

$$\mathrm{Re}(s_i) = \frac{1}{2} \quad \text{for all } s_i \in \sigma(A) = \sigma_p(A).$$

We say that the operator A is *semisimple* if

$$\nu(s_i) = 1 \quad \text{for all } s_i \in \sigma(A) = \sigma_p(A).$$

Here $\nu(s_i)$ is the Riesz index of s_i. For its definition see the paragraph preceding the conditions (OP1)–(OP5) in §2, which A is assumed to satisfy. The conditions (OP1)–(OP5) are satisfied by the operator A obtained from automorphic scattering theory [U], which gives a spectral interpretation of certain Dirichlet L-functions, including the Riemann zeta-function. See Remark 2.1(4) in §2.

In our previous work [BU], we showed $\mathbb{AIT} \Rightarrow$ RH. We also constructed a model $\mathbb{AIT}_{\mathrm{GNS}}$ of abstract intersection theory based on an analogue of the GNS (Gelfand–Naimark–Segal) representation. We call $\mathbb{AIT}_{\mathrm{GNS}}$ a GNS model of abstract intersection theory. We showed that $\mathbb{AIT}_{\mathrm{GNS}} \Leftrightarrow$ RH, assuming the semisimplicity of A [BU, Theorem 3.1].

We observe that there is some flexibility in constructing models of abstract intersection theory to investigate the spectrum of operators in Hilbert space and nontrivial zeros of corresponding Dirichlet L-functions. In this paper we propose a new model $\mathbb{AIT}_m$, which we call a standard model of abstract intersection theory. This model is inspired by the Künneth formula for ℓ-adic cohomology. For this model we show $\mathbb{AIT}_m \Leftrightarrow$ RH & semisimplicity (Theorem 5.2(2)). It is worth pointing out that the assumptions (OP1), (OP2), (OP3-a) and (OP4) on the operator A imply the existence of a standard model of abstract intersection theory which satisfies (AIT1-a)–(AIT1-f), (AIT2) and (AIT3) (see Lemmas 4.1 and 4.2). To prove the RH and semisimplicity of the operator A, we need to add (AIT1-g).

The techniques for proving our results concerning $\mathbb{AIT}_m$ can also be applied to $\mathbb{AIT}_{\mathrm{GNS}}$ from our previous paper [BU]. Namely, we can show $\mathbb{AIT}_{\mathrm{GNS}} \Leftrightarrow$ RH & semisimplicity (Theorem 5.3). Therefore we significantly strengthen our previous results in [BU] for both GNS and standard models, dropping the semisimplicity assumption (the condition (OP3-b) in [BU]). The condition (OP3-b) in this paper is much weaker and is satisfied by operators coming from scattering theory for Dirichlet L-functions [U].

As a corollary of Theorems 5.2 and 5.3 combined with the results of [U] on automorphic scattering, we observe that a Dirichlet L-function, including the Riemann zeta-function, satisfies the RH and its nontrivial zeros are all simple if and only if there is a corresponding standard model $\mathbb{AIT}_m$ (or GNS model $\mathbb{AIT}_{\mathrm{GNS}}$) of abstract intersection theory (Theorem 5.4).

The plan of this paper is as follows.

In §2 we define an analogue of the classical Frobenius morphism for the operator A. The spectrum of this analogue is similar to that of the classical Frobenius morphism if the operator A satisfies the Riemann hypothesis. The introduction of this analogue is also inspired by Weil's explicit formulas [W2].

In §3 we introduce a general notion of abstract intersection theory $\mathbb{AIT}$ and set down its axioms ((AIT1), (AIT2) and (AIT3)).

In §4 we construct a specific example of abstract intersection theory, which we call a standard model $\mathbb{AIT}_m$, using the analogy with the classical Künneth formula for ℓ-adic cohomology.

In §5 we state our main theorems (Theorems 5.2, 5.3 and 5.4).

In §6 we show that there is a strong analogy between Weil's approach to zeta-functions for curves over finite fields and our approach to Dirichlet L-functions. For Weil's intersection theory, see also Grothendieck [Gro], Monsky [Mon] and Serre [S].

We should note that there is a program by Connes and Marcolli (and Consani) [CM] to adapt Weil's proof of RH for function fields to the case of number fields. See also Connes [C]. There is also a conjectural cohomological approach by Deninger [D1, D2] toward the interpretation of Hasse–Weil L-functions in analogy with the étale cohomology interpretation of L-functions of varieties over finite fields.

2. An analogue of the Frobenius morphism for the operator A. Let H be a possibly infinite-dimensional $\mathbb{C}$-Hilbert space. If H is infinite-dimensional we assume that H is separable. Let $A\colon H \supset \mathrm{dom}(A) \to H$ be a possibly unbounded operator on H.

If $s_i \in \sigma(A)$ is an isolated spectrum point, one can take a small enough bounded domain Δ of $\mathbb{C}$ such that $\{s_i\} \Subset \Delta$ (i.e. $\{s_i\} \subset \Delta^\circ$) and $\overline{\Delta} \cap (\sigma(A) - \{s_i\}) = \emptyset$. If A is a closed operator then one can define the *Riesz projection* $P_{\{s_i\}}\colon H \to H$ by

$$P_{\{s_i\}} := \frac{1}{2\pi i} \oint_{\partial\Delta} (sI - A)^{-1}\, ds.$$

Here $I\colon H \to H$ is the identity operator on H. Then $P_{\{s_i\}}$ is a bounded operator on H.

For $s_i \in \sigma_p(A)$, the *Riesz index* $\nu(s_i)$ of s_i is defined as the smallest positive number $\leq \infty$ such that

$$\mathrm{Ker}((s_i I - A)^{\nu(s_i)}) = \mathrm{Image}(P_{\{s_i\}}).$$

Let $\mathrm{mult}(s_i) := \dim_{\mathbb{C}} \mathrm{Image}(P_{\{s_i\}})$, which we call the (algebraic) *multiplicity* of $s_i \in \sigma_p(A)$.

We assume the following properties of A:

(OP1) A is closed.

(OP2) The spectrum $\sigma(A)$ consists only of the point spectrum (i.e. eigenvalues) $\sigma_p(A)$ (i.e. $\sigma(A) = \sigma_p(A)$), which accumulates at most at infinity.

(OP3) (a) $\mathrm{Image}(P_{\{s_i\}})$ is finite-dimensional (i.e. $\mathrm{mult}(s_i) < \infty$) for any $s_i \in \sigma_p(A)$.
(b) $\nu(s_i) = \mathrm{mult}(s_i)$ for any $s_i \in \sigma_p(A)$.

(OP4) $\sigma(A) \subset \Omega_\infty$, where $\Omega_\infty := \{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\}$.

(OP5) $\mathrm{Re}(s_i) < 1/2$ for some $s_i \in \sigma(A)$ if and only if there is $s_j \in \sigma(A)$ such that $\mathrm{Re}(s_j) > 1/2$.

REMARK 2.1. (1) (OP1) is needed when one applies Lemma 2.1 of [BU] on spectral decomposition. The lemma is taken from Gohberg, Goldberg and Kaashoek [GoGK, XV.2, Theorem 2.1, p. 326].

(2) In [BU], the condition (OP3-b) was the semisimplicity $\nu(s_i) = 1$ ($s_i \in \sigma(A)$). The above stated (OP3-b) is a much weaker condition: it says that each eigenvalue of A has just one corresponding Jordan block. Actually this is satisfied in the construction using automorphic scattering theory [U]. See Remark 2.1(4) below.

(3) The above (OP5) is (OP5-a) in [BU]. (OP5-b) of [BU], which is necessary for the construction of GNS models of abstract intersection theory, is not necessary for the construction of standard models in this paper. It is used there to keep the space V an $\mathbb{R}$-linear space in the GNS model. In the standard model we apply the complexification $V_{\mathbb{C}}$ of V instead (see §3). (OP5-b) in [BU] is satisfied by an operator A constructed in [U] (see Remark 2.1(4) below).

(4) Let $\Gamma(N)$ be the principal congruence subgroup of level N such that the modular curve $\Gamma(N)\backslash\mathbb{H} \simeq \Gamma(N)\backslash SL_2(\mathbb{R})/SO(2)$ is noncompact and has one cusp at $i\infty$. Here $\mathbb{H}$ denotes the upper half-plane. Then the scattering matrix in the functional equation of the Eisenstein series essentially contains the Dirichlet L-function $L(s,\chi)$ for a Dirichlet character $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. See Huxley [Hux]. In [U] the second author constructed a scattering theory for automorphic forms on $\Gamma(N)\backslash\mathbb{H}$. Furthermore he constructed an operator A satisfying (OP1)–(OP5) whose (point) spectrum coincides with the nontrivial zeros of the Dirichlet L-function $L(s,\chi)$, counted with multiplicity. That is, $s_i \in \sigma(A)$ $(= \sigma_p(A))$ and $\nu(s_i)$ $(= \mathrm{mult}(s_i)) = m_i$ if and only if s_i

is a nontrivial zero of $L(s,\chi)$ of order m_i. We call $s_i \in \mathbb{C}$ a *nontrivial* zero of the Dirichlet L-function $L(s,\chi)$ if $L(s_i,\chi) = 0$ and $0 < \mathrm{Re}(s_i) < 1$. In automorphic scattering of [U], assumptions (OP1)–(OP5) are consequences of Theorem 4.1 of [U, p. 455]. More specifically, (OP1)–(OP5) are deduced as follows: (OP1) follows from Theorem 4.1(i); (OP2) follows from Theorem 4.1(iii-a); (OP3) follows from Theorem 4.1(iii-b) and (iii-c); (OP4) and (OP5) follow from Theorem 4.1(iv). (OP5-b) in [BU] also follows from Theorem 4.1(iv).

The theory of automorphic scattering was initiated by Pavlov–Faddeev [PavF] and then Lax–Phillips [LP], inspired by Gelfand [Ge]. ∎

Now for $Y > 0$ let

$$\sigma_Y(A) := \{s \in \sigma(A) \mid |\mathrm{Im}(s)| < Y\}.$$

Note that $\sigma_Y(A)$ is a finite set by (OP2) and (OP4). Let the parameter space $\mathcal{Y}$ be defined by

$$\mathcal{Y} := \{Y > 0 \mid \sigma_Y(A) \neq \emptyset\} - \{|\mathrm{Im}(s)| \mid s \in \sigma(A)\}.$$

Fix a function

$$q\colon \mathcal{Y} \to (0,1) \cup (1,\infty).$$

Let $B(X)$ denote the set of bounded operators on a $\mathbb{C}$-Hilbert space X. By definition $T\colon X \supset \mathrm{dom}(T) \to X$ is a bounded operator if $\mathrm{dom}(T) = X$ and the operator norm $\|T\|$ is finite.

Let Σ_H be the set of closed subspaces of H. We will construct maps

$$F_A\colon \mathcal{Y} \to B(H) \quad \text{and} \quad \mathcal{H}\colon \mathcal{Y} \to \Sigma_H$$

such that $F_A(Y)\colon H \to H$ satisfies the following conditions for each $Y \in \mathcal{Y}$:

$$\text{(Frob-a)} \qquad F_A(Y)\mathcal{H}(Y) \subset \mathcal{H}(Y)$$

(i.e. the subspace $\mathcal{H}(Y)$ is invariant for $F_A(Y)$), and

$$\text{(Frob-b)} \quad \begin{cases} \sigma(F_A(Y)|_{\mathcal{H}(Y)}) = \sigma_p(F_A(Y)|_{\mathcal{H}(Y)}) = \{q(Y)^s \mid s \in \sigma_Y(A)\} \\ \qquad\qquad\qquad\qquad \text{(counted with algebraic multplicities)}, \\ \sigma(F_A(Y)) = \sigma_p(F_A(Y)) = \sigma(F_A(Y)|_{\mathcal{H}(Y)}) \cup \{0\}. \end{cases}$$

Note that $\sigma(F_A(Y)|_{\mathcal{H}(Y)})$ is a finite set counted with algebraic multiplicities by (OP1), (OP2), (OP3-a) and (OP4).

The operator $F_A(Y)$ ($Y \in \mathcal{Y}$) is considered to be an analogue of the classical Frobenius morphism, since the spectrum of this analogue is similar to that of the classical Frobenius morphism if the operator A satisfies the Riemann hypothesis. It is also motivated by the spectral side of Weil's explicit formulas [W2] (see §6).

Models $F_{A,m}$ and $\mathcal{H}_m$ of F_A and $\mathcal{H}$. Now we construct models $F_{A,m}\colon \mathcal{Y} \to B(H)$ and $\mathcal{H}_m\colon \mathcal{Y} \to \Sigma_H$ which satisfy (Frob-a) and (Frob-b). These models will constitute parts of a standard model $\mathbb{AIT}_m$ constructed in §4. Let

$$\Omega_Y := \{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1,\ |\mathrm{Im}(s)| < Y\}$$

for $Y \in \mathcal{Y}$. Note that $\Omega_Y \cap \sigma(A) = \sigma_Y(A)$ for $Y \in \mathcal{Y}$ by (OP4). Note also that by (OP2), (OP4) and the definition of $\mathcal{Y}$ we have, for each $Y \in \mathcal{Y}$,

$$\sigma_Y(A) \Subset \Omega_Y \quad (\text{i.e. } \overline{\sigma_Y(A)} = \sigma_Y(A) \subset \Omega_Y^\circ = \Omega_Y),$$
$$\overline{\Omega_Y} \cap (\sigma(A) - \sigma_Y(A)) = \emptyset.$$

Therefore, since we have (OP1), the Riesz projection $P_{\sigma_Y(A)}\colon H \to H$ can be well-defined for $Y \in \mathcal{Y}$ by

$$P_{\sigma_Y(A)} := \frac{1}{2\pi i} \oint_{\partial\Omega_Y} (sI - A)^{-1}\, ds.$$

$P_{\sigma_Y(A)}$ is a bounded operator on H. Let $\mathcal{H}_m\colon \mathcal{Y} \to \Sigma_H$ be defined by

$$\mathcal{H}_m(Y) := \mathrm{Image}(P_{\sigma_Y(A)}).$$

By (OP2) and (OP3-a), $\mathcal{H}_m(Y)$ is finite-dimensional for each $Y \in \mathcal{Y}$.

Given $Y \in \mathcal{Y}$, let

$$F_{A,m}(Y)\colon H \supset \mathrm{dom}(F_{A,m}(Y)) \to H$$

be defined by

$$F_{A,m}(Y)x := \frac{1}{2\pi i}\Big(\oint_{\partial\Omega_Y} q(Y)^s (sI - A)^{-1}\, ds\Big)x$$

for

$$x \in \mathrm{dom}(F_{A,m}(Y)) := \{x \in H \mid F_{A,m}(Y)x \text{ exists in } H\}.$$

Note that $\sigma_Y(A)$ is a bounded set. Thus, by (OP1) we can apply [BU, Lemma 2.1] to have $\mathcal{H}_m(Y) \subset \mathrm{dom}(A)$ and $A\mathcal{H}_m(Y) \subset \mathcal{H}_m(Y)$. Let $A|_{\mathcal{H}_m(Y)}\colon \mathcal{H}_m(Y) \to \mathcal{H}_m(Y) \subset H$ be the restriction of A to $\mathcal{H}_m(Y)$. Since $\mathcal{H}_m(Y)$ is finite-dimensional, $A|_{\mathcal{H}_m(Y)}$ is a bounded operator, i.e. $A|_{\mathcal{H}_m(Y)} \in B(\mathcal{H}_m(Y))$.

Similarly, let $H(s_i) := \mathrm{Image}(P_{\{s_i\}})$. By (OP2) and (OP3-a), $H(s_i)$ is finite-dimensional. Again by [BU, Lemma 2.1], $H(s_i) \subset \mathrm{dom}(A)$ and $AH(s_i) \subset H(s_i)$. Let $A|_{H(s_i)}\colon H(s_i) \to H(s_i) \subset H$ be the restriction of A to $H(s_i)$. By the same argument for $A|_{\mathcal{H}_m(Y)}$, we have $A|_{H(s_i)} \in B(H(s_i))$.

LEMMA 2.1. *Suppose that A satisfies* (OP1), (OP2), (OP3-a) *and* (OP4). *Then*:

(i) *For each $Y \in \mathcal{Y}$,*

$$\mathrm{dom}(F_{A,m}(Y)) = H.$$

Furthermore, $F_{A,m}(Y)$ is a bounded operator on H, i.e. $F_{A,m}(Y) \in B(H)$.

(ii) *The subspace $\mathcal{H}_m(Y)$ is $F_{A,m}(Y)$-invariant:*

$$F_{A,m}(Y)\mathcal{H}_m(Y) \subset \mathcal{H}_m(Y).$$

That is, $F_{A,m}$ satisfies (Frob-a).

(iii) *For each $Y \in \mathcal{Y}$, we have*

$$\sigma(F_{A,m}(Y)|_{\mathcal{H}_m(Y)}) = \sigma_p(F_{A,m}(Y)|_{\mathcal{H}_m(Y)}) = \{q(Y)^s \mid s \in \sigma_Y(A)\}$$

(*counted with algebraic multplicities*) *and*

$$\sigma(F_{A,m}(Y)) = \sigma_p(F_{A,m}(Y)) = \sigma(F_{A,m}(Y)|_{\mathcal{H}_m(Y)}) \cup \{0\}.$$

That is, $F_{A,m}$ satisfies (Frob-b).

(iv) *Let $t\colon \mathcal{Y} \to \mathbb{R}-\{0\}$ be defined by $t(Y) := \log q(Y)$ (i.e. $e^{t(Y)} = q(Y)$) for $Y \in \mathcal{Y}$. For each $Y \in \mathcal{Y}$, we have*

$$\begin{aligned} F_{A,m}(Y) &= e^{t(Y)A|_{\mathcal{H}_m(Y)}} P_{\sigma_Y(A)} \\ &= \sum_{n=0}^{\infty} \frac{t(Y)^n}{n!} A|^n_{\mathcal{H}_m(Y)} P_{\sigma_Y(A)} = \sum_{s_i \in \sigma_Y(A)} e^{t(Y)A|_{H(s_i)}} P_{\{s_i\}}. \end{aligned}$$

(v) *Suppose further that A satisfies* (OP3-b). *Then, with respect to an appropriate basis of $H(s_i)$, $e^{t(Y)A|_{H(s_i)}}$ is written as*

$$e^{t(Y)A|_{H(s_i)}} = N(s_i)$$

with $N(s_i) \in M_{m_i}(\mathbb{C})$ given by

$$\begin{pmatrix} \frac{t(Y)^0 e^{t(Y)s_i}}{0!} & \frac{t(Y)^1 e^{t(Y)s_i}}{1!} & \cdots & \cdots & \frac{t(Y)^{m_i-1} e^{t(Y)s_i}}{(m_i-1)!} \\ & \frac{t(Y)^0 e^{t(Y)s_i}}{0!} & \frac{t(Y)^1 e^{t(Y)s_i}}{1!} & \cdots & \frac{t(Y)^{m_i-2} e^{t(Y)s_i}}{(m_i-2)!} \\ & & \ddots & \ddots & \vdots \\ & & & \frac{t(Y)^0 e^{t(Y)s_i}}{0!} & \frac{t(Y)^1 e^{t(Y)s_i}}{1!} \\ \mathbf{0} & & & & \frac{t(Y)^0 e^{t(Y)s_i}}{0!} \end{pmatrix}.$$

Here $m_i = \nu(s_i)$ $(= \mathrm{mult}(s_i))$.

Proof. Let $K(Y) = \mathrm{Ker}(P_{\sigma_Y(A)})$. Then by [BU, Lemma 2.1], $K(Y)$ is A-invariant in the sense that $A(K(Y) \cap \mathrm{dom}(A)) \subset K(Y)$. Thus one can define $A|_{K(Y)}\colon K(Y) \supset \mathrm{dom}(A|_{K(Y)}) \to K(Y)$. Then we have $\sigma(A|_{K(Y)}) = \sigma(A) - \sigma_Y(A)$ [BU, Lemma 2.1]. We also have

$$A = \begin{pmatrix} A|_{\mathcal{H}_m(Y)} & 0 \\ 0 & A|_{K(Y)} \end{pmatrix}$$

on $H = \mathcal{H}_m(Y) \oplus K(Y)$. Note that the direct sum $\oplus$ does not necessarily mean an orthogonal sum.

By (OP2) and (OP3-a), $(sI - A)^{-1}$ is meromorphic in the whole $\mathbb{C}$-plane. However, since $(sI - A|_{K(Y)})^{-1}$ is holomorphic in Ω_Y, by the functional calculus for the bounded operator $A|_{\mathcal{H}_m(Y)}$ we have

$$\begin{aligned}
F_{A,m}(Y) &= \frac{1}{2\pi i} \oint_{\partial\Omega_Y} q(Y)^s (sI - A)^{-1}\, ds \\
&= \frac{1}{2\pi i} \oint_{\partial\Omega_Y} e^{t(Y)s} (sI - A)^{-1}\, ds \\
&= \frac{1}{2\pi i} \oint_{\partial\Omega_Y} e^{t(Y)s} \left(sI - \begin{pmatrix} A|_{\mathcal{H}_m(Y)} & 0 \\ 0 & A|_{K(Y)} \end{pmatrix} \right)^{-1} ds \\
&= \frac{1}{2\pi i} \oint_{\partial\Omega_Y} e^{t(Y)s} \begin{pmatrix} (sI - A|_{\mathcal{H}_m(Y)})^{-1} & 0 \\ 0 & (sI - A|_{K(Y)})^{-1} \end{pmatrix} ds \\
&= \frac{1}{2\pi i} \oint_{\partial\Omega_Y} e^{t(Y)s} (sI - A|_{\mathcal{H}_m(Y)})^{-1} P_{\sigma_Y(A)}\, ds \\
&= e^{t(Y)A|_{\mathcal{H}_m(Y)}} P_{\sigma_Y(A)} = \begin{pmatrix} e^{t(Y)A|_{\mathcal{H}_m(Y)}} & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

which shows (i) and (ii). By [BU, Lemma 2.1] and (OP4), we have $\sigma(A|_{\mathcal{H}_m(Y)}) = \sigma_Y(A)$. By applying the spectral mapping theorem to the bounded operator $A|_{\mathcal{H}_m(Y)}$ (recall that $\dim_{\mathbb{C}} \mathcal{H}_m(Y) < \infty$), this also shows (iii).

Note that

$$P_{\sigma_Y(A)} = \bigoplus_{s_i \in \sigma_Y(A)} P_{\{s_i\}} \quad \text{and} \quad \mathcal{H}_m(Y) = \bigoplus_{s_i \in \sigma_Y(A)} H(s_i).$$

Here $\bigoplus$ denotes the (not necessarily orthogonal) direct sum. Therefore we have

$$\begin{aligned}
F_{A,m}(Y) &= \frac{1}{2\pi i} \sum_{s_i \in \sigma_Y(A)} \oint_{\partial\Omega_Y} q(Y)^s (sI - A|_{H(s_i)})^{-1} P_{\{s_i\}}\, ds \\
&= \sum_{s_i \in \sigma_Y(A)} e^{t(Y)A|_{H(s_i)}} P_{\{s_i\}}.
\end{aligned}$$

From this (iv) follows.

Note that by [BU, Lemma 2.1] we have $\sigma(A|_{H(s_i)}) = \{s_i\}$. Thus, by (OP3-b), $A|_{H(s_i)}$ is written with respect to an appropriate basis of $H(s_i)$ as

$$A|_{H(s_i)} = M(s_i) = \begin{pmatrix} s_i & 1 & & & \text{\Large 0} \\ & s_i & 1 & & \\ & & \ddots & \ddots & \\ & & & s_i & 1 \\ \text{\Large 0} & & & & s_i \end{pmatrix} \in M_{m_i}(\mathbb{C}).$$

Here $m_i = \nu(s_i)$.

Note that

$$(sI - M(s_i))^{-1} = \begin{pmatrix} \frac{1}{s-s_i} & \frac{1}{(s-s_i)^2} & \cdots & \cdots & \frac{1}{(s-s_i)^{m_i}} \\ & \frac{1}{s-s_i} & \frac{1}{(s-s_i)^2} & \cdots & \frac{1}{(s-s_i)^{m_i-1}} \\ & & \ddots & \ddots & \vdots \\ & & & \frac{1}{s-s_i} & \frac{1}{(s-s_i)^2} \\ \text{\Large 0} & & & & \frac{1}{s-s_i} \end{pmatrix}.$$

Now,

$$q(Y)^s = e^{t(Y)s} = \sum_{n=0}^{\infty} \frac{t(Y)^n e^{t(Y)s_i}}{n!}(s - s_i)^n.$$

From this, (v) follows by using the residue theorem. ∎

3. Abstract intersection theory and its axioms. Let V be an $\mathbb{R}$-linear space endowed with a symmetric $\mathbb{R}$-bilinear form $\beta\colon V \times V \to \mathbb{R}$. Denote by $V_{\mathbb{C}}$ the complexification of V given by $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. To simplify the notation, we identify $v \otimes \alpha$ with αv for $v \in V$ and $\alpha \in \mathbb{C}$. Therefore we have $V \subset V_{\mathbb{C}}$. Then one can define the complexification $\beta_{\mathbb{C}}\colon V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ of β by

$$\beta_{\mathbb{C}}(\alpha_1 v_1, \alpha_2 v_2) := \alpha_1 \overline{\alpha_2} \beta(v_1, v_2) \quad (v_1, v_2 \in V,\ \alpha_1, \alpha_2 \in \mathbb{C}).$$

It is easy to check that $\beta_{\mathbb{C}}(\alpha w_1, w_2) = \alpha \cdot \beta_{\mathbb{C}}(w_1, w_2)$ and $\beta_{\mathbb{C}}(w_2, w_1) = \overline{\beta_{\mathbb{C}}(w_1, w_2)}$ for $w_1, w_2 \in V_{\mathbb{C}}$ and $\alpha \in \mathbb{C}$.

Let $\mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ denote the set of $\mathbb{C}$-linear operators $T\colon V_{\mathbb{C}} \supset \mathrm{dom}(T) \to V_{\mathbb{C}}$ such that $\mathrm{dom}(T) = V_{\mathbb{C}}$. Suppose that there are nonzero vectors v_{01}, v_{10} and h_a in V and maps $v_\delta\colon \mathcal{Y} \to V_{\mathbb{C}}$ and $\Phi_A\colon \mathcal{Y} \to \mathrm{End}_{\mathbb{C}}(V_{\mathbb{C}})$ which satisfy the conditions listed below ((AIT1)–(AIT3)). We call a collection

$$\mathbb{AIT} = (V, v_{01}, v_{10}, v_\delta, h_a, \beta, \Phi_A, F_A, \mathcal{H})$$

that satisfies these conditions an *abstract intersection theory*. The map Φ_A is associated with the operator A in §2. The map $F_A\colon \mathcal{Y} \to B(H)$ along with $\mathcal{H}\colon \mathcal{Y} \to \Sigma_H$ is an analogue of the Frobenius morphism defined in §2, which satisfies (Frob-a) and (Frob-b); F_A is related to Φ_A via the axiom (AIT3).

(AIT1) (a) $\beta(y,x)=\beta(x,y)\in\mathbb{R}$ for $x,y\in V$; $\beta_{\mathbb{C}}(y,x)=\overline{\beta_{\mathbb{C}}(x,y)}\in\mathbb{C}$ for $x,y\in V_{\mathbb{C}}$.
(b) $\beta(v_{01},v_{01})=0$.
(c) $\beta(v_{10},v_{10})=0$.
(d) $\beta(v_{01},v_{10})=1$.

For each $Y\in\mathcal{Y}$ and all $n\geq 0$:

(e) $\beta_{\mathbb{C}}(\Phi_A(Y)^n v_\delta(Y),v_{01})=1$.
(f) $\beta_{\mathbb{C}}(\Phi_A(Y)^n v_\delta(Y),v_{10})=O(q(Y)^n)$.
(g) $\beta_{\mathbb{C}}(\Phi_A(Y)^n v_\delta(Y),\Phi_A(Y)^n v_\delta(Y))=O(q(Y)^n)$.

(AIT2) For $x\in V$, if $\beta(x,h_a)=0$ then $\beta(x,x)\leq 0$.

Note that (AIT1-e)–(AIT1-g) are assumed to hold for each $Y\in\mathcal{Y}$. The Bachmann–Landau notation $O(q(Y)^n)$ in (AIT1) is with respect to $n\gg 0$ for $q(Y)$ with $Y\in\mathcal{Y}$ fixed. We call (AIT2) the *Hodge property*, and h_a a *Hodge vector*.

LEMMA 3.1. *Under the assumptions* (AIT1-a)–(AIT1-d) *and* (AIT2),

$$\beta(x,x)\leq 2\beta(x,v_{01})\beta(x,v_{10})\quad (x\in V).$$

Proof. See [BU, proof of Lemma 3.1]. ∎

Let the $\mathbb{R}$-bilinear form $\langle\cdot,\cdot\rangle_V\colon V\times V\to\mathbb{R}$ be defined by

$$(*)\qquad \langle x,y\rangle_V:=\beta(x,v_{01})\beta(v_{10},y)+\beta(x,v_{10})\beta(v_{01},y)-\beta(x,y)$$

for $x,y\in V$. By Lemma 3.1, $\langle\cdot,\cdot\rangle_V$ is positive semidefinite, i.e. $\langle x,x\rangle_V\geq 0$ for $x\in V$. Indeed, as we will see soon below ((IP-b), (IP-c)), this bilinear form must be positive *semi*definite, not positive definite.

The complexification $\langle\cdot,\cdot\rangle_{V_{\mathbb{C}}}\colon V_{\mathbb{C}}\times V_{\mathbb{C}}\to\mathbb{C}$ of $\langle\cdot,\cdot\rangle_V\colon V\times V\to\mathbb{R}$ is given by

$$\langle\alpha_1 v_1,\alpha_2 v_2\rangle_{V_{\mathbb{C}}}:=\alpha_1\overline{\alpha_2}\langle v_1,v_2\rangle_V$$

for $v_1,v_2\in V$ and $\alpha_1,\alpha_2\in\mathbb{C}$.

LEMMA 3.2. $\langle\cdot,\cdot\rangle_{V_{\mathbb{C}}}$ *is positive semidefinite, i.e.* $\langle x,x\rangle_{V_{\mathbb{C}}}\geq 0$ *for all* $x\in V_{\mathbb{C}}$.

Proof. Since for $x,y\in V$ and $t\in\mathbb{R}$,

$$\langle tx+y,tx+y\rangle_V=\langle x,x\rangle_V t^2+2\langle x,y\rangle_V t+\langle y,y\rangle_V\geq 0,$$

we have the Cauchy–Schwarz inequality for $\langle\cdot,\cdot\rangle_V$,

$$|\langle x,y\rangle_V|\leq\sqrt{\langle x,x\rangle_V\langle y,y\rangle_V}\quad (x,y\in V),$$

provided that $\langle x,x\rangle_V\neq 0$. If $\langle x,x\rangle_V=0$ then $\langle x,y\rangle_V$ must also be zero. Therefore we have the Cauchy–Schwarz inequality for $\langle\cdot,\cdot\rangle_V$ for any $x,y\in V$.

Let $\mathcal{V}$ be a basis of V. Split $\mathcal{V}$ into two disjoint sets $\mathcal{V}=\{u_i\}_{i\in I}\cup\{v_j\}_{j\in J}$ so that $\langle u_i,u_i\rangle_V=0$ and $\langle v_j,v_j\rangle_V\neq 0$. Note that $\mathcal{V}$ is also a basis of $V_{\mathbb{C}}$ with the same properties: $\langle u_i,u_i\rangle_{V_{\mathbb{C}}}=0$ and $\langle v_j,v_j\rangle_{V_{\mathbb{C}}}\neq 0$. Therefore any

$x \in V_{\mathbb{C}}$ can be written as

$$x = \sum_{i \in I_x} \alpha_{x,i} u_i + \sum_{j \in J_x} \alpha_{x,j} v_j$$

for some *finite* subsets $I_x \subset I$ and $J_x \subset J$ with $\alpha_{x,i}, \alpha_{x,j} \in \mathbb{C}$.

Apply the Gram–Schmidt process to $\{v_j\}_{j \in J_x}$ in V to obtain an orthonormal set $\{e_j\}_{j \in J_x}$ in V. Then we have

$$x = \sum_{i \in I_x} \alpha_{x,i} u_i + \sum_{j \in J_x} \alpha'_{x,j} e_j$$

for some $\alpha'_{x,j} \in \mathbb{C}$.

From the Cauchy–Schwarz inequality for $\langle \cdot, \cdot \rangle_V$, we have $\langle u_{i_1}, u_{i_2} \rangle_V = \langle u_{i_1}, u_{i_2} \rangle_{V_{\mathbb{C}}} = 0$ for $i_1, i_2 \in I_x$ and $\langle u_i, e_j \rangle_V = \langle u_i, e_j \rangle_{V_{\mathbb{C}}} = 0$ for $i \in I_x$ and $j \in J_x$. Thus it is easy to see that $\langle x, x \rangle_{V_{\mathbb{C}}} \geq 0$. ■

Note that $\langle \cdot, \cdot \rangle_{V_{\mathbb{C}}}$ is compatible with $\beta_{\mathbb{C}}$, i.e.

$$(**) \qquad \langle x, y \rangle_{V_{\mathbb{C}}} = \beta_{\mathbb{C}}(x, v_{01}) \beta_{\mathbb{C}}(v_{10}, y) + \beta_{\mathbb{C}}(x, v_{10}) \beta_{\mathbb{C}}(v_{01}, y) - \beta_{\mathbb{C}}(x, y).$$

It is easy to see that (AIT1), $(*)$ and $(**)$ imply the following conditions for any $Y \in \mathcal{Y}$:

(IP) (a) $\langle y, x \rangle_V = \langle x, y \rangle_V \in \mathbb{R}$ for $x, y \in V$; $\langle y, x \rangle_{V_{\mathbb{C}}} = \overline{\langle x, y \rangle_{V_{\mathbb{C}}}} \in \mathbb{C}$ for $x, y \in V_{\mathbb{C}}$.
(b) $\langle v_{01}, v_{01} \rangle_V = 0$.
(c) $\langle v_{10}, v_{10} \rangle_V = 0$.
(d) $\langle v_{01}, v_{10} \rangle_V = 0$.

For each $Y \in \mathcal{Y}$ and all $n \geq 0$:

(e) $\langle \Phi_A(Y)^n v_\delta(Y), v_{01} \rangle_{V_{\mathbb{C}}} = 0$.
(f) $\langle \Phi_A(Y)^n v_\delta(Y), v_{10} \rangle_{V_{\mathbb{C}}} = 0$.
(g) $\langle \Phi_A(Y)^n v_\delta(Y), \Phi_A(Y)^n v_\delta(Y) \rangle_{V_{\mathbb{C}}} = O(q(Y)^n)$.

LEMMA 3.3. *For $\langle \cdot, \cdot \rangle_{V_{\mathbb{C}}}$, we have the Cauchy–Schwarz inequality*

$$|\langle x, y \rangle_{V_{\mathbb{C}}}| \leq \sqrt{\langle x, x \rangle_{V_{\mathbb{C}}} \langle y, y \rangle_{V_{\mathbb{C}}}} \qquad (x, y \in V_{\mathbb{C}}).$$

Proof. Let $\lambda = \langle x, x \rangle_{V_{\mathbb{C}}}$. By Lemma 3.2 we have $\lambda \geq 0$. Note that (e.g. MacCluer [Mac, Exercise 1.7, p. 24])

$$0 \leq \langle \lambda y - \langle y, x \rangle_{V_{\mathbb{C}}} x, \lambda y - \langle y, x \rangle_{V_{\mathbb{C}}} x \rangle_{V_{\mathbb{C}}} = \lambda \{ \lambda \langle y, y \rangle_{V_{\mathbb{C}}} - |\langle x, y \rangle_{V_{\mathbb{C}}}|^2 \}.$$

Therefore if $\lambda > 0$ we have the desired inequality. Suppose $\lambda = 0$. For the basis $\mathcal{V}$ in the proof of Lemma 3.2, we have

$$x = \sum_{i \in I_x} \alpha_{x,i} u_i + \sum_{j \in J_x} \alpha_{x,j} v_j$$

for some finite subsets $I_x \subset I$ and $J_x \subset J$ with $\alpha_{x,i}, \alpha_{x,j} \in \mathbb{C}$. Applying the Gram–Schmidt process to $\{v_j\}_{j \in J_x}$ in V, we obtain an orthonormal set

$\{e_j\}_{j\in J_x}$ in V. Then as in the proof of Lemma 3.2 we have, for some $\alpha'_{x,j}$,

$$x = \sum_{i\in I_x} \alpha_{x,i} u_i + \sum_{j\in J_x} \alpha'_{x,j} e_j.$$

Since $\lambda = 0$ we have $\alpha'_{x,j} = 0$. Therefore

$$x = \sum_{i\in I_x} \alpha_{x,i} u_i.$$

Similarly, y can be expressed as

$$y = \sum_{i\in I_y} \alpha_{y,i} u_i + \sum_{j\in J_y} \alpha_{y,j} v_j$$

for some finite subsets $I_y \subset I$ and $J_y \subset J$ with $\alpha_{y,i}, \alpha_{y,j} \in \mathbb{C}$. Since $\langle u_i, u_i\rangle_V = 0$ for $i \in I_x$, we have, by the Cauchy–Schwarz inequality for $\langle\cdot,\cdot\rangle_V$, $\langle u_{i_1}, u_{i_2}\rangle_V = \langle u_{i_1}, u_{i_2}\rangle_{V_\mathbb{C}} = 0$ for $i_1 \in I_x$ and $i_2 \in I_y$ and $\langle u_i, v_j\rangle_V = \langle u_i, v_j\rangle_{V_\mathbb{C}} = 0$ for $i \in I_x$ and $j \in J_y$. Thus we have $\langle x, y\rangle_{V_\mathbb{C}} = 0$. ∎

Now we introduce axiom (AIT3), which we call the *Lefschetz type formula*:

(AIT3) For each $Y \in \mathcal{Y}$ and all $n \geq 0$,

$$\mathrm{tr}(F_A(Y)^n) = \langle \Phi_A(Y)^n v_\delta(Y), v_\delta(Y)\rangle_{V_\mathbb{C}}.$$

Here $\mathrm{tr}(F_A(Y)^n)$ denotes the trace of $F_A(Y)^n$.

4. Standard models of abstract intersection theory. In this section we construct a model

$$\mathbb{AIT}_m = (V_m, v_{01,m}, v_{10,m}, v_{\delta,m}, h_{a,m}, \beta_m, \Phi_{A,m}, F_{A,m}, \mathcal{H}_m)$$

of an abstract intersection theory $\mathbb{AIT}$. We call $\mathbb{AIT}_m$ which satisfies (AIT1)–(AIT3) a *standard model* of abstract intersection theory.

Recall that we have constructed the models $F_{A,m}$ and $\mathcal{H}_m$ of F_A and $\mathcal{H}$ in §2, using conditions (OP1), (OP2), (OP3-a) and (OP4) on the operator A. We will construct the remaining elements of the model below.

Let $\{e_i\}_{i=1}^N$ ($1 \leq N := \dim_\mathbb{C} H \leq \infty$) be an orthonormal basis of the $\mathbb{C}$-Hilbert space H. (Recall that we have assumed that H is separable if it is infinite-dimensional.) Therefore

$$H = \Big\{\sum_{i=1}^N \alpha_i e_i \;\Big|\; \alpha_i \in \mathbb{C},\ \sum_{i=1}^N |\alpha_i|^2 < \infty\Big\}.$$

Let H^1 be the $\mathbb{R}$-Hilbert space defined by

$$H^1 := \Big\{\sum_{i=1}^N \alpha_i e_i \;\Big|\; \alpha_i \in \mathbb{R},\ \sum_{i=1}^N |\alpha_i|^2 < \infty\Big\}.$$

Then we have $H^1_{\mathbb{C}}$ $(:= H^1 \otimes_{\mathbb{R}} \mathbb{C}) = H$ by identifying $e_i \otimes \alpha$ with αe_i for $\alpha \in \mathbb{C}$. Define $\mathbb{R}$-linear spaces H^0 and H^2 by

$$H^0 := \{\alpha f \mid \alpha \in \mathbb{R}\} \quad \text{and} \quad H^2 := \{\alpha g \mid \alpha \in \mathbb{R}\}$$

with

$$\langle f, f\rangle_{H^0} := 0 \quad \text{and} \quad \langle g, g\rangle_{H^2} := 0.$$

Remark 4.1. The reason why $f \in H^0$ and $g \in H^2$ are defined so that they satisfy the above conditions for degenerate inner product is that (IP-b) and (IP-c) in §3 must be satisfied. See (IP-b) and (IP-c) in the proof of Lemma 4.1 below. ■

Then the complexifications $H^0_{\mathbb{C}} := H^0 \otimes_{\mathbb{R}} \mathbb{C}$ and $H^2_{\mathbb{C}} := H^2 \otimes_{\mathbb{R}} \mathbb{C}$ are regarded naturally as

$$H^0_{\mathbb{C}} = \{\alpha f \mid \alpha \in \mathbb{C}\} \quad \text{and} \quad H^2_{\mathbb{C}} = \{\alpha g \mid \alpha \in \mathbb{C}\}$$

by identifying $f \otimes \alpha$ (resp. $g \otimes \alpha$) with αf (resp. αg) for $\alpha \in \mathbb{C}$. Let

$$H^\bullet := H^0 \oplus H^1 \oplus H^2.$$

Here $\oplus$ means the orthogonal direct sum. That is, we assume that f and g are linearly independent and that $\langle f, x\rangle_{H^\bullet} = \langle x, f\rangle_{H^\bullet} = 0$ for $x \in H^1 \oplus H^2$ and $\langle g, x\rangle_{H^\bullet} = \langle x, g\rangle_{H^\bullet} = 0$ for $x \in H^0 \oplus H^1$. The inner product $\langle \cdot, \cdot\rangle_{H^\bullet}$ on $H^\bullet$ is inherited from $\langle \cdot, \cdot\rangle_{H^i}$ $(i = 0, 1, 2)$, that is, $\langle x_i, y_i\rangle_{H^\bullet} := \langle x_i, y_i\rangle_{H^i}$ for $x_i, y_i \in H^i$.

Define an $\mathbb{R}$-linear space V_m by

$$V_m := (H^0 \otimes_{\mathbb{R}} H^2) \oplus (H^1 \otimes_{\mathbb{R}} H^1) \oplus (H^2 \otimes_{\mathbb{R}} H^0)$$

with

$$\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle_{V_m} := \langle x_1, y_1\rangle_{H^\bullet}\langle x_2, y_2\rangle_{H^\bullet}.$$

Since

$$H^1 \otimes_{\mathbb{R}} H^1 = \Big\{ \sum_{i,j=1}^{N} \alpha_{ij} e_i \otimes e_j \;\Big|\; \alpha_{ij} \in \mathbb{R},\ \sum_{i,j=1}^{N} |\alpha_{ij}|^2 < \infty \Big\},$$

we have

$$(H^1 \otimes_{\mathbb{R}} H^1)_{\mathbb{C}} = \Big\{ \sum_{i,j=1}^{N} \alpha_{ij} e_i \otimes e_j \;\Big|\; \alpha_{ij} \in \mathbb{C},\ \sum_{i,j=1}^{N} |\alpha_{ij}|^2 < \infty \Big\} = H^1_{\mathbb{C}} \otimes_{\mathbb{C}} H^1_{\mathbb{C}}$$

by identifying $(e_i \otimes e_j) \otimes \alpha$ with $\alpha e_i \otimes e_j$ for $\alpha \in \mathbb{C}$. Note that $\{e_i \otimes e_j\}_{i,j=1}^N$ is an orthonormal basis of the tensor products $H^1 \otimes_{\mathbb{R}} H^1$ and $H^1_{\mathbb{C}} \otimes_{\mathbb{C}} H^1_{\mathbb{C}}$. Similarly, by identifying $(f \otimes g) \otimes \alpha$ (resp. $(g \otimes f) \otimes \alpha$) with $\alpha f \otimes g$ (resp. $\alpha g \otimes f$) for $\alpha \in \mathbb{C}$, we have

$$(H^0 \otimes_{\mathbb{R}} H^2)_{\mathbb{C}} = \{\alpha f \otimes g \mid \alpha \in \mathbb{C}\} = H^0_{\mathbb{C}} \otimes_{\mathbb{C}} H^2_{\mathbb{C}}$$

and

$$(H^2 \otimes_{\mathbb{R}} H^0)_{\mathbb{C}} = \{\alpha g \otimes f \mid \alpha \in \mathbb{C}\} = H^2_{\mathbb{C}} \otimes_{\mathbb{C}} H^0_{\mathbb{C}}.$$

Note that generally we have $(X \otimes_{\mathbb{R}} Y)_{\mathbb{C}} = X_{\mathbb{C}} \otimes_{\mathbb{C}} Y_{\mathbb{C}}$. Therefore

$$(V_m)_{\mathbb{C}} = V_m \otimes_{\mathbb{R}} \mathbb{C} = (H^0_{\mathbb{C}} \otimes_{\mathbb{C}} H^2_{\mathbb{C}}) \oplus (H^1_{\mathbb{C}} \otimes_{\mathbb{C}} H^1_{\mathbb{C}}) \oplus (H^2_{\mathbb{C}} \otimes_{\mathbb{C}} H^0_{\mathbb{C}})$$

with

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle_{(V_m)_{\mathbb{C}}} = \langle x_1, y_1 \rangle_{H^\bullet_{\mathbb{C}}} \langle x_2, y_2 \rangle_{H^\bullet_{\mathbb{C}}},$$

where

$$H^\bullet_{\mathbb{C}} = H^0_{\mathbb{C}} \oplus H^1_{\mathbb{C}} \oplus H^2_{\mathbb{C}}$$

as the orthogonal direct sum. Note that the complexification $\langle \cdot, \cdot \rangle_{H^\bullet_{\mathbb{C}}}$ of the inner product $\langle \cdot, \cdot \rangle_{H^\bullet}$ is given by $\langle \alpha_1 x_1, \alpha_2 x_2 \rangle_{H^\bullet_{\mathbb{C}}} := \alpha_1 \overline{\alpha_1} \langle x_1, x_2 \rangle_{H^\bullet}$ for $x_1, x_2 \in H^\bullet$ and $\alpha_1, \alpha_2 \in \mathbb{C}$.

Extend the operator A on $H^1_{\mathbb{C}}$ $(= H)$ to an operator A on $H^\bullet_{\mathbb{C}}$ by

$$Af = A|_{H^0_{\mathbb{C}}} f := 0 \quad \text{and} \quad Ag = A|_{H^2_{\mathbb{C}}} g := g.$$

Accordingly, we extend the map $F_{A,m} \colon \mathcal{Y} \to B(H)$ to $F_{A,m} \colon \mathcal{Y} \to \mathrm{End}_{\mathbb{C}}(H^\bullet_{\mathbb{C}})$ so that

$$F_{A,m}(Y)f := e^{t(Y)A|_{H^0_{\mathbb{C}}}} f = f \quad \text{and} \quad F_{A,m}(Y)g := e^{t(Y)A|_{H^2_{\mathbb{C}}}} g = q(Y)g$$

for $Y \in \mathcal{Y}$. Here $\mathrm{End}_{\mathbb{C}}(H^\bullet_{\mathbb{C}})$ denotes the set of $\mathbb{C}$-linear operators $T \colon H^\bullet_{\mathbb{C}} \supset \mathrm{dom}(T) \to H^\bullet_{\mathbb{C}}$ with $\mathrm{dom}(T) = H^\bullet_{\mathbb{C}}$.

Let

$$v_{01,m} := f \otimes g \in H^0 \otimes_{\mathbb{R}} H^2 \subset H^0_{\mathbb{C}} \otimes_{\mathbb{C}} H^2_{\mathbb{C}},$$
$$v_{10,m} := g \otimes f \in H^2 \otimes_{\mathbb{R}} H^0 \subset H^2_{\mathbb{C}} \otimes_{\mathbb{C}} H^0_{\mathbb{C}}.$$

Recall that $\mathcal{H}_m(Y) := \mathrm{Image}(P_{\sigma_Y(A)}) \subset H^1_{\mathbb{C}}$, and also $F_{A,m}(Y)\mathcal{H}_m(Y) \subset \mathcal{H}_m(Y)$ (i.e. (Frob-a)) by Lemma 2.1(ii). Recall that, by (OP2) and (OP3-a), $\mathcal{H}_m(Y)$ is finite-dimensional. Let $g(Y) := \frac{1}{2} \dim_{\mathbb{C}} \mathcal{H}_m(Y)$. Let $\{e_i^Y\}_{i=1}^{2g(Y)}$ be an orthonormal basis of $\mathcal{H}_m(Y)$.

Recall that by (OP4), (Frob-b) also holds.

For each $Y \in \mathcal{Y}$ let

$$v_{\delta,m}(Y) := \Big(\sum_{i=1}^{2g(Y)} e_i^Y \otimes e_i^Y \Big) + v_{01,m} + v_{10,m} \in (V_m)_{\mathbb{C}}.$$

Let $\Phi_{A,m}(Y) := I \otimes F_{A,m}(Y)$, where I denotes the identity operator on $H^\bullet_{\mathbb{C}} = H^0_{\mathbb{C}} \oplus H^1_{\mathbb{C}} \oplus H^2_{\mathbb{C}}$.

Lemma 4.1. *Suppose that* $A \colon H \supset \mathrm{dom}(A) \to H$ *is an operator that satisfies* (OP1), (OP2), (OP3-a) *and* (OP4). *Then the above construction satisfies*:

(i) *The conditions* (IP-a)–(IP-f).
(ii) *The Lefschetz type formula* (AIT3).

Proof. (i) (IP-a) is obvious from the definition.

(IP-b): $\langle v_{01,m}, v_{01,m}\rangle_{V_m} = \langle f\otimes g, f\otimes g\rangle_{V_m} = \langle f,f\rangle_{H^\bullet}\langle g,g\rangle_{H^\bullet} = 0$.

(IP-c): $\langle v_{10,m}, v_{10,m}\rangle_{V_m} = \langle g\otimes f, g\otimes f\rangle_{V_m} = \langle g,g\rangle_{H^\bullet}\langle f,f\rangle_{H^\bullet} = 0$.

(IP-d): $\langle v_{01,m}, v_{10,m}\rangle_{V_m} = \langle f\otimes g, g\otimes f\rangle_{V_m} = \langle f,g\rangle_{H^\bullet}\langle g,f\rangle_{H^\bullet} = 0$.

Since $F_{A,m}(Y)^n f = f$, $F_{A,m}(Y)^n g = q(Y)^n g$ and $\mathcal{H}_m(Y)$ is $F_{A,m}(Y)$-invariant, we have

$$\begin{aligned}
\Phi_{A,m}(Y)^n v_{\delta,m}(Y) &= I\otimes F_{A,m}(Y)^n\Big\{\sum_{i=1}^{2g(Y)} e_i^Y\otimes e_i^Y + f\otimes g + g\otimes f\Big\}\\
&= \sum_{i=1}^{2g(Y)} e_i^Y\otimes F_{A,m}(Y)^n e_i^Y + f\otimes F_{A,m}(Y)^n g + g\otimes F_{A,m}(Y)^n f\\
&= \sum_{i=1}^{2g(Y)} e_i^Y\otimes F_{A,m}(Y)^n e_i^Y + f\otimes q(Y)^n g + g\otimes f\\
&= \sum_{i=1}^{2g(Y)} e_i^Y\otimes F_{A,m}(Y)^n e_i^Y + q(Y)^n f\otimes g + g\otimes f.
\end{aligned}$$

(IP-e) and (IP-f) follow from this since $H^0\perp H^1\perp H^2$ and $\langle f,f\rangle_{H^0} = \langle g,g\rangle_{H^2} = 0$.

(ii) To show (AIT3) note that

$$\begin{aligned}
&\langle \Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{\delta,m}(Y)\rangle_{(V_m)_{\mathbb{C}}}\\
&\quad= \Big\langle \sum_{i=1}^{2g(Y)} e_i^Y\otimes F_{A,m}(Y)^n e_i^Y + q(Y)^n f\otimes g + g\otimes f,\\
&\qquad\qquad \sum_{j=1}^{2g(Y)} e_j^Y\otimes e_j^Y + f\otimes g + g\otimes f\Big\rangle_{(V_m)_{\mathbb{C}}}\\
&\quad= \Big\langle \sum_{i=1}^{2g(Y)} e_i^Y\otimes F_{A,m}(Y)^n e_i^Y, \sum_{j=1}^{2g(Y)} e_j^Y\otimes e_j^Y\Big\rangle_{H^1_{\mathbb{C}}\otimes_{\mathbb{C}} H^1_{\mathbb{C}}}\\
&\quad= \sum_{i=1}^{2g(Y)}\sum_{j=1}^{2g(Y)} \langle e_i^Y, e_j^Y\rangle_{H^1_{\mathbb{C}}}\langle F_{A,m}(Y)^n e_i^Y, e_j^Y\rangle_{H^1_{\mathbb{C}}}\\
&\quad= \sum_{i=1}^{2g(Y)} \langle F_{A,m}(Y)^n e_i^Y, e_i^Y\rangle_{H^1_{\mathbb{C}}} = \mathrm{tr}(F_{A,m}(Y)^n).
\end{aligned}$$

Note that by Lemma 2.1(iii), $\sigma(F_{A,m}(Y)|_{\mathcal{H}_m(Y)}) = \sigma_p(F_{A,m}(Y)|_{\mathcal{H}_m(Y)})$ is a finite set counted with algebraic multiplicities. Thus $\mathrm{tr}(F_{A,m}(Y)^n)$ is well defined. This completes the proof of (AIT3). ∎

LEMMA 4.2. *In the same situation as in Lemma 4.1 and in its proof, there is a bilinear form* $\beta_m \colon V_m \times V_m \to \mathbb{R}$ *and a Hodge vector* $h_{a,m} \in V$ *which satisfy* (AIT1-a)–(AIT1-f), (AIT2), $(*)$ *and* $(**)$.

Proof. First we prove (AIT1-a)–(AIT1-f), $(*)$ and $(**)$. Recall that

$$V_m = (H^0 \otimes_{\mathbb{R}} H^2) \oplus (H^1 \otimes_{\mathbb{R}} H^1) \oplus (H^2 \otimes_{\mathbb{R}} H^0),$$
$$v_{01,m} = f \otimes g \in H^0 \otimes_{\mathbb{R}} H^2, \qquad v_{10,m} = g \otimes f \in H^2 \otimes_{\mathbb{R}} H^0.$$

Therefore we can set

$$\beta_m(v_{01,m}, v_{01,m}) := 0, \qquad \beta_m(v_{10,m}, v_{10,m}) := 0,$$
$$\beta_m(v_{01,m}, v_{10,m}) = \beta_m(v_{10,m}, v_{01,m}) := 1,$$

which are (AIT1-b), (AIT1-c) and (AIT1-d), respectively. Furthermore we can set

$$\beta_m(x, v_{01,m}) = \beta_m(v_{01,m}, x) := 0,$$
$$\beta_m(x, v_{10,m}) = \beta_m(v_{10,m}, x) := 0$$

for all $x \in H^1 \otimes_{\mathbb{R}} H^1$. Then

$$(\beta_m)_{\mathbb{C}}(x, v_{01,m}) = (\beta_m)_{\mathbb{C}}(v_{01,m}, x) = 0,$$
$$(\beta_m)_{\mathbb{C}}(x, v_{10,m}) = (\beta_m)_{\mathbb{C}}(v_{10,m}, x) = 0$$

for all $x \in (H^1 \otimes_{\mathbb{R}} H^1)_{\mathbb{C}} = H^1_{\mathbb{C}} \otimes_{\mathbb{C}} H^1_{\mathbb{C}}$.

Now for each $Y \in \mathcal{Y}$ let

$$v_{\delta 1,m}(Y) := \sum_{i=1}^{2g(Y)} e_i^Y \otimes e_i^Y \in (H^1 \otimes_{\mathbb{R}} H^1)_{\mathbb{C}} = H^1_{\mathbb{C}} \otimes_{\mathbb{C}} H^1_{\mathbb{C}}.$$

Note that $v_{\delta,m}(Y) = v_{\delta 1,m}(Y) + v_{01,m} + v_{10,m}$. Recall from the proof of Lemma 4.1 that

$$\begin{aligned}\Phi_{A,m}(Y)^n v_{\delta,m}(Y) &= \sum_{i=1}^{2g(Y)} e_i^Y \otimes F_{A,m}(Y)^n e_i^Y + q(Y)^n f \otimes g + g \otimes f \\ &= \sum_{i=1}^{2g(Y)} e_i^Y \otimes F_{A,m}(Y)^n e_i^Y + q(Y)^n v_{01,m} + v_{10,m} \\ &= \Phi_{A,m}(Y)^n v_{\delta 1,m}(Y) + q(Y)^n v_{01,m} + v_{10,m}.\end{aligned}$$

Thus, since $\Phi_{A,m}(Y)^n v_{\delta 1,m}(Y) \in (H^1 \otimes_{\mathbb{R}} H^1)_{\mathbb{C}}$, we have

$$(\beta_m)_{\mathbb{C}}(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{01,m}) = 1,$$
$$(\beta_m)_{\mathbb{C}}(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{10,m}) = q(Y)^n = O(q(Y)^n),$$

which are (AIT1-e) and (AIT1-f), respectively. Now that we are given $\beta_m(x, v_{01,m})$, $\beta_m(x, v_{10,m})$, $\beta_m(v_{10,m}, y)$ and $\beta_m(v_{01,m}, y)$, and $\langle x, y\rangle_{V_m}$, we can define $\beta_m(x, y)$ for $x, y \in V_m$ by

$$\beta_m(x,y) := \beta_m(x,v_{01,m})\beta_m(v_{10,m},y) + \beta_m(x,v_{10,m})\beta_m(v_{01,m},y) - \langle x,y\rangle_{V_m}.$$

Then we see that (AIT1-a), (∗) and (∗∗) are satisfied.

Finally, we prove (AIT2). Let $h_{a,m} := v_{01,m} + v_{10,m}$. If $\beta_m(x,h_{a,m}) = 0$, then $\beta_m(x,v_{10,m}) = -\beta_m(x,v_{01,m})$. Hence

$$\begin{aligned}\beta_m(x,x) &= 2\beta_m(x,v_{01,m})\beta_m(x,v_{10,m}) - \langle x,x\rangle_{V_m}\\ &= -2\beta_m(x,v_{01,m})^2 - \langle x,x\rangle_{V_m} \le 0.\end{aligned}$$

Therefore $h_{a,m}$ is a Hodge vector. ∎

Lemma 4.3. *In the same situation as in Lemmas* 4.1 *and* 4.2 *and in their proofs, suppose that* (IP-g) *further holds. Then* (AIT1-g) *is also satisfied.*

Proof. We have

$$\begin{aligned}&(\beta_m)_{\mathbb{C}}(\varPhi_{A,m}(Y)^n v_{\delta,m}(Y), \varPhi_{A,m}(Y)^n v_{\delta,m}(Y))\\ &\quad= (\beta_m)_{\mathbb{C}}(\varPhi_{A,m}(Y)^n v_{\delta,m}(Y), v_{01,m}) \cdot (\beta_m)_{\mathbb{C}}(v_{10,m}, \varPhi_{A,m}(Y)^n v_{\delta,m}(Y))\\ &\quad\quad+ (\beta_m)_{\mathbb{C}}(\varPhi_{A,m}(Y)^n v_{\delta,m}(Y), v_{10,m}) \cdot (\beta_m)_{\mathbb{C}}(v_{01,m}, \varPhi_{A,m}(Y)^n v_{\delta,m}(Y))\\ &\quad\quad- \langle \varPhi_{A,m}(Y)^n v_{\delta,m}(Y), \varPhi_{A,m}(Y)^n v_{\delta,m}(Y)\rangle_{(V_m)_{\mathbb{C}}}.\end{aligned}$$

(AIT1-g) follows from this and (IP-g). ∎

Remark 4.2. Note that, given an inner product $\langle\cdot,\cdot\rangle_{H^1_{\mathbb{C}}}$ for $H^1_{\mathbb{C}} = H$, the choice of β_m is not unique in our construction of standard models. ∎

5. Main theorems. We use the following lemma (see e.g. [Mon, Lemma 2.2, p. 20]) in the proof of Theorem 5.2 below.

Lemma 5.1. *Let λ_i $(1 \le i \le N < \infty)$ be complex numbers. Then there exist infinitely many integers $n \ge 1$ such that $|\lambda_1|^n \le |\sum_{i=1}^N \lambda_i^n|$.*

Theorem 5.2. *Let $A\colon H \supset \mathrm{dom}(A) \to H$ be an operator satisfying* (OP1), (OP2), (OP3-a), (OP4) *and* (OP5).

1. *If there exists an abstract intersection theory $\mathbb{AIT}$ (in the sense of §3) for A, then the Riemann hypothesis holds for A.*
2. *Suppose further that A satisfies* (OP3-b). *Then there exists a standard model $\mathbb{AIT}_m$ for A if and only if the Riemann hypothesis holds for A and A is semisimple.*

Proof. (1) Suppose that the RH for A does not hold. Then by (OP5) one can find $Y \in \mathcal{Y}$ such that $\sigma_Y(A)$ contains $s_\alpha, s_\beta \in \sigma(A)$ with $\mathrm{Re}(s_\alpha) < 1/2$, $\mathrm{Re}(s_\beta) > 1/2$. Therefore $\sigma_Y(A)$ contains s_1 such that $q(Y)^{\mathrm{Re}(s_1)} > q(Y)^{1/2}$. Indeed, if $0 < q(Y) < 1$ set $s_1 = s_\alpha$, while if $q(Y) > 1$ set $s_1 = s_\beta$.

Recall that $\sigma_Y(A)$ is a finite set. Let s_i $(2 \le i \le 2g(Y) := \dim_{\mathbb{C}} \mathcal{H}(Y))$ be all the other eigenvalues of A in $\sigma_Y(A)$, counted with algebraic multiplicities. Let $\lambda_i = q(Y)^{s_i}$ $(1 \le i \le 2g(Y))$. Then by Lemma 5.1, $\nu_n := \sum_{i=1}^{2g(Y)} \lambda_i^n$

is not $O(q(Y)^{n/2})$, since we could choose s_1 so that $|\lambda_1|^n = |q(Y)^{s_1}|^n = q(Y)^{n/2}(1+\epsilon)^n$ for some $\epsilon > 0$.

By (Frob-b), we have

$$\begin{aligned}\sigma(F_A(Y)^n) = \sigma_p(F_A(Y)^n) &= \{q(Y)^{ns} \mid s \in \sigma_Y(A)\} \cup \{0\} \\ &= \{\lambda_i^n \mid 1 \le i \le 2g(Y)\} \cup \{0\}\end{aligned}$$

(counted with algebraic multplicities). By (AIT3) and Lemma 3.3 (the Cauchy–Schwarz inequality), we have

$$\begin{aligned}|\nu_n| &= |\mathrm{tr}(F_A(Y)^n)| = |\langle \Phi_A(Y)^n v_\delta(Y), v_\delta(Y)\rangle_{V_{\mathbb{C}}}| \\ &\le \sqrt{|\langle v_\delta(Y), v_\delta(Y)\rangle_{V_{\mathbb{C}}}| \cdot |\langle \Phi_A(Y)^n v_\delta(Y), \Phi_A(Y)^n v_\delta(Y)\rangle_{V_{\mathbb{C}}}|}.\end{aligned}$$

Therefore, by (IP-g), we see that ν_n is $O(q(Y)^{n/2})$. However, this is a contradiction.

Now, we prove the "if" part of (2). By Lemma 4.1, we have (IP-a)–(IP-f) and (AIT3) for V_m and $\Phi_{A,m}(Y)$. By Lemma 4.2, we have (AIT1-a)–(AIT1-f) and (AIT2). Therefore all we have to do is to verify (IP-g) in order to apply Lemma 4.3 to obtain (AIT1-g). Since the RH for the operator A is assumed to hold, each eigenvalue λ_ℓ ($1 \le \ell \le 2g(Y)$), counted with algebraic multiplicity, of $F_{A,m}(Y)$ can be written as $\lambda_\ell = q(Y)^{1/2} e^{i\theta_\ell}$ ($\theta_\ell \in \mathbb{R}$). By the semisimplicity assumption on A, one can choose eigenvectors w_ℓ associated with λ_ℓ so that $F_{A,m}(Y) w_\ell = \lambda_\ell w_\ell$. Recall that $\{e_i^Y\}_{i=1}^{2g(Y)}$ ($g(Y) := \frac{1}{2}\dim_{\mathbb{C}} \mathcal{H}_m(Y)$) is an orthonormal basis of $\mathcal{H}_m(Y)$ (see §4). Now one can write $e_i^Y = \sum_{\ell=1}^{2g(Y)} \alpha_{i\ell} w_\ell$ for some $\alpha_{i\ell} \in \mathbb{C}$. Then

$$\begin{aligned}&\langle \Phi_{A,m}(Y)^n v_{\delta,m}(Y), \Phi_{A,m}(Y)^n v_{\delta,m}(Y)\rangle_{(V_m)_{\mathbb{C}}} \\ &= \Big\langle \sum_{i=1}^{2g(Y)} e_i^Y \otimes F_{A,m}(Y)^n e_i^Y + q(Y)^n f \otimes g + g \otimes f, \\ &\qquad\qquad \sum_{j=1}^{2g(Y)} e_j^Y \otimes F_{A,m}(Y)^n e_j^Y + q(Y)^n f \otimes g + g \otimes f \Big\rangle_{(V_m)_{\mathbb{C}}} \\ &= \Big\langle \sum_{i=1}^{2g(Y)} e_i^Y \otimes F_{A,m}(Y)^n e_i^Y, \sum_{j=1}^{2g(Y)} e_j^Y \otimes F_{A,m}(Y)^n e_j^Y \Big\rangle_{H^1_{\mathbb{C}} \otimes_{\mathbb{C}} H^1_{\mathbb{C}}} \\ &= \sum_{i=1}^{2g(Y)} \sum_{j=1}^{2g(Y)} \langle e_i^Y, e_j^Y\rangle_{H^1_{\mathbb{C}}} \langle F_{A,m}(Y)^n e_i^Y, F_{A,m}(Y)^n e_j^Y\rangle_{H^1_{\mathbb{C}}} \\ &= \sum_{i=1}^{2g(Y)} \langle F_{A,m}(Y)^n e_i^Y, F_{A,m}(Y)^n e_i^Y\rangle_{H^1_{\mathbb{C}}} \\ &= \sum_{i=1}^{2g(Y)} \Big\langle \sum_{\ell=1}^{2g(Y)} \alpha_{i\ell} F_{A,m}(Y)^n w_\ell, \sum_{m=1}^{2g(Y)} \alpha_{im} F_{A,m}(Y)^n w_m \Big\rangle_{H^1_{\mathbb{C}}}.\end{aligned}$$

Since $F_{A,m}(Y)^n w_\ell = \lambda_\ell^n w_\ell$, we have (IP-g). Therefore by Lemma 4.3, we have (AIT1-g).

We now prove the "only if" part of (2). By Lemma 2.1(i)–(iii), $F_{A,m}$ and $\mathcal{H}_m$ satisfy (Frob-a) and (Frob-b). Therefore $\mathbb{AIT}_m \Rightarrow$ RH can be proved as in the proof of (1).

Let us now show $\mathbb{AIT}_m \Rightarrow$ semisimplicity. Suppose that on the contrary we have $\mathbb{AIT}_m$ but A is not semisimple. Then one can find $Y \in \mathcal{Y}$ such that

$$\sigma_Y(A) = \{s_1, \ldots, s_N\}$$

with

$$|\mathrm{Im}(s_1)| < \cdots < |\mathrm{Im}(s_N)|$$

and

$$\nu(s_1) = \cdots = \nu(s_{N-1}) = 1 \quad \text{and} \quad \nu(s_N) > 1.$$

We want to calculate

$$\langle \Phi_{A,m}(Y)^n v_{\delta,m}(Y), \Phi_{A,m}(Y)^n v_{\delta,m}(Y) \rangle_{(V_m)_\mathbb{C}}$$

and show that it is not of order $O(q(Y)^n)$, which contradicts (IP-g).

We use the notation in Lemma 2.1 and its proof. Then $m_i = \dim_\mathbb{C} H(s_i) = \nu(s_i) = \mathrm{mult}(s_i)$ $(1 \le i \le N)$ by (OP3-b). We regard $H(s_i)$ as $\mathbb{C}^{m_i}$. Then we can take a basis $w_{i,\ell} \in \mathbb{C}^{m_i}$ $(1 \le \ell \le m_i = \dim_\mathbb{C} H(s_i))$ of the form

$$w_{i,\ell} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ell$$

so that $e^{t(Y)A|_{H(s_i)}}$ can be written in the matrix form $N(s_i)$ as in Lemma 2.1(v). In other words, $w_{i,\ell}$ $(1 \le \ell \le m_i)$ are generalized eigenvectors of $M(s_i) = A|_{H(s_i)}$. Let

$$J_i := \begin{pmatrix} 0 & 1 & & & \text{\Large 0} \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ \text{\Large 0} & & & & 0 \end{pmatrix} \in M_{m_i}(\mathbb{C}).$$

Since $J_i^m = 0$ for $m \geq m_i$, $N(s_i)$ in Lemma 2.1(v) can be written as

$$(5.1)\quad N(s_i) = e^{t(Y)s_i} \begin{pmatrix} \frac{t(Y)^0}{0!} & \frac{t(Y)^1}{1!} & \cdots & \cdots & \frac{t(Y)^{m_i-1}}{(m_i-1)!} \\ & \frac{t(Y)^0}{0!} & \frac{t(Y)^1}{1!} & \cdots & \frac{t(Y)^{m_i-2}}{(m_i-2)!} \\ & & \ddots & \ddots & \vdots \\ & & & \frac{t(Y)^0}{0!} & \frac{t(Y)^1}{1!} \\ \mathbf{0} & & & & \frac{t(Y)^0}{0!} \end{pmatrix}$$

$$= e^{t(Y)s_i} \sum_{k=0}^{m_i-1} \frac{(t(Y)J_i)^k}{k!} = e^{t(Y)s_i} \sum_{k=0}^{\infty} \frac{(t(Y)J_i)^k}{k!} = e^{t(Y)s_i} e^{t(Y)J_i}.$$

Note that

$$(5.2)\qquad e^{nt(Y)J_i} w_{i,\ell} = \begin{pmatrix} \frac{(nt(Y))^{\ell-1}}{(\ell-1)!} \\ \frac{(nt(Y))^{\ell-2}}{(\ell-2)!} \\ \vdots \\ \frac{(nt(Y))^0}{0!} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sum_{k=1}^{\ell} \frac{(nt(Y))^{k-1}}{(k-1)!} w_{i,\ell-k+1}$$

for $1 \leq i \leq N$ and $1 \leq \ell \leq m_i$.

Recall that $\{e_\mu^Y\}_{\mu=1}^{2g(Y)}$ is an orthonormal basis of $\mathcal{H}_m(Y)$. Thus

$$e_\mu^Y = \sum_{i=1}^{N} \sum_{\ell=1}^{m_i} \alpha_{i,\ell}^\mu w_{i,\ell}$$

for some $\alpha_{i,\ell}^\mu \in \mathbb{C}$. Then by Lemma 2.1(iv)–(v) we have

$$\begin{aligned} F_{A,m}(Y)^n e_\mu^Y &= \sum_{i=1}^{N} F_{A,m}(Y)^n \sum_{\ell=1}^{m_i} \alpha_{i,\ell}^\mu w_{i,\ell} = \sum_{i=1}^{N} N(s_i)^n \sum_{\ell=1}^{m_i} \alpha_{i,\ell}^\mu w_{i,\ell} \\ &= \sum_{i=1}^{N} \sum_{\ell=1}^{m_i} \alpha_{i,\ell}^\mu N(s_i)^n w_{i,\ell}. \end{aligned}$$

Recall from the proof of the "if" part of (2) that

$$\langle \Phi_{A,m}(Y)^n v_{\delta,m}(Y), \Phi_{A,m}(Y)^n v_{\delta,m}(Y) \rangle_{(V_m)_{\mathbb{C}}} = \sum_{\mu=1}^{2g(Y)} \langle F_{A,m}(Y)^n e_\mu^Y, F_{A,m}(Y)^n e_\mu^Y \rangle_{H^1_{\mathbb{C}}}.$$

Now using (5.1) and (5.2) we have

$$\begin{aligned}
&\langle F_{A,m}(Y)^n e_\mu^Y, F_{A,m}(Y)^n e_\mu^Y\rangle_{H^1_{\mathbb{C}}} \\
&\quad = \Big\langle \sum_{i=1}^{N}\sum_{\ell=1}^{m_i} \alpha_{i,\ell}^\mu N(s_i)^n w_{i,\ell}, \sum_{j=1}^{N}\sum_{m=1}^{m_i} \alpha_{j,m}^\mu N(s_j)^n w_{j,m}\Big\rangle_{H^1_{\mathbb{C}}} \\
&\quad = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{m_i}\sum_{m=1}^{m_i} \alpha_{i,\ell}^\mu \overline{\alpha_{j,m}^\mu}\langle N(s_i)^n w_{i,\ell}, N(s_j)^n w_{j,m}\rangle_{H^1_{\mathbb{C}}} \\
&\quad = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{m_i}\sum_{m=1}^{m_i} \alpha_{i,\ell}^\mu \overline{\alpha_{j,m}^\mu}\langle e^{nt(Y)s_i} e^{nt(Y)J_i} w_{i,\ell}, e^{nt(Y)s_j} e^{nt(Y)J_j} w_{j,m}\rangle_{H^1_{\mathbb{C}}} \\
&\quad = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{m_i}\sum_{m=1}^{m_i} \alpha_{i,\ell}^\mu \overline{\alpha_{j,m}^\mu} e^{nt(Y)(s_i+\overline{s_j})}\langle e^{nt(Y)J_i} w_{i,\ell}, e^{nt(Y)J_j} w_{j,m}\rangle_{H^1_{\mathbb{C}}} \\
&\quad = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{m_i}\sum_{m=1}^{m_i} \alpha_{i,\ell}^\mu \overline{\alpha_{j,m}^\mu} e^{nt(Y)(s_i+\overline{s_j})} \\
&\qquad\qquad \times \Big\langle \sum_{a=1}^{\ell} \frac{(nt(Y))^{a-1}}{(a-1)!} w_{i,\ell-a+1}, \sum_{b=1}^{m} \frac{(nt(Y))^{b-1}}{(b-1)!} w_{j,m-b+1}\Big\rangle_{H^1_{\mathbb{C}}}.
\end{aligned}$$

Let $M_\mu := \max\{\ell \mid \alpha_{N,\ell}^\mu \neq 0\}$. Then $\alpha_{N,\ell}^\mu \overline{\alpha_{N,m}^\mu} = 0$ if $\ell > M_\mu$ or $m > M_\mu$. Note that $\mathrm{Re}(s_i) = 1/2$ (for all i) since the RH holds by $\mathbb{AIT}_m \Rightarrow$ RH. Recall that $q(Y) = e^{t(Y)}$. Therefore

$$\begin{aligned}
&\langle F_{A,m}(Y)^n e_\mu^Y, F_{A,m}(Y)^n e_\mu^Y\rangle_{H^1_{\mathbb{C}}} \\
&\quad = \alpha_{N,M_\mu}^\mu \overline{\alpha_{N,M_\mu}^\mu} e^{2\mathrm{Re}(s_N)nt(Y)} \frac{(nt(Y))^{2(M_\mu-1)}}{\{(M_\mu-1)!\}^2}\langle w_{N,1}, w_{N,1}\rangle_{H^1_{\mathbb{C}}} \\
&\qquad + O(e^{nt(Y)}(nt(Y))^{2M_\mu-3}) \\
&\quad = |\alpha_{N,M_\mu}^\mu|^2 q(Y)^n \frac{(n\log q(Y))^{2(M_\mu-1)}}{\{(M_\mu-1)!\}^2}\|w_{N,1}\|^2_{H^1_{\mathbb{C}}} + O(q(Y)^n(nt(Y))^{2M_\mu-3}) \\
&\quad = C_\mu q(Y)^n n^{2(M_\mu-1)} + O(q(Y)^n n^{2M_\mu-3}),
\end{aligned}$$

where

$$C_\mu = |\alpha_{N,M_\mu}^\mu|^2 \frac{(\log q(Y))^{2(M_\mu-1)}}{\{(M_\mu-1)!\}^2}\|w_{N,1}\|^2_{H^1_{\mathbb{C}}} > 0.$$

Let $M := \max\{M_\mu \mid 1 \leq \mu \leq 2g(Y)\}$. Since e_μ^Y $(1 \leq \mu \leq 2g(Y))$ form a basis of $\mathcal{H}_m(Y)$, $\alpha_{N,m_N}^\mu \neq 0$ for at least one μ. Hence $M = m_N > 1$. Now we have

$$\sum_{\mu=1}^{2g(Y)} \langle F_{A,m}(Y)^n e_\mu^Y, F_{A,m}(Y)^n e_\mu^Y \rangle_{H^1_{\mathbb{C}}}$$
$$= \Big(\sum_{M_\mu = m_N} C_\mu \Big) q(Y)^n n^{2(m_N-1)} + O(q(Y)^n n^{2m_N-3}) \neq O(q(Y)^n),$$

which contradicts (IP-g). This completes the proof. ∎

In our previous paper [BU] we constructed a model of abstract intersection theory based on an analogue of the GNS (Gelfand–Naimark–Segal) representation. Let us call this model which satisfies (INT1)–(INT3) in [BU] a GNS model and denote it as $\mathbb{AIT}_{\mathrm{GNS}}$. The method of the proof of the above theorem also applies to this model. Therefore we have the following theorem.

THEOREM 5.3. *Let* $A\colon H \supset \mathrm{dom}(A) \to H$ *be an operator satisfying* (OP1)–(OP5). *Suppose further that* A *satisfies* (OP5-b) *of* [BU]. *Then there exists a GNS model* $\mathbb{AIT}_{\mathrm{GNS}}$ *for* A *if and only if the Riemann hypothesis holds for* A *and* A *is semisimple.*

We say that $L(s,\chi)$ *satisfies the Riemann hypothesis* if any nontrivial zero s_i of $L(s,\chi)$ satisfies $\mathrm{Re}(s_i) = 1/2$. We say that a nontrivial zero s_i of $L(s,\chi)$ is *simple* if it is a zero of order one.

Combining Theorems 5.2 and 5.3 with [U, Theorem 4.1(iv)] (see Remark 2.1(4)) we obtain the following theorem.

THEOREM 5.4. *Let* $A\colon H \supset \mathrm{dom}(A) \to H$ *be an operator constructed in* [U] *for the principal congruence subgroup* $\Gamma(N)$. *Let* $L(s,\chi)$ *be the Dirichlet L-function for a Dirichlet character* $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. *Then*

(1) $L(s,\chi)$ *satisfies the Riemann hypothesis and its nontrivial zeros are all simple if and only if there exists a standard model* $\mathbb{AIT}_m$ *for* A.
(2) $L(s,\chi)$ *satisfies the Riemann hypothesis and its nontrivial zeros are all simple if and only if there exists a GNS model* $\mathbb{AIT}_{\mathrm{GNS}}$ *for* A.

REMARK 5.1. In the above theorem, if $N = 1$ (i.e. $\Gamma(1) = SL_2(\mathbb{Z})$) then the Dirichlet L-function $L(s,\chi)$ reduces to the Riemann zeta-function $\zeta(s)$. ∎

6. Analogy with the classical theory. Recall that Weil's explicit formula (according to Patterson [Pat]) reads

$$\underbrace{\phi(0)+\phi(1)-\sum_\rho \phi(\rho)}_{\text{spectral term}} = \underbrace{W_\infty(f) + \sum_{p\colon \text{prime}} \log p \sum_{n=1}^{\infty} \{f(p^n)+f(p^{-n})\} p^{-n/2}}_{\text{geometric term}}.$$

Here f is a fast decreasing function on $\mathbb{R}_+$, ϕ is the Mellin transform of f, W_∞ is an appropriate functional of f, and ρ runs over the nontrivial zeros of the Riemann zeta-function (or the L-function), counted with multiplicity. For the original work of Weil, see [1952b] and [1972] of [W2]. See also [C] and [CM, p. 344].

The idea of introducing the model $F_{A,m}(Y)$ of an analogue of the Frobenius morphism in this paper is suggested by the spectral side of the above formula. By [BU, p. 702, Lemma 2.2] there is a function $\phi_Y(s)$ ($Y \in \mathcal{Y}$) which is analytic in an open set $\Supset \Omega_\infty$ such that

(i) $\phi_Y(0) = 1$,
(ii) $\phi_Y(1) = q(Y)$,
(iii) $\phi_Y(s_i) = q(Y)^{s_i}$ if $s_i \in \sigma_Y(A)$,
(iv) $\lim_{s\to s_i} \phi_Y(s)/(s-s_i)^{m_i} = c_{Y,i} \in \mathbb{C}$ for some $c_{Y,i} \neq 0$ if $s_i \in \sigma(A) - \sigma_Y(A)$ with $\nu(s_i) = m_i$.

For this $\phi_Y(s)$, let $\phi_Y(A)\colon H \supset \mathrm{dom}(\phi_Y(A)) \to H$ be defined by

$$\phi_Y(A)x := \lim_{\substack{T\to\infty \\ T\in\mathcal{Y}}} \frac{1}{2\pi i}\Big(\oint_{\partial\Omega_T} \phi_Y(s)(sI-A)^{-1}\,ds\Big)x$$

for

$$x \in \mathrm{dom}(\phi_Y(A)) := \{x \in H \mid \text{the limit } \phi_Y(A)x \text{ exists in } H\}.$$

Then it is easy to prove that $\mathrm{dom}(\phi_Y(A)) = H$ and

$$\phi_Y(A) = F_{A,m}(Y).$$

For the proof use $(sI - M(s_i))^{-1}$ from the proof of Lemma 2.1. It is also easy to see that

$$\mathrm{tr}(\phi_Y(A)) = \sum_{s_i\in\sigma_Y(A)} \mathrm{mult}(s_i)\phi_Y(s_i).$$

Let C be a smooth projective curve (one-dimensional scheme) over a finite field $\mathbb{F}_q$. Let Frob be the Frobenius morphism on C. Then $F_A(Y)$ in §2 is an analogue of Frob.

For the surface $S = C\times C$ over $\mathbb{F}_q$, let $\mathrm{Pic}(S)$ be its Picard group, which we regard as a $\mathbb{Z}$-module, so as to preserve the analogy with Weil divisors. The $\mathbb{R}$-linear space V in §3 is modeled on $\mathrm{Pic}(S)\otimes_{\mathbb{Z}}\mathbb{R}$. The $\mathbb{R}$-bilinear form $\beta(\cdot,\cdot)$ in §3 is modeled on the $\mathbb{R}$-tensored intersection pairing $i(\cdot,\cdot)$ on $\mathrm{Pic}(S)\otimes_{\mathbb{Z}}\mathbb{R}$.

The operator $\Phi_A(Y)$ in (AIT1) is an analogue of the linear map on $\mathrm{Pic}(S)\otimes_{\mathbb{Z}}\mathbb{R}$ induced by the morphism $\mathrm{id}\times\mathrm{Frob}$ on S. Then one may regard v_{01}, v_{10}, $v_\delta(Y)$ and $\Phi_A(Y)^n v_\delta(Y)$ in (AIT1) as analogues of the cycles $\mathrm{pt}\times C$, $C\times\mathrm{pt}$, Δ and Γ_{Frob^n} in $\mathrm{Pic}(S)$, respectively. Here Δ is the diagonal, and Γ_{Frob^n} is the graph of Frob^n. So here is the dictionary:

$\mathrm{Pic}(S)\otimes_{\mathbb{Z}}\mathbb{R}$	V
$\mathrm{pt}\times C$	v_{01}
$C\times \mathrm{pt}$	v_{10}
Δ	$v_\delta(Y)$
Γ_{Frob^n}	$\Phi_A(Y)^n v_\delta(Y)$

The cycles $\mathrm{pt}\times C$, $C\times\mathrm{pt}$, Δ and Γ_{Frob^n} have the following properties:

(i) $i(\mathrm{pt}\times C,\mathrm{pt}\times C)=0$,
(ii) $i(C\times\mathrm{pt},C\times\mathrm{pt})=0$,
(iii) $i(\mathrm{pt}\times C,C\times\mathrm{pt})=1$,
(iv) $i(\Gamma_{\mathrm{Frob}^n},\mathrm{pt}\times C)=1$,
(v) $i(\Gamma_{\mathrm{Frob}^n},C\times\mathrm{pt})=q^n$,
(vi) $i(\Gamma_{\mathrm{Frob}^n},\Gamma_{\mathrm{Frob}^n})=q^n$.

The axioms of (AIT1) are analogues of these properties.

The Hodge property in (AIT2) comes from the classical Hodge index theorem. A Hodge vector h_a corresponds to an ample hyperplane section of S, thereby $\beta(\cdot,h_a)$ gives an analogue of the degree function $\deg\otimes_{\mathbb{Z}}1\colon \mathrm{Pic}(S)\otimes_{\mathbb{Z}}\mathbb{R}\to\mathbb{R}$. Lemma 3.1 is an analogue of the inequality of Castelnuovo–Severi.

The construction of V_m of a standard model in §4 is suggested by the Künneth formula for the étale cohomology. The Tate conjecture for $S=C\times C$ and codimension one is equivalent to the map $\mathrm{Pic}\,S\otimes\mathbb{Q}_\ell\to H^2_{\text{ét}}(S,\mathbb{Q}_\ell(1))$ being bijective (Tate [T2, Proposition (4.3)]). Note that $H^2_{\text{ét}}(S,\mathbb{Q}_\ell(1))=H^2_{\text{ét}}(\bar S,\mathbb{Q}_\ell(1))^{\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$, where $\bar S=S\times_{\mathbb{F}_q}\overline{\mathbb{F}}_q$ (see [T2]). Tate [T1] himself has proven his conjecture for abelian varieties over finite fields for the case of codimension one. From this the Tate conjecture follows also for $S=C\times C$ in the codimension one case. By the Künneth formula for ℓ-adic cohomology we have

$$\begin{aligned}H^2_{\text{ét}}(\bar S,\mathbb{Q}_\ell)\simeq{}&\big(H^0_{\text{ét}}(\bar C,\mathbb{Q}_\ell)\otimes H^2_{\text{ét}}(\bar C,\mathbb{Q}_\ell)\big)\oplus\big(H^1_{\text{ét}}(\bar C,\mathbb{Q}_\ell)\otimes H^1_{\text{ét}}(\bar C,\mathbb{Q}_\ell)\big)\\&\oplus\big(H^2_{\text{ét}}(\bar C,\mathbb{Q}_\ell)\otimes H^0_{\text{ét}}(\bar C,\mathbb{Q}_\ell)\big).\end{aligned}$$

Here $\bar C=C\times_{\mathbb{F}_q}\overline{\mathbb{F}}_q$. The definition of the $\mathbb{R}$-linear space V_m is modeled on this. For the Künneth formula for ℓ-adic cohomology see Milne [Mil, Chap. 6, §8].

For a morphism $\varphi\colon C\to C$, the Lefschetz fixed-point formula for the ℓ-adic étale cohomology group $H^i_{\text{ét}}=H^i_{\text{ét}}(\bar C,\mathbb{Q}_\ell)$ is

$$\mathrm{tr}(\varphi^{*n}|_{H^0_{\text{ét}}})-\mathrm{tr}(\varphi^{*n}|_{H^1_{\text{ét}}})+\mathrm{tr}(\varphi^{*n}|_{H^2_{\text{ét}}})=i(\Gamma_{\varphi^n},\Delta),$$

where Γ_{φ^n} is the graph of φ^n. If $\varphi=\mathrm{Frob}$, then it turns out that

$$\begin{aligned}\mathrm{tr}(\varphi^{*n}|_{H^0_{\text{ét}}})&=1=i(\Gamma_{\varphi^n},\mathrm{pt}\times C)i(\Delta,C\times\mathrm{pt}),\\\mathrm{tr}(\varphi^{*n}|_{H^2_{\text{ét}}})&=q^n=i(\Gamma_{\varphi^n},C\times\mathrm{pt})i(\Delta,\mathrm{pt}\times C).\end{aligned}$$

So the Lefschetz fixed-point formula for $\varphi^n = \mathrm{Frob}^n$ reads

$$\begin{aligned}\mathrm{tr}(\varphi^{*n}|_{H^1_{\text{ét}}}) &= i(\Gamma_{\varphi^n}, \mathrm{pt}\times C)i(\Delta, C\times \mathrm{pt})\\ &\quad + i(\Gamma_{\varphi^n}, C\times \mathrm{pt})i(\Delta, \mathrm{pt}\times C) - i(\Gamma_{\varphi^n}, \Delta)\\ &=: \langle \Gamma_{\varphi^n}, \Delta\rangle_{\mathrm{Pic}(S)\otimes_{\mathbb{Z}}\mathbb{R}}.\end{aligned}$$

(AIT3) is modeled on this. Consider the operators A and $F_{A,m}(Y)$ $(Y \in \mathcal{Y})$ which are extended to $H^\bullet_{\mathbb{C}} = H^0_{\mathbb{C}} \oplus H^1_{\mathbb{C}} \oplus H^2_{\mathbb{C}}$ as in §4. Then we have

$$\begin{aligned}\phi_Y(A)f &= F_{A,m}(Y)f = f = \phi_Y(0)f,\\ \phi_Y(A)g &= F_{A,m}(Y)g = q(Y)g = \phi_Y(1)g,\end{aligned}$$

for $f \in H^0_{\mathbb{C}}$ and $g \in H^2_{\mathbb{C}}$. The operator $\phi_Y(A)$ acting on $H^i_{\mathbb{C}}$ is an analogue of Frob^* acting on $H^i_{\text{ét}}$ $(i = 0, 1, 2)$. Since

$$\begin{aligned}&\Phi_{A,m}(Y)^n v_{\delta,m}(Y)\\ &\quad = \sum_{i=1}^{2g(Y)} e_i^Y \otimes F_{A,m}(Y)^n e_i^Y + f\otimes F_{A,m}(Y)^n g + g\otimes F_{A,m}(Y)^n f\\ &\quad = \sum_{i=1}^{2g(Y)} e_i^Y \otimes F_{A,m}(Y)^n e_i^Y + \phi_Y(1)^n v_{01,m} + \phi_Y(0)^n v_{10,m}\end{aligned}$$

(see the proof of Lemma 4.2), by the proof of Lemma 4.2 we have

$$\begin{aligned}\mathrm{tr}(\phi_Y(A)^n|_{H^0_{\mathbb{C}}}) &= \phi_Y(0)^n\\ &= (\beta_m)_{\mathbb{C}}(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{01,m}) \cdot (\beta_m)_{\mathbb{C}}(v_{10,m}, v_{\delta,m}(Y)),\\ \mathrm{tr}(\phi_Y(A)^n|_{H^2_{\mathbb{C}}}) &= \phi_Y(1)^n\\ &= (\beta_m)_{\mathbb{C}}(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{10,m}) \cdot (\beta_m)_{\mathbb{C}}(v_{01,m}, v_{\delta,m}(Y)).\end{aligned}$$

Therefore $(**)$ yields

$$\begin{aligned}&\mathrm{tr}(\phi_Y(A)^n|_{H^0_{\mathbb{C}}}) - \mathrm{tr}(\phi_Y(A)^n|_{H^1_{\mathbb{C}}}) + \mathrm{tr}(\phi_Y(A)^n|_{H^2_{\mathbb{C}}})\\ &\qquad = (\beta_m)_{\mathbb{C}}(\Phi_{A,m}(Y)^n v_{\delta,m}(Y), v_{\delta,m}(Y)),\end{aligned}$$

which is equivalent to (AIT3).

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