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Network connectivity dynamic modelling

Keywords

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Abstract

The chapter presents an approach based on basic notions issued from the graph theory and from the system reliability theory. The approach seeks to describe the transitions in the connectivity state of non-directional graphs induced by systemic losses of nodes and edges. Each component loss (node or edge) represents a transition and results in a connectivity degradation. Some of the transitions are classified as non-critical while others are critical. Degradations are measured using the notion of topological graph diameter issued from the graph theory. The critical transition notion is issued from the system reliability theory. The approach determines the degraded graph diameter corresponding to each possible transition and subsequently the criticality of the transition. The criticality threshold is determined by the highest acceptable connectivity order which is a function of the degraded graph diameter. A network with 9 nodes and 15 edges is used as an academic study case to illustrate the applicability of the approach. Nodes are supposed to be identical, as well as the edges. All network components (nodes and edges) are mutually independent. These precedent hypotheses are intended to evacuate all sources of numerical useless complexity. As our main objective is to highlight the original characteristic of the proposed approach.

1. Introduction

The chapter presents an approach to describe the transitions in the global connectivity state of non-directional graphs induced by systemic losses of nodes and edges. Systemic component losses exclude all losses induced by man or natural hazards. Component losses (node or edge) result in degradations in the network overall connectivity. Network transitions from one connectivity level to another can be determined. Some of these transitions are classified as non-critical while others are critical. The topological graph diameter $\mathfrak{D}_G$. Is a natural candidate to measure network connectivity and the criticality of the connectivity transitions. Then the proposed approach determines the graph diameter corresponding to each possible transition.

The criticality is determined by a threshold dependent on the highest acceptable diameter of a degraded graph.

A network with 9 nodes and 15 edges is used as an academic study case to illustrate the applicability of the approach. In this study case, nodes are identical, as well as the edges. Besides, nodes and edges are mutually independent. These working hypotheses aim at evacuating all sources of numerical useless complexity. The chapter's main objective is to highlight the original characteristic of the proposed approach.

The two main streams describing network connectivity are briefly presented in Section 2, known as: the topological and the algebraic connectivity. In Section 3, the Binary Topological Model (BTM) is presented. It provides the numerical tool to determine the graph diameter at different connectivity states: degraded or not. In Section 4, the academic case study is presented as well as the iterative procedure to determine the graph diameter. The criticality is defined and

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the critical states are identified. The probabilistic dynamic model to determine the connectivity state of the network is established, as well. Finally, some numerical results are illustrated. Section 5 ends the chapter by a synthetic presentation of the main features of the presented approach.

2. Network connectivity

Network connectivity modelling and analysis is an issue that is receiving strong intention from academia, specialised research institutions, industry, and utilities. This multiplicity of interests gave rise to a large amount of literature of all kinds of complexity and application. It varies from purely theoretically oriented to standardisation and regulation passing by system engineering R&D activities. No one connectivity model or connectivity assessment methodology could satisfy that wide range of interests characterising the place of the networks in our modern societies (Rose, 2015). One can schematically distinguish two main streams in modelling the networks connectivity: conceptually oriented or engineering oriented. The two streams are not opposite to each other but rather complementary.

In the conceptual stream, mathematical knowledge and assets are the most required. The graph theory is by so far, un excellent underline asset in that stream. The models that can be expected from that stream are very often descriptive but global.

The engineering stream does rather require much more applied mathematics and good insight of networks operational conditions and performance measure in many sectors, such as: communication, transport, energy transmission, pipelines, etc. The models that can be expected from that stream are schematically classified in two subclasses: *systemic and qualitative* or *analytic and quantitative*.

Nevertheless, network resilience modelling presents even much more challenging concerns. On one hand, what does really *resilience* of a system mean? And specifically, what could the resilience of a network really mean?

The author thinks that a solution may come up if we can develop a meaningful and measurable *concept* describing through a performance dynamic model the network performance.

But above all, can one develop a network performance dynamic model without modelling the network global connectivity?

If not! And it is *not* from our standpoint!

Can one introduce *time* in some way to describe dynamically the global connectivity, given that resilience cannot be but a dynamic notion? In fact, the *connectivity* notion as introduced in the graph theory is static.

We will briefly present the connectivity as described by the graph theory and then be proceeding to a quick presentation of the Binary Topological Model – BTM before proposing the connectivity dynamic modelling approach.

2.1. Graph theory and connectivity

The use of graphs to represent and solve mathematical problems can be tracked down to the Swiss mathematician Leonhard Euler. In 1736, L. Euler published the solution to the Königsberg bridge problem (how to traverse each of the bridges of the Prussian city of Königsberg only once). However, we are still far from talking about the *graph theory* – GT as proposed by König. In his treatise, König (König, 1936) treated four selected recreational mathematical problems: Königsberg bridges, a problem of describing polygons, a problem of configuring dominoes and the problem of circulating in mazes. König was in fact investigating on the recreational problems, (Mizuno, 2011), not more!

The GT provides certainly the most consistent and coherent formal frame to describe, characterise, and classify networks. Yet, the concept of *connectivity* does not exist, strictly speaking, in the GT. However, the *graph diameter* is a fundamental concept of the GT that many researchers and engineers consented to use as a measure of the graph global *connectivity*.

Why not?

Following the notations of the GT, a graph G is composed of N nodes (vertices) connected through a set of E edges. Each couple of nodes in the graph may be connected or not. The binary square matrix that describes the node-to-node (N-N) connectivity is called the adjacent matrix A. The matrix's elements are binary: 1 for directly N-N connected otherwise null. We call it a *1st order connectivity*. Obviously, not all the couples of nodes are directly connected in a given graph. Still, they can be connected through paths composed of more than one edge. We call that ***higher*** *order N-N connectivity*.

There are many useful metrics to measure graph connectivity. Among the well-known are the degree distribution (Barabasi, 1999), the characteristic path length (Watts, 1998), the clustering coefficient (Albert, 2002) and the graph diameter (König, 1936; Wilson, 1998; Bondy, 2008). We have chosen to use the graph diameter. A graph diameter is the longest of the shortest *N-N* paths in a graph. The issue is that the graph diameter is a static notion by conception and cannot directly be described by a dynamic model.

However, does exist any contradiction between this latest claim and the important R&D activities on dynamic graph modelling using spectral analysis and Laplacian matrix eigenvalues, since decades? Let us, then put some light on this and get closer to the: graph spectra and Laplacian matrix.

2.2. Graph spectra and Laplacian matrix

The graph spectra and the Laplacian matrix were developed in response to a specific problem in chemistry (the application in a theory of unsaturated conjugated hydrocarbons known as the Hückel molecular orbital theory (Hükel, 1931). That is also confirmed by Cvetković et al. (Cvetković et al., 1979) who is one of the pioneers having formalised the graph spectra concept and the Laplacian matrix spectra analysis in its modern form. That specific need in chemistry required the introduction of linear algebra in order to investigate the characteristics of the adjacent matrix and gave birth to the concepts: *Laplacian matrices* and *eigenvalues*. Strictly speaking, neither the *Laplacian matrix* nor the *eigen values* are fundamental concepts of the GT.

The eigenvalues of the Laplacian matrix *A* are usually denoted by $l_1, l_2, l_3,..., l_n$. These eigenvalues are also called the *spectra of the graph G* and are often denoted in the increasing order: $l_1 < l_2 < l_3 < ... < l_n$. The largest eigenvalue l_n is called the *index of the graph*, (Cvetković, 2000, 2010), and the spectral radius of the graph G, $r(G)$, as well. This eigenvalue has received the highest attention in this context, since the number of walks of length k is, approximately, proportional to $r^k(G)$. The least smallest eigenvalue l_2 has attractive properties, as well (Mohar, 1991; Cvetković, 2011; Van Mieghem, 2011). It receives a specific interest because of its relation to numerous graph invariants including connectivity, expanding properties, isoperimetric number, maximum cut, independence number, genus, diameter, mean distance, and bandwidth-type parameters of a graph (Mohar, 1991). Fiedler (Fiedler, 1973) has investigated several of their properties. This eigenvalue is known as the *algebraic connectivity* of a graph.

Notice the emergence of the term *connectivity*. Underling the definition as *algebraic connectivity* is obviously to distinguish this connectivity from the "topological connectivity" consentingly measured using the graph diameter $\mathfrak{D}_G$.

The highest success of the Laplacian matrix and the graph spectra analyse in systems engineering is in the field of the computer networks. This is because of the important role that the highest eigenvalue l_n plays in modelling virus propagation in computer networks, (Cvetković, 2011; Van Mieghem, 2011). The smaller the highest eigenvalue, the higher is the robustness of the network against the spread of viruses. In fact, it was shown by Wang Y et al. (Wang Y et al., 2003) that the epidemic threshold in spreading viruses is proportional to $1/l_2$.

The pioneering work of Cvetković et al. (Cvetković et al., 1979) has definitively given to that *connectivity* its full *algebraic* nature. This *algebraic* nature rendered the *algebraic connectivity* extremely popular in fields such as: membrane vibration modelling, dimer problem (thermodynamic properties of the diatomic molecules adsorbed on the surface of a crystal) and random walks of various kinds in a lattice graph. A complete list of problems in physics, chemistry, and computer sciences where the graph spectra approach was of great usefulness is given in (Cvetković, 2000). Many of these applications are dynamic. This wide success gave a strong authority to that notion of Algebraic connectivity that many engineers and researchers struggling to transpose to the network resilience dynamic modelling.

Does exist any recent development in the *algebraic connectivity* that may indicate a universal transposability to all types of network application and use, including network resilience?

Yes, one excellent example is the *effective graph resistance* (Ellens, 2011; Klein, 1993). As all the algebraic approaches, it investigates *algebraic* *connectivity*. It can be extended to model resilience if it integrates the connectivity states transition dynamics.

2.3. Topological vs algebraic connectivity

The determination of the algebraic connectivity is based on the use of linear algebra concepts and tools. Calculating the eigenvalues for weighted and binary adjacent matrices of large size graphs is a source of many awesome numerical problems. Adding to that, the algebraic connectivity is a fuzzy definition, at least from engineering systems standpoint. The same can be told regarding the *eigen values*.
Switching to the topological connectivity which is simply measured by the graph diameter $\mathfrak{D}_G$. allow us to get rid of these fuzzy definitions of the algebraic connectivity in favour of a physical definition based on the number of edges linking each couple of nodes.
But, if we remove linear algebra contribution (Laplacian matrix and eigenvalues), how will the BTM introduce the dynamic aspect?

3. Network binary topological model

Following the GT, the elements of the *adjacency matrix* – A are binary with value: 1 for directly connected couples of nodes otherwise null. The Binary Topological Model (BTM) refers to it as an adjacency matrix of 1st order, denoted by A^1. The BTM extends the notion of the *adjacency matrix* to higher orders of connectivity. The BTM uses the notation A^n for the adjacency matrix of the n^{th} order and whose binary elements are e_{ij}^n with a value equal to 1 if the couple (i, j) is connected through *n*-edges otherwise null (Eid, 2017, Eid et al., 2012). The adjacency matrices of higher orders are determined as following. First, one constructs the matrix U^{n+1} defined as:

$$U^{n+1} = A^1 \cdot A^n. \tag{1}$$

Then, one constructs the adjacency matrix A^{n+1} and its elements e_{ij}^n as following:

$$e_{ij}^n = \begin{cases} 0 & \text{if} \quad i = j \\ 0 & \text{if} \quad u_{ij}^n = 0 \\ 1 & \text{if} \quad u_{ij}^n > 0. \end{cases} \tag{2}$$

Then, one keeps iterating between equations (1) and (2) until all the elements e_{ij}^n become equal to unity except the diagonal ones. That means when each couple of nodes in the graph is connected. The corresponding order n will then be called ***the nominal connectivity order*** – NCO. The BTM uses the NCO as a measure of the network connectivity.
The NCO is the minimal number of connections required such as each node in the network in connected with all the others. It is the minimal dimension of the topological space in which each node sees all the others. As if they were distributed on the same spherical surface in an n-dimensional space. Obviously, the NCO is significantly different from the algebraic graph diameter but the closest to the classical graph diameter $\mathfrak{D}_G$.
The determination of the NCO or the graph diameter $\mathfrak{D}_G$ is immediate and does not require any use of advanced linear algebra concepts such as the eigenvalues. We just use iterative matrix products with no use of any determinant analysis. Still, we should introduce another concept which is the connectivity states and the transitions. This is the last step before stepping forward to the dynamic modelling.

3.1. Connectivity states and transitions

The NCO characterises the network nominal operational state as it has been designed by the designer to achieve a required performance level. This is the operational state which is accepted by the operators and that has the safety/security level required by the competent authorities. The NCO is a *nominal* and *global* characteristic measure.
Any failure of a node or an edge produces a degradation in the network global connectivity. It impacts negatively on the network performance level. We may talk about performances degradation. Local failures (nodes or edges) will immediately produce a degradation in the network connectivity expressed by the adjacency matrix. The degraded adjacency matrix will show a new graph diameter ${\mathfrak{D}_G}^*$ associated to a higher connectivity order. The Degraded Connectivity Order (DCO) is obviously higher than the NCO.
There are then many possible degraded connectivity levels, depending on the size and complexity of the network. Determining the complete set of all DCO is carried on using the above it-

erative matrices product procedure, based on equations (1) and (2), after switching off all possible edges and nodes one by one. One can, then, construct the graph of all possible connectivity states (nominal and degraded) and the corresponding transitions between each couple of states. We get then the spectra of the network operational states. This spectral graph of the operational states characterises the network. The spectra are not algebraic but topological ones.

The Topological Connectivity Spectral Analysis (TCSA) allows us to construct a dynamic model describing the operational life of a network. It allows us to:

- setup a conventional threshold at a given degradation state and declare that the network is unavailable at a DCO higher or equal to the threshold value, i.e. beyond this degraded graph diameter,
- once after, the critical and the non-critical transitions are identified (all transitions resulting in a DCO higher than the threshold are critical),
- given the well-established association between all failure modes and transitions (critical or not), one can determine the probability of the network to be unavailable according to the conventional threshold.

Thanks to the TCSA, the problem is transformed to a classical multi-state system modelling and can be treated using the system reliability theory (SRT), (Natvig, 2011; Lisnianski et al., 2010). As required by the SRT, some of the degraded states are acceptable while others are not. The network *service supply rupture* will be declared if the network connectivity state at time *t* belongs to one of the states that are considered as *unacceptable* / *rupture*.

The acceptable connectivity states include the nominal connectivity state and all other non-critical degraded states. The conventional threshold fixes the connectivity order above which the service supply rupture is declared. Different thresholds may even be defined to signal the passage of the network from one awareness level to another during the evolution of a given degradation scenario. The degradation process can also be dynamically described.

The network transitions are governed by a system of a first order differential equations describing the probability of being in each state at time *t*. This system of differential equations can be linear or non-linear. The inputs that one needs are: the failure and repair rates of each node and *N*-*N* link, the dependency between failures and the initial conditions. Besides, one needs the network topological graph.

Given the availability of the input data, one can proceed to the modelling of the dynamic behaviour of the network connectivity. Calculations can be carried on: analytically or numerically (Markov, semi-Markov, Monte-Carlo simulation), dependant on the inputs' complexity, (Natvig, 2011; Lisnianski et al., 2010).

4. The case study

In Figure 1, one presents a network with 9 nodes and 15 edges.

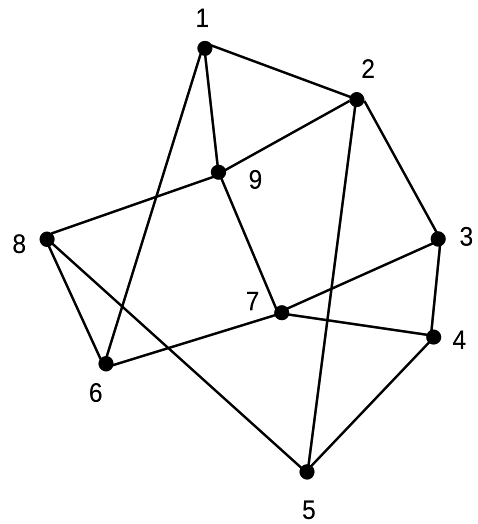

Figure 1. Schematic presentation of the studied network.

Each of the nodes is directly connected to only three others except for the nodes {2,7,9} that are directly connected to four others. The adjacency matrix is given in Table 1. The adjacency matrix describes the 1st order connectivity state of the network, i.e., *N*-*N* connectivity through only one edge. Applying the procedure described in Section 3 using (1) and (2), one can follow the progression of the *N*-*N* connectivity at higher orders. Table 1 shows the state of the connectivity of the 1st order. This is the network adjacency matrix A^1. We may underline that the network connectivity is irregular as some nodes are connected to four nodes while some others are connected to only three other nodes.

Table 1. The adjacency matrix A^1 of the studied network (the 1st order connectivity state)

	1	2	3	4	5	6	7	8	9	
1	0	1	0	0	0	1	0	0	1	3
2	1	0	1	0	1	0	0	0	1	4
3	0	1	0	1	0	0	1	0	0	3
4	0	0	1	0	1	0	1	0	0	3
5	0	1	0	1	0	0	0	1	0	3
6	1	0	0	0	0	0	1	1	0	3
7	0	0	1	1	0	1	0	0	1	4
8	0	0	0	0	1	1	0	0	1	3
9	1	1	0	0	0	0	1	1	0	4

Table 2 shows the state of the network connectivity of the 2nd order. We notice that the network connectivity has improved. For example, the node {1} is now connected to four more nodes (7 – 3 = 4). The nodes {2,5,6,7,9} have already access to the other eight nodes.

Table 2. The 2nd order connectivity state of the network

	1	2	3	4	5	6	7	8	9	
1	0	1	1	0	1	1	1	1	1	7
2	1	0	1	1	1	1	1	1	1	8
3	1	1	0	1	1	1	1	0	1	7
4	0	1	1	0	1	1	1	1	1	7
5	1	1	1	1	0	1	1	1	1	8
6	1	1	1	1	1	0	1	1	1	8
7	1	1	1	1	1	1	0	1	1	8
8	1	1	0	1	1	1	1	0	1	7
9	1	1	1	1	1	1	1	1	0	8

Table 3 shows the state of the network connectivity of the 3rd order. At the 3rd order, the network is regular and complete. Each node has access to all the others.

Table 3. The 3rd order connectivity state of the network

	1	2	3	4	5	6	7	8	9	
1	0	1	1	1	1	1	1	1	1	8
2	1	0	1	1	1	1	1	1	1	8
3	1	1	0	1	1	1	1	1	1	8
4	1	1	1	0	1	1	1	1	1	8
5	1	1	1	1	0	1	1	1	1	8
6	1	1	1	1	1	0	1	1	1	8
7	1	1	1	1	1	1	0	1	1	8
8	1	1	1	1	1	1	1	0	1	8
9	1	1	1	1	1	1	1	1	0	8

Now, it is useless to continue to follow the connectivity progression for orders higher than three. All higher order connectivity matrices will show the same characteristics: regular and complete. The NCO of this network is then equal to three (NCO = 3).

The minimal order of the regular and complete connectivity state is then determined and characterises the network connectivity diameter ($\mathfrak{D}_G$ = 3).

The loss of some nodes and edges may or may not increase the NCO. If the losses result in a transition in the NCO to a higher connectivity order, the loss and the corresponding transition will be called and signalled "critical" if higher than a given threshold. The transition of the network connectivity to a higher connectivity order is considered as a functional degradation.

4.1. Losses and connectivity degradation

The loss of one or more of the edges or the nodes would increase the graph diameter $\mathfrak{D}_G$ and subsequently the network connectivity order will become higher than the NCO. This increase corresponds to a functional degradation in the network connectivity. The departure from the NCO will be used to measure the functional degradation of a network resulted by the edge and node losses. The departure from the NCO will be judged acceptable or not in the light of some operational and safety requirements. To better illustrate that we propose to follow up with the systemic degradation of the study case.

Let $l_{n,m}$ describe the availability of the direct link between two nodes (n, m). The loss of this direct link $\overline{l_{n,m}}$ can logically be described by the following Boolean expression:

$$\overline{l_{n,m}} = \overline{e_{n,n}} + \overline{e_{n,m}} + \overline{e_{m,m}} \tag{3}$$

where, the set $\{\overline{e_{n,n}}, \overline{e_{m,m}}\}$ describes the loss (unavailability) of the nodes n and m and the set $\{\overline{e_{n,m}}\}$ describes the loss of the edge (n, m). The operator '+' is the Boolean operator OR (in this expression and all the following Boolean ones).

The 1st order possible losses are distinguished in link-losses $\mathcal{L}^1(E)$ and node-losses $\mathcal{L}^1(N)$, such as:

$$\mathcal{L}^1(E) = \{\overline{l_{1,2}}, \overline{l_{1,6}}, \overline{l_{1,9}}, \overline{l_{2,3}}, \overline{l_{2,5}}, \overline{l_{2,9}}, \overline{l_{3,4}}, \overline{l_{3,7}}, \overline{l_{4,5}}, \overline{l_{4,7}}, \overline{l_{5,8}}, \overline{l_{6,7}}, \overline{l_{6,8}}, \overline{l_{7,9}}, \overline{l_{8,9}}\} \quad (4)$$

$$\mathcal{L}^1(N) = \{\overline{e_{1,1}}, \overline{e_{2,2}}, \overline{e_{3,3}}, \overline{e_{4,4}}, \overline{e_{5,5}}, \overline{e_{6,6}}, \overline{e_{7,7}}, \overline{e_{8,8}}, \overline{e_{9,9}}\} \quad (5)$$

But not all these possible losses given in (4) and (5) will result in a departure from the NCO of the network. One should then use the iterative matrix product procedure described in (1) and (2) to determine which of the losses will degrade the connectivity of the network.
The global set $\mathcal{L}^1$ of 1st order losses is then described by:

$$\mathcal{L}^1 = \mathcal{L}^1(E) + \mathcal{L}^1(N) \quad (6)$$

Considering the loss $\overline{l_{n,m}}$ as described by (3), one can reduce expression (6) to:

$$\mathcal{L}^1 = \mathcal{L}'^1(E) + \mathcal{N}^1 \quad (7)$$

Where the sets $\mathcal{L}'^1(E)$ and $\mathcal{N}^1$ are given in (8) and (9) below.
The possible 1st order global losses may then be distinguished in an edge-losses set $\mathcal{L}'^1(E)$ and an independent node-losses set $\mathcal{L}^1(N)$, such as:

$$\mathcal{L}'^1(E) = \{\overline{e_{1,2}}, \overline{e_{1,6}}, \overline{e_{1,9}}, \overline{e_{2,3}}, \overline{e_{2,5}}, \overline{e_{2,9}}, \overline{e_{3,4}}, \overline{e_{3,7}}, \overline{e_{4,5}}, \overline{e_{4,7}}, \overline{e_{5,8}}, \overline{e_{6,7}}, \overline{e_{6,8}}, \overline{e_{7,9}}, \overline{e_{8,9}}\} \quad (8)$$

$$\mathcal{N}^1 = \{\overline{e_{1,1}}, \overline{e_{2,2}}, \overline{e_{3,3}}, \overline{e_{4,4}}, \overline{e_{5,5}}, \overline{e_{6,6}}, \overline{e_{7,7}}, \overline{e_{8,8}}, \overline{e_{9,9}}\} \quad (9)$$

Still, what is a critical transition? The departure from the NCO may be considered as a critical transition if it brings the network connectivity to an undesired state. However, additional criteria may be considered as well, as we will see later.
The procedure to identify critical losses amongst those identified in equations (8) and (9) is:

1. Identify all possible node-losses and edge-losses, and all their combinations.
2. Put ***i*** = 0
3. Consider the loss (i + 1)
4. Update the network adjacency matrix.
5. Determine the degraded connectivity order DCO, using the iterative procedure described by (1) and (2).
6. Is the degraded network connectivity considered as critical?
7. If non, Go-to step #3.
8. If yes, score the loss (i + 1) as a critical one and the correspondent graph diameter $\mathfrak{D}_G$ (i + 1).
9. Go-to step #3 as many times as the number of the identified losses of all orders (in step #1).
10. STOP.

4.2. Critical transitions

In the present case the critical connectivity state is determined according to the following conventional criteria:

1. “all losses that result in an increase in the NCO” (all losses associated with $\mathfrak{D}_G > 3$), or
2. “all losses of an order equal to or higher than three”, whatever is $\mathfrak{D}_G$.

Applying the 1st criteria and separating the set of the node-losses ($\mathcal{N}^n$) and the edge-losses ($\mathcal{E}^n$) of order (n), one can determine the following: the set of 1st order critical edge-losses $\mathcal{E}^1$ is empty:

$$\mathcal{E}^1 = \emptyset \quad (10)$$

none of the 1st order edge-losses, (8), resulted a critical transition.
While the set of the 1st order critical node-losses $\mathcal{N}^1$ is determined such as:

$$\mathcal{N}^1 = \overline{e_{11}} + \overline{e_{22}} + \overline{e_{33}} + \overline{e_{44}} + \overline{e_{55}} + \overline{e_{66}} + \overline{e_{77}} + \overline{e_{88}} + \overline{e_{99}} \quad (11)$$

all the 1st order node-losses, (9), result in critical transitions.
The set of 2nd order critical edge-losses is:

$$\mathcal{E}^2 = \{(\overline{l_{1,2}}, \overline{l_{1,9}}), (\overline{l_{1,2}}, \overline{l_{2,9}}), (\overline{l_{1,2}}, \overline{l_{4,7}}), (\overline{l_{1,2}}, \overline{l_{6,7}}), (\overline{l_{1,6}}, \overline{l_{1,9}}), (\overline{l_{1,6}}, \overline{l_{6,7}}), (\overline{l_{2,3}}, \overline{l_{3,7}}), (\overline{l_{2,3}}, \overline{l_{4,5}}), (\overline{l_{2,3}}, \overline{l_{6,7}}), (\overline{l_{2,3}}, \overline{l_{7,9}}), (\overline{l_{2,5}}, \overline{l_{4,5}}), (\overline{l_{2,5}}, \overline{l_{5,8}}), (\overline{l_{2,9}}, \overline{l_{5,8}}), (\overline{l_{3,4}}, \overline{l_{4,5}}), (\overline{l_{3,4}}, \overline{l_{4,7}}), (\overline{l_{3,7}}, \overline{l_{4,7}}), (\overline{l_{4,5}}, \overline{l_{4,7}}), (\overline{l_{4,5}}, \overline{l_{6,7}}), (\overline{l_{4,7}}, \overline{l_{5,8}}), (\overline{l_{5,8}}, \overline{l_{6,7}}), (\overline{l_{5,8}}, \overline{l_{8,9}}), (\overline{l_{6,7}}, \overline{l_{6,8}}), (\overline{l_{6,7}}, \overline{l_{7,9}}), (\overline{l_{6,8}}, \overline{l_{8,9}})\}. \quad (12)$$

There are only 24 critical second order cut-sets out of the possible 105 cut-sets $\left(\binom{15}{2} = 105\right)$.

Cut sets with orders higher than two, $\mathcal{L}^{n>2}$, are all considered as critical in application of the 2nd criteria mention above. They result in network fragmentation, i.e., the loss of the connectivity completeness in the best case and the isolation of all the nodes in the worst case.

One can then write:

$$\mathcal{L}^{n>2} = \mathcal{L}^{n>2}(E) + \mathcal{L}^{n>2}(N).$$

The global Boolean expression of critical losses of all orders $\mathcal{L}$ is then given by the following Boolean expression:

$$\mathcal{L} = \mathcal{N}^1 + \mathcal{E}^2 + \mathcal{L}^{n>2}(E) + \mathcal{L}^{n>2}(N)$$
$$= \mathcal{N}^1 + \left(\mathcal{E}^2 + \mathcal{L}^{n>2}(E)\right). \quad (13)$$

We underline that the set $\mathcal{L}$ contains two independent subsets $\{\mathcal{N}^1\}$ and $\{\mathcal{E}^2, \mathcal{L}^{n>2}(E)\}$. The first subset contains exclusively node-losses and the second contains exclusively edge-losses. Additionally, the subsets $\{\mathcal{E}^2\}$ and $\{\mathcal{L}^{n>2}(E)\}$ are disjoint subsets.

Once the undesired states are defined and the correspondent critical transitions are determined, one can proceed to developing the degradation dynamic model of the network with respect to the defined failure criteria.

4.3. Connectivity time-degradation

Having identified the critical connectivity states according to the given criteria, we would like to determine the probability to be in any of these critical states. In other words, to determine the network unavailability.

Two basic pieces of knowledge are still required: the unavailability of each node ($u_n(t)$) and of each edge ($u_e(t)$), as a function of time. Notice that a lost component is an unavailable one. We consider only the systemic unavailability. That excludes man-induced or environment-induced component losses.

The unavailability of any of the components (nodes, edges) can be issued from operational feedback data, empirical models, or analytical models. As we opt for the analytical models, the required data are the failure rate and the repair rate of the nodes and the edges, see Table 4.

Table 4. Components failure and repair data

	Failure rate (λ) (h^{-1})	Repair rate (μ) (h^{-1})
Node	10^{-3}	$4\cdot10^{-2}$
Edge	$2.5\cdot10^{-2}$	10^{-1}

The transition rates are supposed constant with time and mutually independent. To avoid all numerical useless complexity, we assumed that all nodes are identical as well as the edges.

The probability to be in one of the critical connectivity-states is the probability to be in the set $\mathcal{L}$, (13). An equivalent expression to (13) is:

$$\mathcal{L} = \mathcal{N}^1 + \overline{\mathcal{N}^1}\cdot\left(\mathcal{E}^2 + \mathcal{L}^{n>2}(E)\right), \quad (14)$$

where the operator '·' is the Boolean 'AND'. expression (14) is easier regarding numerical use. The probability $Q_{\mathcal{L}}(t)$ that the network be in a critical state, will then be determined as:

$$Q_{\mathcal{L}}(t) = Q_{\mathcal{N}^1}(t)$$
$$+ \left(1 - Q_{\mathcal{N}^1}(t)\right)\left[Q_{\mathcal{E}^2}(t) + Q_{\mathcal{L}^{n>2}(E)}(t)\right]. \quad (15)$$

Notice that (15) is a numerical algebraic equation not a Boolean expression.

Where $Q_{\mathcal{N}^1}(t)$, $Q_{\mathcal{E}^2}(t)$, and $Q_{\mathcal{L}^{n>2}(E)}(t)$ are the probabilities to be in the critical sets $\mathcal{N}^1$, $\mathcal{E}^2$, and $\mathcal{L}^{n>2}(E)$, respectively. Given that nodes are identical, edges are identical, these probabilities are determined by as following:

$$Q_{\mathcal{N}^1}(t) = 1 - [(1 - q_n(t))]^9, \quad (15)$$

$$Q_{\mathcal{E}^2}(t) = 24\cdot(q_e(t))^2\cdot(1 - q_e(t))^{15-2} \quad (16)$$

$$Q_{\mathcal{L}^{n>2}(E)}(t),$$

$$= \sum_{j=3}^{15}\left[\binom{15}{j}(q_e(t))^j(1 - q_e(t))^{15-j}\right], \quad (17)$$

where q_n(t) and $q_e(t)$ are the unavailability of a node and an edge, respectively.

We can now proceed to the quantification of the probability of being in a critical state and the partial contribution of node-losses or edge-losses into the critical state of the network.

4.4. Numerical results and discussion

Accordingly, the unavailability $q_c(t)$ of each component 'c' will be determined by:

$$q_c(t) = \frac{\lambda_c}{\lambda_c+\mu_c}\left(1 - e^{-(\lambda_c+\mu_c)t}\right), \tag{18}$$

where the suffix c refers to nodes it is n and to edges if it is e.
The time-profiles of the node and the edge unavailability are illustrated in Figure 2.

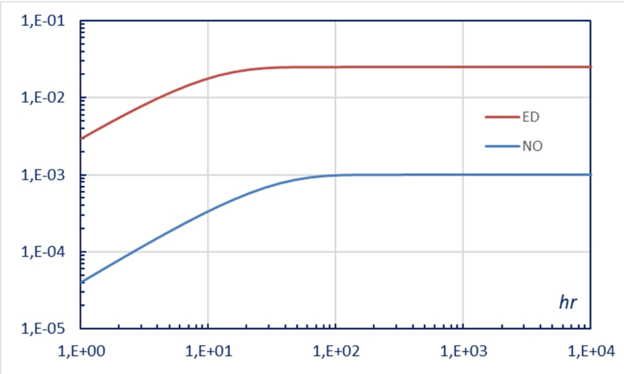

Figure 2. Components' unavailability vs time (ED: edges, NO: nodes).

The intermediate probabilities $Q_{\mathcal{N}^1}(t)$, $Q_{\mathcal{E}^2}(t)$, $Q_{\mathcal{L}^{n>2}(E)}(t)$ can then be directly calculated using equations (15)–(17).
Finally, one can determine the probability $Q_{\mathcal{L}}(t)$, (14), that the network is in a critical state and the contribution of the node-losses and edge-losses into $Q_{\mathcal{L}}(t)$, Figure 3.

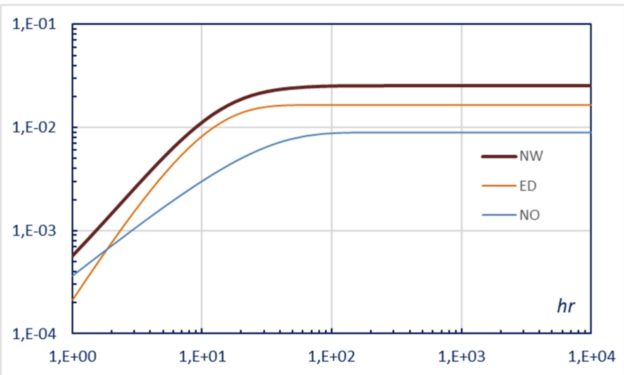

Figure 3. The probabilities $Q_{\mathcal{L}}(t)$, $Q_{\mathcal{N}^1}(t)$, and $\left[Q_{\mathcal{E}^2}(t) + Q_{\mathcal{L}^{n>2}(E)}(t)\right]$ vs time (NW: network, ED: edges, NO: nodes).

We may also be interested in determining the conditional probability of edge-losses and of node-losses, given that the network is in a critical state, Figure 4. That presents the relative contribution versus time of each type of losses in the overall network critical state.

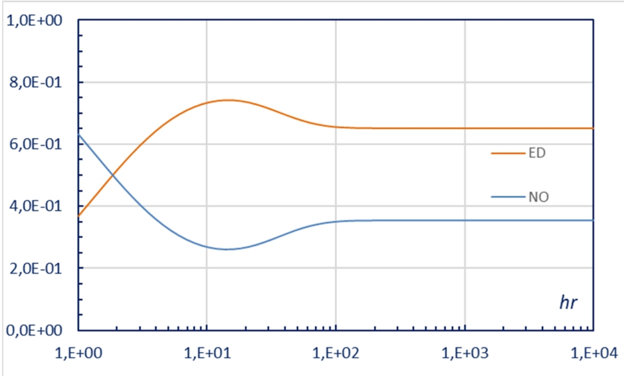

Figure 4. The relative contribution of different type of losses into the critical state of the network vs time (ED: edges, NO: nodes).

5. Conclusion

In this chapter we presented a dynamic probabilistic approach to assess the global connectivity of networks.
For this purpose, we used graph diameter $\mathfrak{D}_G$ to measure the connectivity, as defined by the graph theory. However, we propose a new procedure to determine graphs diameters $\mathfrak{D}_G$. Still, the formalism of the graph theory is static.
As some of the node-losses and the edge-losses may result in a departure from the nominal graph diameter $\mathfrak{D}_G$, they will be identified and tagged. The degraded network will have a new diameter $\mathfrak{D}$ higher then $\mathfrak{D}_G$.
The couples (losses, $\mathfrak{D}$) will be assessed with respect to a criticality threshold established in view of some requirements of different origin: operational, safety, economic, strategic, etc.
Subsequently, the couples (loss, $\mathfrak{D}$) that produce critical connectivity transitions will be isolated.
Finally, the probability of these critical transitions can be determined. The partial contributions of node-losses and of edge-losses can also be determined.
To demonstrate the applicability of the demarche, an academic study case is used. The sources of numerical complexity are eliminated, as we supposed that nodes are identical and edges as well. Besides, the components' unavailability is supposed analytically described.
Our main objective is to evidence the main char-

acteristic of the demarche. These characteristics are recalled as: topological and dynamic.

This topological characteristic is directly issued from the graph theory. Precisely, the use of the graph diameter as the unique measure of the graph global connectivity state.

The dynamic description of the connectivity degradation is achieved through the discrimination between the acceptable connectivity degradations and the critical ones. The notion of critical states and transition is directly issued from the system reliability theory.

The discrimination between critical and non-critical states is decided upon a threshold determined by a critical diameter: the maximum acceptable degraded diameter, amongst others.

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