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OSCILLATION OF SOLUTIONS TO NON-LINEAR DIFFERENCE EQUATIONS WITH SEVERAL ADVANCED ARGUMENTS

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Abstract. This work concerns the oscillation and asymptotic properties of solutions to the non-linear difference equation with advanced arguments

$$x_{n+1} - x_n = \sum_{i=1}^{m} f_{i,n}(x_{n+h_{i,n}}).$$

We establish sufficient conditions for the existence of positive, and negative solutions. Then we obtain conditions for solutions to be bounded, convergent to positive infinity and to negative infinity and to zero. Also we obtain conditions for all solutions to be oscillatory.

Keywords: advanced difference equation, non-oscillatory solution.

Mathematics Subject Classification: 39A11.

1. INTRODUCTION

In recent years there has been a lot of research concerning the oscillation of solutions to difference and differential equations with advanced arguments. These equations appear in mathematical models in which the present state depends on future states [1, 4, 7, 19]. The strong interest in these equations arises from having applications such as population dynamics where a difference equation with constant advanced arguments can serve as a mathematical model that includes a k-th generation [3]. Nowadays there exists an extensive literature on the oscillation theory of advanced type differential and difference equations. See, for example, the references in this article, and the references therein.

In this article we study the oscillation and asymptotic properties of solutions to the advanced difference equation

$$x_{n+1} - x_n = \sum_{i=1}^{m} f_{i,n}(x_{n+h_{i,n}}), \tag{1.1}$$

where $\{f_{i,n}\}_{n=1}^{\infty}$ are sequences of real-valued functions and $\{h_{i,n}\}_{n=1}^{\infty}$ are sequences of positive integers.

In Section 2, we present some conditions for the existence of positive, and of negative solutions to a linear version of (1.1). In Section 3, by extending a result in [5], we obtain conditions for all oscillations to be bounded, and to tend to zero. In Section 4, we obtain conditions for every solution to be oscillatory. Also we compare our conditions with those obtained in [13] for constant advances. Also we illustrate our results with examples.

To study the oscillation of solutions, we assume that solutions exist and are defined for all n large enough. Quite frequently solutions are obtained as fixed points of contraction mappings, which is the case in Theorem 3.4 below. In general it is not clear how to formulate initial-value problems for advanced difference and differential equations. However, in special cases we can obtain a unique of solution. Consider the difference equation

$$x_{n+1} - x_n = f_n(x_n, x_{n+1}, \dots, x_{n+m}) , \tag{1.2}$$

with $m > 1$. For each fixed set of values $n, a_1, \dots, a_m$, we assume that the function $f_n(a_1, a_2, \dots, a_m, \cdot)$ is one-to-one and onto from $\mathbb{R}$ to $\mathbb{R}$. Then using the initial data $x_1, x_2, \dots, x_m$ we solve for x_{m+1} in the equation $x_2 - x_1 = f_1(x_1, x_2, \dots, x_{m+1})$. Then using the data $x_2, \dots, x_{m+1}$ we solve for x_{m+2} in the equation $x_3 - x_2 = f_1(x_2, x_3, \dots, x_{m+2})$. Then solve for x_{m+3} in $x_4 - x_3 = f_1(x_3, x_4, \dots, x_{m+3})$, etc. This way we construct the solution by defining one entry at the time. This is known as the method of steps.

2. EXISTENCE OF NON-OSCILLATORY SOLUTIONS

By a solution $\{x_n\}$ we mean a sequence of real numbers that satisfies (1.1) for all n large enough.

A solution is said to be oscillatory if for every positive integer n_0 there exist $n_1, n_2 \geq n_0$ such that $x_{n_1} x_{n_2} \leq 0$. A non-oscillatory solution is either eventually positive or eventually negative.

In this section we restrict our attention to the particular case of (1.1), when $f_{i,n}(x) = a_{i,n} x$; thus we have the linear difference equation

$$x_{n+1} - x_n = \sum_{i=1}^{m} a_{i,n} x_{n+h_{i,n}} . \tag{2.1}$$

When $p < n$, we use the conventions $\prod_{i=n}^{p} x_i = 1$ and $\sum_{i=n}^{p} x_i = 0$.

Theorem 2.1. *Assume that $a_{i,n} \geq 0$ and that there exists a sequence $\{u_n\}$ of non-negative terms satisfying*

$$u_{n+1} \geq 1 + \sum_{i=1}^{m} a_{i,n} \prod_{j=1}^{h_{i,n}} u_{n+j}, \quad n \geq n_0. \tag{2.2}$$

Then (2.1) *has positive solutions, and negative solutions.*

Proof. Let $\{u_n\}$ be a solution of (2.2), and let $v_{1,n} := u_n$. Then for $n \geq n_0$ and $\ell \geq 0$, we define a double indexed sequence $\{v_{\ell,n}\}$ recursively by

$$v_{\ell+1,n+1} = 1 + \sum_{i=1}^{m} a_{i,n} \prod_{j=1}^{h_{i,n}} v_{\ell,n+j}$$

Then, by (2.2),

$$0 \leq v_{2,n+1} = 1 + \sum_{i=1}^{m} a_{i,n} \prod_{j=1}^{h_{i,n}} v_{1,n+j} \leq v_{1,n+1}, \quad n \geq n_0.$$

By induction, we can show that $0 \leq \cdots \leq v_{\ell+1,n+1} \leq v_{\ell,n+1} \leq \cdots \leq v_{1,n+1}$. Then, for each fixed $n+1$, the limit $\lim_{\ell \to +\infty} v_{\ell,n+1} =: v_{n+1}$ exists. This limit satisfies

$$v_{n+1} = 1 + \sum_{i=1}^{m} a_{i,n} \prod_{j=1}^{h_{i,n}} v_{n+j}.$$

Then for any x_{n_0}, the sequence

$$x_n = x_{n_0} \prod_{i=n_0}^{n} v_i$$

is a solution of (2.1). When $x_{n_0} > 0$ this is a positive solution and when $x_{n_0} < 0$ this is a negative solution. □

Now we consider the difference equation whose coefficients and advanced arguments are smaller than those of (2.1),

$$y_{n+1} - y_n = \sum_{i=1}^{m} b_{i,n} y_{n+g_{i,n}}. \tag{2.3}$$

Theorem 2.2. *Assume the conditions of Theorem 2.1 hold, and that $1 \leq g_{i,n} \leq h_{i,n}$, $0 \leq b_{i,n} \leq a_{i,n}$ for $n \geq 1$. Then (2.3) has positive solutions and negative solutions.*

Proof. Let $\{u_n\}$ be a non-negative solution of (2.2). Then $u_n \geq 1$ and

$$u_{n+1} \geq 1 + \sum_{i=1}^{m} a_{i,n} \prod_{j=1}^{h_{i,n}} u_{n+j} \geq 1 + \sum_{i=1}^{m} b_{i,n} \prod_{j=1}^{h_{i,n}} u_{n+j} \geq 1 + \sum_{i=1}^{m} b_{i,n} \prod_{j=1}^{g_{i,n}} u_{n+j}.$$

Using this inequality in Theorem 2.1, we have the existence of positive solutions and negative solutions. □

Corollary 2.3. *Assume that the sequences $\{a_{i,n}\}_n$ and $\{h_{i,n}\}_n$ are bounded as follows: $0 \leq a_{i,n} \leq a_i$ and $1 \leq h_{i,n} \leq h_i$ for $1 \leq i \leq m$ and $n \geq n_0$. Also assume that the inequality*

$$\lambda \geq 1 + \sum_{i=1}^{m} a_i \lambda^{h_i} \tag{2.4}$$

has a non-negative solution λ. Then (2.1) has positive solutions and negative solutions.

Proof. We consider the equation with constant coefficients and constant advances

$$y_{n+1} - y_n = \sum_{i=1}^{m} a_i y_{n+h_i}. \tag{2.5}$$

Note that using (2.4), we can show that the constant sequence $u_n = \lambda$ satisfies (2.2) with $a_{i,n} = a_i$. Then by Theorem 2.1, we have positive solutions and negative solutions to (2.5). Since $a_{i,n} \le a_i$ and $h_{i,n} \le h_i$, by applying Theorem 2.2, we have positive solutions and negative solutions to (2.1). □

Now we consider an equation with positive and negative coefficients,

$$x_{n+1} - x_n = \sum_{i=1}^{m} \Big(a_{i,n} x_{n+h_{i,n}} - b_{i,n} x_{n+g_{i,n}} \Big). \tag{2.6}$$

Theorem 2.4. *Suppose that* $0 \le b_{i,n} \le a_{i,n}$, *and* $1 \le g_{i,n} \le h_{i,n}$ *for* $1 \le i \le m$ *and* $n \ge n_0$. *If inequality* (2.2) *has a nonnegative solution, then* (2.6) *has positive solutions and negative solutions.*

Proof. Let $\{u_n\}$ be a nonnegative solution of (2.2) and let $v_{1,n} = u_n$. We define the double indexed sequence $\{v_{\ell,n}\}$ recursively by

$$v_{\ell+1,n+1} = 1 + \sum_{i=1}^{m} \Big(a_{i,n} \prod_{j=1}^{h_{i,n}} v_{\ell,n+j} - b_{i,n} \prod_{j=1}^{g_{i,n}} v_{\ell,n+j} \Big), \quad \text{for } n \ge n_0,\ \ell \ge 1.$$

By (2.2), we have

$$\begin{aligned} v_{1,n+1} &\ge 1 + \sum_{i=1}^{m} a_{i,n} \prod_{j=1}^{h_{i,n}} v_{1,n+j} \\ &\ge 1 + \sum_{i=1}^{m} \Big(a_{i,n} \prod_{j=1}^{h_{i,n}} v_{1,n+j} - b_{i,n} \prod_{j=1}^{g_{i,n}} v_{1,n+j} \Big) = v_{2,n+1}. \end{aligned}$$

Since $0 \le b_{i,n} \le a_{i,n}$ and $1 \le g_{i,n} \le h_{i,n}$ and $1 \le v_{1,n}$, it follows that $1 \le v_{2,n}$.

Next, we use induction: Assuming that $1 \le v_{\ell-1,n}$, from $0 \le b_{i,n} \le a_{i,n}$ it follows that $1 \le v_{\ell,n}$. Now assuming that $v_{\ell,n} \le v_{\ell-1,n}$, we wish to show that $v_{\ell+1,n} \le v_{\ell,n}$ which by definition has the form

$$\begin{aligned} &1 + \sum_{i=1}^{m} \Big(a_{i,n} \prod_{j=1}^{h_{i,n}} v_{\ell,n+j} - b_{i,n} \prod_{j=1}^{g_{i,n}} v_{\ell,n+j} \Big) \\ &\le 1 + \sum_{i=1}^{m} \Big(a_{i,n} \prod_{j=1}^{h_{i,n}} v_{\ell-1,n+j} - b_{i,n} \prod_{j=1}^{g_{i,n}} v_{\ell-1,n+j} \Big). \end{aligned}$$

The above inequality is equivalent to

$$\sum_{i=1}^{m}\Big[\prod_{j=1}^{g_{i,n}} v_{\ell,n+j}\Big(a_{i,n}\prod_{j=g_{i,n}+1}^{h_{i,n}} v_{\ell,n+j}-b_{i,n}\Big)\Big] \\ \leq \sum_{i=1}^{m}\Big[\prod_{j=1}^{g_{i,n}} v_{\ell-1,n+j}\Big(a_{i,n}\prod_{j=g_{i,n}+1}^{h_{i,n}} v_{\ell-1,n+j}-b_{i,n}\Big)\Big].$$

This inequality follows from the assumptions $1 \leq v_{\ell,n} \leq v_{\ell-1,n}$ and $0 \leq b_{i,n} \leq a_{i,n}$. Therefore, $1 \leq v_{\ell+1,n} \leq v_{\ell,n}$. Consequently, for each fixed n the limit $\lim_{\ell\to\infty} v_{\ell,n} := v_n$ exists and is non-negative. The sequence defined with these limits satisfies

$$0 \leq v_{n+1} = 1 + \sum_{i=1}^{m}\Big(a_{i,n}\prod_{j=1}^{h_{i,n}} v_{j+n} - b_{i,n}\prod_{j=1}^{g_{i,n}} v_{j+n}\Big), \quad n \geq n_0.$$

Then for any x_{n_0}, the sequence

$$x_n = x_{n_0}\prod_{i=n_0}^{n} v_i$$

is a solution of (2.1). When $x_{n_0} > 0$ this is a positive solution and when $x_{n_0} < 0$ this is a negative solution. □

3. ASYMPTOTIC BEHAVIOR

In this section we study the behavior of solutions to (1.1), as $n \to \infty$. Some of our results are analog to those in [15] for continuous variables. Our first result uses the assumption

(H1) There exists constants $a_{i,n} \geq 0$ such that for $i = 1, \dots, m$ and $n \geq 1$:

$$f_{i,n}(x)\begin{cases} \geq a_{i,n}x & \text{if } x \geq 0,\\ \leq a_{i,n}x & \text{if } x < 0.\end{cases}$$

An example of function satisfying the above condition is $f_{i,n}(x) = x(2+\sin(x))$, with $a_{i,n} = 1$.

Theorem 3.1. *Let* $\{x_n\}$ *be a solution of* (1.1). *Assume* (H1) *and*

$$\sum_{i=n_0}^{+\infty}\sum_{j=1}^{m} a_{j,i} = +\infty. \tag{3.1}$$

If $\{x_n\}$ *is eventually positive then* $\lim_{n\to\infty} x_n = +\infty$. *If* $\{x_n\}$ *is eventually negative then* $\lim_{n\to\infty} x_n = -\infty$.

Proof. Let $x_n > 0$ for $n \geq n_0$, then by (1.1) and (H1), $\{x_n\}$ is non-decreasing and

$$x_{n+1} - x_n \geq \sum_{i=1}^{m} a_{i,n} x_{n+h_{i,n}} \geq x_{n_0} \sum_{i=1}^{m} a_{i,n}.$$

Summing both sides of this inequality from n_0 to n, we obtain

$$x_{n+1} \geq x_{n_0} \Big(1 + \sum_{i=n_0}^{n} \sum_{j=1}^{m} a_{i,j} \Big).$$

When $n \to \infty$, by (3.1), we obtain $\lim_{n\to+\infty} x_n = +\infty$. The negative case has a similar proof. □

For the next result we use the assumption

(H2) There exists constants $a_{i,n} \leq 0$ such that for $i = 1, \dots, m$ and $n \geq 1$:

$$f_{i,n}(x) \begin{cases} \leq a_{i,n} x & \text{if } x \geq 0, \\ \geq a_{i,n} x & \text{if } x < 0. \end{cases}$$

Theorem 3.2. *Let $\{x_n\}$ be a solution of* (1.1). *Assume* (H2) *and*

$$\sum_{i=n_0}^{+\infty} \sum_{j=1}^{m} a_{j,i} = -\infty. \tag{3.2}$$

If $\{x_n\}$ is non-oscillatory then $\lim_{n\to\infty} x_n = 0$.

Proof. Assume that $\{x_n\}$ is a positive solution of (1.1). Then by (1.1) and (H2), $\{x_n\}$ is non-increasing. So, $\{x_n\}$ has a nonnegative limit, $\alpha = \lim_{n\to+\infty} x_n$. If $\alpha > 0$, then by (H2),

$$x_{n+1} - x_n \leq \sum_{i=1}^{m} a_{i,n} x_{n+h_{i,n}} \leq \alpha \sum_{i=1}^{m} a_{i,n}.$$

Summing both sides of this inequality from n_0 to n and using (3.2), we get $\lim_{n\to\infty} x_n = -\infty$. This contradicts $\{x_n\}$ begin eventually positive; therefore $\alpha = 0$. The eventually negative case has a similar proof. □

We use the forward difference operator $\Delta x_n = x_{n+1} - x_n$ in the following lemma. In the proof of the lemma we essentially follow the lines of a method for solving a difference equation, an area of considerable recent interest (see, for example [16–18] and the references therein, where closely related methods were used).

Lemma 3.3. *Assume that*

$$A_n := \sum_{i=1}^{m} a_{i,n} \neq -1.$$

Then a sequence $\{x_n\}$ is a solution of (2.1) *if and only if it is a solution of*

$$x_n = x_{n_0} \prod_{i=n_0}^{n-1} (1 + A_i) + \sum_{\ell=n_0}^{n-1} \Big(\sum_{i=1}^{m} a_{i,\ell} \sum_{j=\ell}^{\ell+h_{i,\ell}-1} \Delta x_j \Big) \prod_{i=\ell+1}^{n-1} (1 + A_i). \tag{3.3}$$

Proof. Using the telescoping property $x_p - x_n = \sum_{i=n}^{p-1} \Delta x_i$, and (2.1), we have that $\{x_n\}$ is a solution of (2.1) if and only if

$$\Delta x_n = \sum_{i=1}^{m} a_{i,n} x_{n+h_{i,n}} = \sum_{i=1}^{m} a_{i,n}\Big(x_n + \sum_{j=n}^{n+h_{i,n}-1} \Delta x_j\Big).$$

Hence,

$$\Delta x_n - x_n A_n = \sum_{i=1}^{m} a_{i,n} \sum_{j=n}^{n+h_{i,n}-1} \Delta x_j.$$

Multiplying by $\prod_{i=n_0}^{n}(1 + A_i)^{-1}$ both sides of the last equality, we obtain

$$\Delta\Big(x_n \prod_{i=n_0}^{n-1}(1 + A_i)^{-1}\Big) = \Big(\sum_{i=1}^{m} a_{i,n} \sum_{j=n}^{n+h_{i,n}-1} \Delta x_j\Big) \prod_{i=n_0}^{n}(1 + A_i)^{-1}.$$

Changing the variable n by ℓ and summing both sides form $\ell = n_0$ to $\ell = n - 1$, we have

$$x_n \prod_{i=n_0}^{n-1}(1 + A_i)^{-1} - x_{n_0} = \sum_{\ell=n_0}^{n-1} \Big(\sum_{i=1}^{m} a_{i,\ell} \sum_{j=\ell}^{\ell+h_{i,\ell}-1} \Delta x_j\Big) \prod_{i=n_0}^{\ell}(1 + A_i)^{-1}.$$

Multiplying it by $\prod_{i=n_0}^{n-1}(1+A_i)$, we obtain (3.3). Now starting from (3.3) and retracing the steps above we obtain equation (2.1). This completes the proof. $\square$

Theorem 3.4. *Assume the conditions of Lemma 3.3 are satisfied and that there are constants α and n_0 such that*

$$\sum_{\ell=n_0}^{n-1} \Big(\sum_{i=1}^{m} |a_{i,\ell}| \sum_{j=\ell}^{\ell+h_{i,\ell}-1} \sum_{k=1}^{m} |a_{k,j}|\Big) \prod_{i=\ell+1}^{n-1} |1 + A_i| \le \alpha < 1 \quad \textit{for } n \ge n_0. \tag{3.4}$$

Also assume that

$$\prod_{i=n_0}^{n-1} |1 + A_i| \textit{ remains bounded as } n \to \infty. \tag{3.5}$$

Then for each initial value x_{n_0}, there is a unique solution to (2.1)*; furthermore, this solution is bounded.*

Proof. Let B be the collection of bounded sequences that have a common value at $n = n_0$. Then B is closed subset of a complete metric space under the supremum norm. Based on Lemma 3.3, we define the operator $T : B \to B$ by

$$Tx_n = x_{n_0} \prod_{i=n_0}^{n-1}(1 + A_i) + \sum_{\ell=n_0}^{n-1} \Big(\sum_{i=1}^{m} a_{i,\ell} \sum_{j=\ell}^{\ell+h_{i,\ell}-1} \Delta x_j\Big) \prod_{i=\ell+1}^{n-1}(1 + A_i). \tag{3.6}$$

Note that when $n = n_0$, there are no terms in the product and no terms in the summation; therefore $Tx_{n_0} = x_{n_0}$. To estimate Tx_n, we note that by (2.1),

$$|\Delta x_j| \le \sum_{k=1}^{m} |a_{k,j}|\,|x_{j+h_{k,j}}| \le \sup_{j\le p} |x_p| \sum_{k=1}^{m} |a_{k,j}|. \tag{3.7}$$

Then by (3.6), we have

$$|Tx_n| \le |x_{n_0}| \prod_{i=n_0}^{n-1} |1+A_i| + \sup_{n_0\le p} |x_p| \sum_{\ell=n_0}^{n-1} \Big(\sum_{i=1}^{m} |a_{i,\ell}| \sum_{j=\ell}^{\ell+h_{i,\ell}-1} \sum_{i=k}^{m} |a_{k,j}| \Big) \prod_{i=\ell+1}^{n-1} |1+A_i|. \tag{3.8}$$

Since $\{x_n\}$ is bounded, by (3.4) and (3.5), the sequence $\{Tx_n\}$ is also bounded. Now we show that T is a contraction. Let $\{x_n\}$ and $\{y_n\}$ be sequences in B. The same process as for (3.8) yields

$$|Tx_n - Ty_n| \le \sup_{p\ge n_0} |x_p - y_p| \sum_{\ell=n_0}^{n-1} \Big(\sum_{i=1}^{m} |a_{i,\ell}| \sum_{j=\ell}^{\ell+h_{i,\ell}-1} \sum_{i=k}^{m} |a_{k,j}| \Big) \prod_{i=\ell+1}^{n-1} |1+A_i|.$$

Using the norm $\|x\| = \sup_{n\ge n_0} |x_n|$ and condition (3.4), we have

$$\|Tx - Ty\| \le \alpha \|x - y\|.$$

Therefore T is a contraction on B and has a unique fixed point, which by Lemma 3.3 is the unique solution of (2.1) with the given initial value x_{n_0}. $\square$

Note that under the assumptions of Theorem 3.4, all solutions must be bounded. For any solution we use its value x_{n_0} as the initial value in Theorem 3.4.

Now we provide an example where the assumptions of Theorem 3.4 are satisfied.

Example 3.5. Let $m = 5$, $f_{i,n}(x) = a_{i,n}x$ with $a_{i,n} = (-1)^{i+n}/(i+n)^2$. Note that

$$|A_n| = |\sum_{i=1}^{5} a_{i,n}| \le \sum_{i=1}^{5} |a_{i,n}| \le \frac{5}{(n+1)^2} \tag{3.9}$$

which satisfies the assumption in Lemma 3.3. Then

$$\prod_{i=n_0}^{n-1} |1+A_i| \le \prod_{i=n_0}^{n-1} \Big(1 + \frac{5}{(i+1)^2}\Big) =: p.$$

To estimate p we use that $\ln(1+x) \le x$ for $x > -1$.

$$\begin{aligned} \ln(p) &= \sum_{i=n_0}^{n-1} \ln\Big(1 + \frac{5}{(i+1)^2}\Big) \le \sum_{i=n_0}^{n-1} \frac{5}{(i+1)^2} \\ &\le \int_{n_0}^{n} \frac{5}{x^2}\,dx \le 5\Big(\frac{1}{n_0} - \frac{1}{n}\Big) \le \frac{5}{n_0}. \end{aligned}$$

Therefore,

$$\prod_{i=n_0}^{n-1} |1 + A_i| \le \exp\left(\frac{5}{n_0}\right), \tag{3.10}$$

which indicates that (3.5) is satisfied. Using (3.9) and that $h_{i,n} \le 4$, we have

$$\sum_{i=1}^{5} |a_{i,\ell}| \sum_{j=\ell}^{\ell+h_{i,\ell}-1} \sum_{k=1}^{m} |a_{k,j}| \le \frac{4(5)^2}{(\ell+1)^4}.$$

By (3.10),

$$\sum_{\ell=n_0}^{n-1} \frac{4(5)^2}{(\ell+1)^4} \exp\left(\frac{5}{\ell}\right) \le \frac{4(5)^2}{5} \exp\left(\frac{5}{n_0}\right) \sum_{\ell=n_0}^{n-1} \frac{1}{(\ell+1)^4} \le \frac{4(5)^2}{5} \exp\left(\frac{5}{n_0}\right) \frac{1}{3n_0^3}.$$

In the above inequality we used integration from n_0 to n, as for (3.10). Finally for $n_0 = 5$ the above expression is less than 1, and (3.4) is satisfied. Then by Theorem 3.4, for each initial value x_{n_0}, there is a unique solution.

4. OSCILLATION OF ALL SOLUTIONS TO (1.1)

Li [13] used elaborate estimates to obtain the oscillation of solutions to

$$x_n - x_{n-1} = \sum_{i=1}^{m} a_{i,n} x_{n+h_i}.$$

Here, we use simple estimates to obtain conditions for the oscillation of all solutions to (1.1). Then we compare our conditions with those in [13].

First we extend a result in [5] from an equation with single and constant advance to multiple and variable advances.

Theorem 4.1. *Assume* (H1) *and*

$$2 \le \min_{1\le i\le m} \inf_{1\le n} h_{i,n} =: h_0\,, \tag{4.1}$$

$$\liminf_{n\to\infty} \sum_{i=1}^{m} a_{i,n} > \frac{1}{h_0}\Big(1 - \frac{1}{h_0}\Big)^{h_0-1}. \tag{4.2}$$

Then every solution of (1.1) *is oscillatory.*

Proof. We assume that there is an eventually positive solution $\{x_n\}$ of (1.1) and obtain a contradiction. The proof for eventually negative solutions is similar and is omitted. Let $x_n > 0$ for all $n \ge n_0$. From (H1) we have

$$x_{n+1} - x_n \ge \sum_{i=1}^{m} a_{i,n} x_{n+h_{i,n}}. \tag{4.3}$$

By (H1), $\{x_n\}$ is a non-decreasing sequence; thus $x_{n+h_0} \le x_{n+h_{i,n}}$. Let $r_n = x_n/x_{n+1}$. Then $0 < r_n \le 1$ for all $n \ge n_0$. From (4.3) we have

$$1 - r_n \ge \sum_{i=1}^{m} a_{i,n} \frac{1}{r_{n+1} r_{n+2} \cdots r_{n-1+h_0}}\,, \quad \text{for } n \ge n_0.$$

Let c be the average between the two sides of inequality (4.2). Then

$$1 - r_n > c \frac{1}{r_{n+1} r_{n+2} \cdots r_{n-1+h_0}}.$$

Let $\gamma = \sup_{n \ge n_0} r_n$. Then $0 < \gamma \le 1$, and taking the infimum over n, in both sides, we have $1 - \gamma \ge c/\gamma^{h_0-1}$, which implies

$$(1-\gamma)\gamma^{h_0-1} \ge c.$$

The left-hand side attains its maximum when $\gamma = (h_0 - 1)/h_0$. Therefore

$$\frac{1}{h_0}\Big(1 - \frac{1}{h_0}\Big)^{h_0-1} \ge (1-\gamma)\gamma^{h_0-1} \ge c > \frac{1}{h_0}\Big(1 - \frac{1}{h_0}\Big)^{h_0-1}.$$

this contradiction competes the proof. □

Theorem 4.2. *Assume* (H1), (4.1), *and*

$$\limsup_{n\to\infty} \sum_{i=1}^{m} a_{i,n} > 1. \tag{4.4}$$

Then every solution of (1.1) *is oscillatory.*

Proof. We assume that there is an eventually positive solution $\{x_n\}$ of (1.1) and obtain a contradiction. The proof for eventually negative solutions is similar and is omitted. As in the proof of Theorem 4.1, we divide (4.1) by x_{n+1} and let $r_n = x_n/x_{n+1}$. Since $0 < r_n \le 1$ we have

$$1 > 1 - r_n \ge \sum_{i=1}^{m} a_{i,n} \frac{1}{r_{n+1} r_{n+2} \cdots r_{n-1+h_0}} \ge \sum_{i=1}^{m} a_{i,n} \quad \text{for } n \ge n_0.$$

This contradicts (4.4) and completes the proof. □

We remark that condition (7) in [13] and condition (4.4) here are independent of each other. Our values $a_{i,n}$ and $h_{i,n}$ correspond to $p_i(n)$ and k_i respectively in [13]. Let $f_{i,n}(x) = a_{i,n}x$, $m = 1$, $h_{i,n} = 2$ and

$$p_i(n) = a_{i,n} = \begin{cases} 3/2 & \text{if } n \text{ is a multiple of 3,} \\ 0 & \text{otherwise.} \end{cases}$$

When $p_i(n) = 0$, the summand in (7) is zero. When $p_i(n) \neq 0$, the terms $q_1, q_2, \ldots$ are zero in (7); thus the summands in (7) are zero. In both cases (7) is not satisfied while (4.4) is satisfied.

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