Opuscula Math. 37, no. 6 (2017), 779–794
http://dx.doi.org/10.7494/OpMath.2017.37.6.779 Opuscula Mathematica

ON THE STEKLOV PROBLEM INVOLVING THE $p(x)$-LAPLACIAN WITH INDEFINITE WEIGHT

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Communicated by Vicentiu D. Radulescu

Abstract. Under suitable assumptions, we study the existence of a weak nontrivial solution for the following Steklov problem involving the $p(x)$-Laplacian

$$\begin{cases} \Delta_{p(x)} u = a(x)|u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda V(x)|u|^{q(x)-2}u & \text{on } \partial\Omega. \end{cases}$$

Our approach is based on min-max method and Ekeland's variational principle.

Keywords: $p(x)$-Laplace operator, Steklov problem, variable exponent Sobolev spaces, variational methods, Ekeland's variational principle.

Mathematics Subject Classification: 35J48, 35J66.

1. INTRODUCTION

The purpose of this paper is to study the following Steklov problem involving the $p(x)$-Laplacian

$$(\mathbf{P}_\lambda) \quad \begin{cases} \Delta_{p(x)} u = a(x)|u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda V(x)|u|^{q(x)-2}u & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, λ is a positive parameter, $a \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_\Omega a > 0$, $p, q \in C(\bar{\Omega})$, $V \in L^{s(x)}(\partial\Omega)$ such that $\frac{N-1}{p(x)-1} < s(x)$, for all $x \in \partial\Omega$ and ν is the outer unit normal to $\partial\Omega$.

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. The study of this kind of operator have been an interesting topic like electrorheological fluids (see [30]), elastic mechanics (see [33]), stationary thermo-rheological viscous flows of non-Newtonian fluids and image processing (see [7]) and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium (see [3]). Many results have been obtained on this kind of problems, for instance we here cite [2, 5, 8, 9, 15, 18, 19, 27, 28].

The inhomogeneous Steklov problems involving the p-Laplacian has been the object of study in many paper (one can see [24]), in which the authors have studied this class of inhomogeneous Steklov problems in the cases of $p(x) \equiv p = 2$ and $p(x) \equiv p > 1$, respectively.

In the following, let us recall that, S.G. Deng in [10], studied problem $(\mathbf{P}_\lambda)$ in the particular case when $V(x) \equiv 1$, $a(x) \equiv 1$ and $p(x) \equiv q(x)$, the author proved the existence of infinitely many eigenvalues sequences and he present a sufficient conditions for the infimum eigenvalues is zero and positive.

Inspired by the above-mentioned papers, we study problem $(\mathbf{P}_\lambda)$. In this new situation we will show, firstly and under appropriates conditions, that for any $\lambda > 0$ the problem $(\mathbf{P}_\lambda)$ has a weak nontrivial solution with negative energy. Moreover, by using Ekeland's variational principle (see [12]), we showed the existence of continuous spectrum. The paper is organized as follows. In Section 2, we recall the definition of variable exponent Lebesgue spaces, $L^{p(x)}(\Omega)$, as well as Sobolev spaces, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. In Section 3, we give the main results. Finally their proofs are presented in Section 4.

2. NOTATIONS AND PRELIMINARIES

In this section, we recall some definitions and basic properties of the generalized Lebesgue Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ (for details, see [14, 23, 25]).

Set

$$C_+(\overline{\Omega}) := \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ such that

$$1 < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty, \tag{2.1}$$

we denote

$$L^{p(x)} = \Big\{u : u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)} dx < \infty\Big\}.$$

We recall the following so-called Luxemburg norm on this space defined by the formula

$$|u|_{p(x)} = \inf\Big\{\mu > 0 : \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{p(x)} dx \le 1\Big\}.$$

Variable exponent Lebesgue spaces are like classical Lebesgue spaces in many respects: they are Banach spaces, they are reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$ then there exists a continuous embedding $L^{p_2}(x)(\Omega) \hookrightarrow L^{p_1}(x)(\Omega)$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\Big|\int_\Omega uv dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)|u|_{p(x)}|v|_{p'(x)} \tag{2.2}$$

holds true (see [14] and [23]).

Moreover, if h_1, h_2 and $h_3 : \overline{\Omega} \to (1, \infty)$ are three Lipschitz continuous functions such that $1/h_1(x) + 1/h_2(x) + 1/h_3(x) = 1$, then for any $u \in L^{h_1(x)}(\Omega), v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$, the following inequality holds ([13, Proposition 2.5]):

$$\Big|\int_\Omega uvw\, dx\Big| \leq \Big(\frac{1}{h_1^-} + \frac{1}{h_2{}^-} + \frac{1}{h_3{}^-}\Big)|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)}. \tag{2.3}$$

The modular on the space $L^{p(x)}(\Omega)$ is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx,$$

and it satisfies the following proposition.

Proposition 2.1 ([23])**.** *For all $u, v \in L^{p(x)}(\Omega)$, we have*

(1) $|u|_{p(x)} < 1\,(resp. = 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\,(resp. = 1, > 1)$,
(2) $\min(|u|^{p^-}_{p(x)}, |u|^{p^+}_{p(x)}) \leq \rho_{p(x)}(u) \leq \max(|u|^{p^-}_{p(x)}, |u|^{p^+}_{p(x)})$,
(3) $\rho_{p(x)}(u - v) \to 0 \Leftrightarrow |u - v|_{p(x)} \to 0$.

Proposition 2.2 ([11])**.** *Let p and q be two measurable functions such that $1 \leq p(x)q(x) \leq \infty$ for a.e. $x \in \Omega$ and $p \in L^\infty(\Omega)$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$\min(|u|^{p^+}_{p(x)q(x)}, |u|^{p^-}_{p(x)q(x)}) \leq ||u|^{p(x)}|_{q(x)} \leq \max(|u|^{p^-}_{p(x)q(x)}, |u|^{p^+}_{p(x)q(x)}).$$

The Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) := \Big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\Big\},$$

and equipped with the norm

$$\|u\|_{1,p(x)} := |u|_{p(x)} + |\nabla u|_{p(x)}.$$

It is well known [16] that, in view of (2.1), both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, equipped respectively with the above norms, are separable, reflexive and uniformly convex Banach spaces. When $a \in L^\infty(\Omega)$ with $ess\inf_\Omega a > 0$, for any $u \in W^{1,p(x)}(\Omega)$, define

$$\|u\|_a := \inf\Big\{\lambda > 0 : \int_\Omega \Big(\Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} + a(x)\Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\Big)dx \leq 1\Big\}.$$

Then, it is easy to see that $\|u\|_a$ is a norm on $W^{1,p(x)}(\Omega)$ equivalent to $\|u\|_{1,p(x)}$. In what follows, we will use $\|u\|_a$ instead of $\|u\|_{1,p(x)}$ on $E = W^{1,p(x)}(\Omega)$.

We have the following proposition.

Proposition 2.3 ([16])**.** *Put*

$$\rho_{a(x),p(x)}(u) = \int_\Omega (|\nabla u|^{p(x)} + a(x)|u|^{p(x)})dx.$$

For $u \in E$, we have

(1) $\|u\|_a \geq 1 \Leftrightarrow \|u\|_a^{p^-} \leq \rho_{a(x),p(x)}(u) \leq \|u\|_a^{p^+}$,

(2) $\|u\|_a \leq 1 \Leftrightarrow \|u\|_a^{p^+} \leq \rho_{a(x),p(x)}(u) \leq \|u\|_a^{p^-}$.

For a given measurable function $a : \partial\Omega \to \mathbb{R}$, we define the weighted variable exponent Lebesgue space by

$$L^{p(x)}_{a(x)}(\partial\Omega) := \Big\{ u \,|\, u : \partial\Omega \to \mathbb{R} \,\text{is a mesurable and} \int\limits_{\partial\Omega} |a(x)||u(x)|^{p(x)} d\sigma_x < +\infty \Big\}$$

with the norm

$$|u|_{(p(x),a(x))} = |u|_{L^{p(x)}_{a(x)}(\partial\Omega)} = \inf \Big\{ \tau > 0 : \int\limits_{\partial\Omega} |a(x)| \Big| \frac{u(x)}{\tau} \Big|^{p(x)} d\sigma_x \leq 1 \Big\},$$

where $d\sigma_x$ is the measure on the boundary. Then, $L^{p(x)}_{a(x)}(\partial\Omega)$ is a Banach space. In particular, when $a(x) \equiv 1$ on $\partial\Omega$, $L^{p(x)}_{a(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$.

Proposition 2.4. *Let*

$$\rho(u) = \int_{\partial\Omega} |a(x)||u(x)|^{p(x)} d\sigma.$$

For $u, u_k \in L^{p(x)}_{a(x)}(\partial\Omega)$ $(k = 1, 2, \ldots)$, we have

(1) $|u|_{(p(x),a(x))} \geq 1 \Leftrightarrow |u|^{p^-}_{(p(x),a(x))} \leq \rho(u) \leq |u|^{p^+}_{(p(x),a(x))}$,

(2) $|u|_{(p(x),a(x))} \leq 1 \Leftrightarrow |u|^{p^+}_{(p(x),a(x))} \leq \rho(u) \leq |u|^{p^-}_{(p(x),a(x))}$,

(3) $|u_k|_{(p(x),a(x))} \to 0 \Leftrightarrow \rho(u_k) \to 0$,

(4) $|u_k|_{(p(x),a(x))} \to \infty \Leftrightarrow \rho(u_k) \to \infty$.

For $A \subset \overline{\Omega}$, denote by $p^-(A) = \inf\limits_{x\in A} p(x), p^+(A) := \sup\limits_{x\in A} p(x)$. Define

$$p^\partial(x) = (p(x))^\partial = \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N, \end{cases}$$

$$p^\partial_{r(x)}(x) := \frac{r(x)-1}{r(x)} p^\partial(x),$$

where $x \in \partial\Omega, r \in C(\partial\Omega)$ with $r^- = \inf\limits_{x\in\partial\Omega} r(x) > 1$

Throughout this paper, we assume the following conditions:

(H) $1 \leq q^+ < p^-$, $p^-(\partial\Omega) < p^\partial(x)$, $\frac{N-1}{p(x)-1} < s(x)$, for all $x \in \partial\Omega$ and $V \in L^{s(x)}(\partial\Omega)$ such that $V > 0$ in $\Omega_0 \subset\subset \partial\Omega$ with $\text{meas}_\sigma(\Omega_0) > 0$.

In the following, we recall an important theorem which will be needed throughout this paper.

Theorem 2.5 ([10, Theorem 2.1]). *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$. Suppose that $a \in L^{r(x)}(\partial\Omega), r \in C(\partial\Omega)$ with $r(x) > \dfrac{p^\partial(x)}{p^\partial(x) - 1}$ for all $x \in \partial\Omega$. If $q \in C(\partial\Omega)$ and*

$$1 \leq q(x) < p^\partial_{r(x)}(x) \quad \textit{for all } x \in \partial\Omega. \tag{2.4}$$

Then, there exists a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{a(x)}(\partial\Omega)$. In particular, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q_0(x)}(\partial\Omega)$, where $1 \leq q_0(x) < p^\partial(x)$ for all $x \in \partial\Omega$.

Definition 2.6. We say that $u \in E$ is weak solution of $(\mathbf{P}_\lambda)$ if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega a(x)|u|^{p(x)-2}uv dx - \lambda \int_{\partial\Omega} V(x)|u|^{q(x)-2}uv d\sigma = 0,$$

for any $v \in E$.

We cite the very recent monograph by Kristály *et al.* [22] as a general reference for the basic notions used in the paper.

3. THE MAIN RESULTS AND AN AUXILIARY RESULTS

Our first result established using min-max method is the following.

Theorem 3.1. *Assuming that assumption* (**H**) *holds, then for all $\lambda > 0$, problem $(\boldsymbol{P}_\lambda)$ has at least one non trivial weak solution with negative energy.*

The second result is obtained, using Ekeland's variational principal.

Theorem 3.2. *Under assumption* (**H**)*, there exists λ^* such that, for all $\lambda \in (0, \lambda^*)$, problem $(\mathbf{P}_\lambda)$ has a non trivial weak solution.*

We denote by s' the conjugate exponent of the function s and we put

$$r(x) := \frac{s(x)q(x)}{s(x) - q(x)}.$$

Remark 3.3. Under assumption (**H**), we have $s'(x)q(x) < p^\partial(x)$ and $r(x) < p^\partial(x)$ for all $x \in \partial\Omega$, so, due to theorem 2.5, $W^{1,p(x)}(\Omega) \hookrightarrow L^{s'(x)q(x)}(\partial\Omega)$ and $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ are compact and continuous.

We mention also the following proposition that will be needed later.

Proposition 3.4. *For u, $v \in L^{s_1'(x)q(x)}(\Omega)$, one has*

$$|u-v|_{q(x)s_1'(x)} \to 0 \Rightarrow ||u|^{q(x)} - |v|^{q(x)}|_{s_1'(x)} \to 0.$$

Proof. Fix $x \in \Omega$. By the Lagrange theorem applied to $f(u) = |u|^{q(x)}$, there exists $C_0(x)$ somewhere between $u(x)$ and $v(x)$ such that

$$\frac{f(u(x) - f(v(x))}{u(x)-v(x)} = f^{'}(C_0(x)),$$

thus,

$$\int_\Omega ||u|^{q(x)} - |v|^{q(x)}|^{s_1^{'}(x)} \le \int_\Omega \left[|u-v|\frac{N}{2}\max(|u|,|v|)^{q(x)-1}\right]^{s_1^{'}(x)}.$$

So, by using the Hölder inequality and Proposition 2.2 one has

$$\int_\Omega ||u|^{q(x)} - |v|^{q(x)}|^{s_1^{'}(x)} \le C(x)||u-v|^{s_1^{'}(x)}|_{q(x)} \le C(x)|u-v|^c_{s_1^{'}(x)q(x)}.$$

Finally, it remains to use Proposition 2.1 to finish the proof. □

For $u \in E$, the energy functional associated to problem $(\mathbf{P}_\lambda)$ is defined as:

$$\Phi_\lambda(u) = \Psi(u) - \lambda J(u),$$

where $\Psi(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + a(x)|u|^{p(x)})dx$ and $J(u) = \int_{\partial\Omega} \frac{V(x)}{q(x)}|u|^{q(x)}d\sigma.$

In order to formulate the variational problem $(\mathbf{P}_\lambda)$, we mention, using Remark 3.3, that J is well defined as we have for all $u \in E$,

$$|J(u)| \le \frac{1}{q^-}|V|_{L^{s(x)}(\partial\Omega)}||u|^{q(x)}|_{L^{s'(x)}(\partial\Omega)} \le \frac{1}{q^-}|V|_{s(x)}|u|^{c_1}_{L^{s'(x)q(x)}(\partial\Omega)},$$

where c_1 is a positive constant.

Proposition 3.5. *Under assumption* $(\mathbf{H})$, $\Phi_\lambda \in C^1(E,\mathbb{R})$ *and* $u \in E$ *is a critical point of* Φ_λ *if and only if* u *is a weak solution for the problem* $(\mathbf{P}_\lambda)$.

Proof. To show that $\Phi_\lambda \in C^1(E,\mathbb{R})$, we show that for all $\varphi \in E$,

$$\lim_{t\to 0^+} \frac{\Phi_\lambda(u+t\varphi) - \Phi_\lambda(u)}{t} = \langle d\Phi_\lambda(u), \varphi\rangle,$$

and $d\Phi_\lambda : E \to E^*$ continuous, where we denote by E^* the dual space of E. For all $\varphi \in E$, we have

$$\begin{aligned}
\lim_{t\to 0^+} \frac{J(u+t\varphi)-J(u)}{t} &= \frac{d}{dt} J(u+t\varphi)|_{t=0} = \frac{d}{dt} \int\limits_{\partial\Omega} \frac{V(x)}{q(x)} |u+t\varphi|^{q(x)} d\sigma|_{t=0} \\
&= \int\limits_{\partial\Omega} \frac{\partial}{\partial t} \left(\frac{V(x)}{q(x)} |u+t\varphi|^{q(x)} \right) |_{t=0} d\sigma \\
&= \int\limits_{\partial\Omega} V(x)|u+t\varphi|^{q(x)-1} sgn(u+t\varphi)\varphi|_{t=0} d\sigma \\
&= \int\limits_{\partial\Omega} V(x)|u+t\varphi|^{q(x)-2}(u+t\varphi)\varphi|_{t=0} d\sigma \\
&= \int\limits_{\partial\Omega} V(x)|u|^{q(x)-2} u\varphi d\sigma = \langle dJ(u), \varphi\rangle.
\end{aligned}$$

The differentiation under the integral is allowed for t close to zero. Indeed, for $|t| < 1$, we have

$$|V(x)|u+t\varphi|^{q(x)-2}(u+t\varphi)\varphi| \le |V(x)|(|u|+|\varphi|)^{q(x)-1}|\varphi| \in L^1(\partial\Omega).$$

Since $u, \varphi \in E$, we have

$$|u|, |\varphi| \in E \hookrightarrow L^{q(x)}(\partial\Omega) \text{ and } \ |\varphi| \in E \hookrightarrow L^{r(x)}(\partial\Omega).$$

Due to the fact that $V \in L^{s(x)}(\partial\Omega)$, the conclusion is an immediate consequence of inequality (2.3). For $u \in E$ chosen, we show that $dJ(u) \in W^{-1,p'(x)}(\Omega) = E^*$, where $1/p(x) + 1/p'(x) = 1$. It is easy to see that $dJ(u)$ is linear.

Since there is a continuous embedding $E \hookrightarrow L^{r(x)}(\partial\Omega)$, there exists a constant $M > 0$ such that

$$|v|_{L^{r(x)}(\partial\Omega)} \le M\|v\|_a \quad \text{for all } v \in E. \tag{3.1}$$

Using (2.3) and (3.1) we obtain

$$\begin{aligned}
|\langle dJ(u), \varphi\rangle| &= |\int\limits_{\partial\Omega} V(x)|u|^{q(x)-2}u\varphi d\sigma| \\
&\le \int\limits_{\partial\Omega} |V(x)||u|^{q(x)-1}|\varphi| d\sigma \\
&\le |V|_{L^{s(x)}(\partial\Omega)} ||u|^{q(x)-1}|_{L^{\frac{q(x)}{q(x)-1}}(\partial\Omega)} |\varphi|_{L^{r(x)}(\partial\Omega)} \\
&\le M|V|_{L^{s(x)}(\partial\Omega)} ||u|^{q(x)-1}|_{L^{\frac{q(x)}{q(x)-1}}(\partial\Omega)} \|\varphi\|_a.
\end{aligned}$$

Hence there exists $M_1 = |V|_{L^{s(x)}(\partial\Omega)} ||u|^{q(x)-1}|_{L^{\frac{q(x)}{q(x)-1}}(\partial\Omega)} > 0$ such that

$$|\langle dJ(u), \varphi\rangle| \le M_1 \|\varphi\|_a.$$

Using the linearity of $dJ(u)$ and the above inequality we deduce that $dJ(u) \in E^* = W^{-1,p'(x)}(\Omega)$.

Lemma 3.6 ([4]). *The map*

$$L^{q(x)}(\partial\Omega) \ni u \mapsto |u|^{q(x)-2}u \in L^{\frac{q(x)}{q(x)-1}}(\partial\Omega)$$

is continuous.

We conclude that J is Fréchet differentiable.

We can also prove (see [1]), that $\Psi \in C^1(E,\mathbb{R})$. So $\Phi_\lambda \in C^1(E,\mathbb{R})$ because $\Psi,\ J \in C^1(E,\mathbb{R})$. Moreover,

$$\langle d\Phi_\lambda(u), v\rangle = \int\limits_\Omega \big(|\nabla u|^{p(x)-2}\nabla u \nabla v + a(x)|u|^{p(x)-2}uv\big)dx - \lambda \int\limits_{\partial\Omega} V(x)|u|^{q(x)-2}uv d\sigma$$

for all $v \in E$. Let u be a critical point of Φ_λ. Then we have $d\Phi_\lambda(u) = 0_{E^*}$, that is

$$\langle d\Phi_\lambda(u), v\rangle = 0 \text{ for all } v \in E,$$

which yields

$$\int\limits_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla v + a(x)|u|^{p(x)-2}uv\Big)\, dx - \lambda \int\limits_{\partial\Omega} V(x)|u|^{q(x)-2}uv d\sigma = 0,$$

for all $v \in E$. It follows that u is a weak solution for the problem $(\mathbf{P}_\lambda)$.

By Definition 2.6, we have

$$\int\limits_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla v + a(x)|u|^{p(x)-2}uv\Big)\, dx - \lambda \int\limits_{\partial\Omega} V(x)|u|^{q(x)-2}uv d\sigma = 0,$$

for all $v \in E$. That is $\langle d\Phi_\lambda(u), v\rangle = 0$ for all $v \in E$. We obtain $d\Phi_\lambda(u) = 0_{E^*}$. Hence u is a critical point of Φ_λ. This completes the proof of Proposition 3.5. □

4. PROOF OF OUR RESULTS

4.1. PROOF OF THEOREM 3.1

To prove Theorem 3.1, we remark firstly that, under Remark 3.3, there exists $C_2 > 0$ such that

$$|u|_{L^{s'(x)q(x)}(\partial\Omega)} \le C_2\|u\|_a \quad \text{for all } u \in E. \tag{4.1}$$

Now, we are in a position to show that Φ_λ possesses a nontrivial global minimum point in E.

Lemma 4.1. *Under assumption* $(\mathbf{H})$*, the functional* Φ_λ *is coercive on* E.

Proof. First, we recall that in view of assumption (**H**), inequality (2.2), Remark 3.3, Propositions 2.2 and 2.3, one has for every $u \in E$ with $\|u\|_a > 1$

$$\begin{aligned}\Phi_\lambda(u) &\geq \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}dx + a(x)|u|^{p(x)}) - \frac{\lambda}{q^-}\int_{\partial\Omega} |V(x)||u|^{q(x)}d\sigma \\ &\geq \frac{1}{p^+}\rho_{a(x),p(x)}(u) - k\frac{\lambda}{q^-}|V|_{s(x)}|u^{q(x)}|_{L^{s'(x)}(\partial\Omega)} \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \frac{k\lambda}{q^-}|V|_{s(x)}\min(C_2^{q^-}\|u\|_a^{q^-}, C_2^{q^+}\|u\|_a^{q^+}),\end{aligned}$$

where k is a positive constant. Since $q^+ < p^-$, we infer that $\Phi_\lambda(u) \to \infty$ as $\|u\| \to \infty$, in other words I_λ is coercive on E. □

In the sequel, put $q_0^- = \inf_{x\in\overline{\Omega}_0} q(x)$ and $p_0^- = \inf_{x\in\overline{\Omega}_0} p(x)$.

Lemma 4.2. *Under assumption* (**H**)*, there exists $u_0 \in E$ such that $u_0 \geq 0, u_0 \neq 0$ and $\Phi_\lambda(tu_0) < 0$, for $t > 0$ small enough.*

Proof. Since $q_0^- < p_0^-$, then let $\epsilon_0 > 0$ be such that $q_0^- + \epsilon_0 < p_0^-$. On the other hand, since $q \in C(\overline{\Omega}_0)$ it follows that there exists an open set $\Omega_1 \subset\subset \Omega_0 \subset\subset \partial\Omega$ such that $|q(x) - q_0^-| < \epsilon_0$ for all $x \in \Omega_1$. Thus, we conclude that $q(x) \leq q_0^- + \epsilon_0 < p_0^-$ for all $x \in \Omega_1$.

Let $u_0 \in C_0^\infty(\Omega)$ such that $\operatorname{supp}(u_0) \subset \Omega_1 \subset \Omega_0$, $u_0 = 1$ in a subset $\Omega'_1 \subset \operatorname{supp}(u_0)$, $0 \leq u_0 \leq 1$ in Ω_1. Then we have

$$\begin{aligned}\Phi_\lambda(tu_0) &= \int_\Omega \frac{t^{p(x)}}{p(x)}(|\nabla u_0|^{p(x)} + a(x)|u_0|^{p(x)}dx) - \lambda\int_{\partial\Omega} \frac{t^{q(x)}}{q(x)}V(x)|u_0|^{q(x)}d\sigma \\ &= \int_{\Omega_0} \frac{t^{p(x)}}{p(x)}(|\nabla u_0|^{p(x)} + a(x)|u_0|^{p(x)}dx) - \lambda\int_{\Omega_1} \frac{t^{q(x)}}{q(x)}V(x)|u_0|^{q(x)}d\sigma \\ &\leq \frac{t^{p_0^-}}{p_0^-}\int_{\Omega_0}(|\nabla u_0|^{p(x)} + a(x)|u_0|^{p(x)}dx) - \frac{\lambda t^{q_0^-+\epsilon_0}}{q_0^-}\int_{\Omega_1} V(x)|u_0|^{q(x)}d\sigma.\end{aligned}$$

Therefore

$$\Phi_\lambda(tu_0) < 0,$$

for $t < \delta^{1/(p_0^- - q_0^- - \epsilon_0)}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda p_0^- \int_{\Omega_1} V(x)|u_0|^{q(x)}d\sigma}{q_0^+ \int_{\Omega_0}(|\nabla u_0|^{p(x)} + a(x)|u_0|^{p(x)}dx)}\right\}.$$

Finally, we point out that

$$\int_{\Omega_0}\left(|\nabla u_0|^{p(x)} + a(x)|u_0|^{p(x)}dx\right)dx > 0.$$

In fact, if

$$\int_{\Omega_0} \Big(|\nabla u_0|^{p(x)} + a(x)|u_0|^{p(x)} dx \Big)\, dx = 0,$$

then $\|u_0\|_a = 0$ and consequently $u_0 = 0$ in Ω which is a contradiction. The proof of the lemma is complete. □

In the sequel, put $m_\lambda = \inf_{u \in E} \Phi_\lambda(u)$, then we have the following result.

Proposition 4.3. *Assume that assumption* (**H**) *holds, then* Φ_λ *attains his global minimizer in* E*, that is, there exists* $u_* \in E$ *such that* $\Phi_\lambda(u_*) = m_\lambda < 0$.

Proof. Let $\{u_n\}$ be a minimizing sequence, that is to say $\Phi_\lambda(u_n) \to m_\lambda$. Suppose $\{u_n\}$ is not bounded, so $\|u_n\|_a \to +\infty$ as $n \to +\infty$. Since Φ_λ is coercive, then

$$\Phi_\lambda(u_n) \to +\infty \ \text{ as } \|u_n\|_a \to +\infty.$$

This contradicts the fact that $\{u_n\}$ is a minimizing sequence, so $\{u_n\}$ is bounded in E and therefore up to a subsequence, there exists $u_* \in E$ such that $u_n \rightharpoonup u_*$ (weakly) in E and $u_n \to u_*$ (strongly) in $L^{q(x)}(\partial\Omega)$, where $1 \le q(x) < p^\partial(x)$ for all $x \in \partial\Omega$.

Since $\Psi : E \to \mathbb{R}$ is sequentially weakly lower semicontinous (one can see [1]), then we have

$$\Psi(u_*) \le \liminf_{n\to+\infty} \Psi(u_n). \tag{4.2}$$

Now, let us prove that if $\{u_n\} \subset E$ is a sequence which converges weakly to u_* on E, then we have

$$J(u_n) \to J(u_*) \quad \text{as} \quad n \to +\infty. \tag{4.3}$$

For this purpose, recall that the compact embedding $E \hookrightarrow L^{s'(x)q(x)}(\partial\Omega)$. In addition, using the Hölder type inequality we have

$$J(u_n) - J(u_*) \le \frac{k_1}{q^-} |V|_{L^{s(x)}(\partial\Omega)} \big| |u_n|^{q(x)} - |u_*|^{q(x)} \big|_{L^{s'(x)}(\partial\Omega)}.$$

By using Proposition 4.5, the convergence (4.3) holds true, so Φ_λ is weakly lower semicontinuous and consequently

$$m_\lambda \le \Phi_\lambda(u_*) \le \liminf_{n\to+\infty} \Phi_\lambda(u_n) = m_\lambda.$$

This completes the proof of Proposition 4.3. □

Then, Theorem 3.1 is true.

4.2. PROOF OF THEOREM 3.2

In this section, we aim to prove Theorem 3.2 by using Ekeland's variational principle. To this aim, we need the following lemma.

Lemma 4.4. *Under the same hypothesis as in Theorem 3.1, for all $\rho \in (0,1)$ there exist $\lambda_* > 0$ and $b > 0$ such that for all $u \in E$ with $\|u\|_a = \rho$ we have*

$$\Phi_\lambda(u) \geq b > 0 \ \text{ for all } \ \lambda \in (0, \lambda_*).$$

Proof. Let $u \in E$ with $\|u\|_a = \rho$, then, it follows from inequalities (2.2), (4.1) and proposition (2.3) that

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \frac{k\lambda}{q^-}|V|_{L^{s(x)}(\partial\Omega)} \min(C_2^{q^-}\|u\|_a^{q^-}, C_2^{q^+}\|u\|_a^{q^+}),\\ &= \frac{k|V|_{L^{s(x)}(\Omega)}}{q^-}\min((C_2\rho)^{q^-}, (C_2\rho)^{q^+})\\ &\quad \times \left(\frac{q^-\rho^{p^-}}{kp^+|V|_{L^{s(x)}(\partial\Omega)}\min((C_2\rho)^{q^-}, (C_2\rho)^{q^+})} - \lambda\right).\end{aligned} \tag{4.4}$$

By the above inequality, we remark that if we define

$$\lambda_* = \frac{q^-\rho^{p^-}}{kp^+|V|_{L^{s(x)}(\partial\Omega)}\min((C_2\rho)^{q^-}, (C_2\rho)^{q^+})}, \tag{4.5}$$

then, the result of Lemma 4.4 follows. □

Before proving our main result, we give several propositions that will be used later.

Proposition 4.5.

$$\lim_{n\to\infty}\int_{\partial\Omega} V(x)|u_n|^{q(x)-2}u_n(u_n - u)d\sigma = 0.$$

Proof. Using inequality (2.3) we have

$$\int_{\partial\Omega} V(x)|u_n|^{q(x)-2}u_n(u_n - u)dx \leq |V|_{L^{s(x)}(\partial\Omega)}||u_n|^{q(x)-2}u_n|_{L^{\frac{q(x)}{q(x)-1}}(\partial\Omega)}|u_n - u|_{L^{r(x)}(\partial\Omega)}.$$

Then if

$$||u_n|^{q(x)-2}u_n|_{L^{\frac{q(x)}{q(x)-1}}(\partial\Omega)} > 1,$$

by Proposition 2.2, we get

$$||u_n|^{q(x)-2}u_n|_{L^{\frac{q(x)}{q(x)-1}}(\partial(\Omega)} \leq |u_n|^{q^+}_{L^{q(x)}(\partial\Omega)}.$$

The compact embedding $E \hookrightarrow L^{q(x)}(\partial\Omega)$ ends the proof. □

Proposition 4.6 ([31, Proposition 3.1])**.** *Ψ' is a mapping of S_+ type, that is, if $u_n \rightharpoonup u$ weakly in E and*

$$\limsup_{n\longrightarrow\infty} \prec \Psi'(u_n) - \Psi'(u), u_n - u \succ \leq 0,$$

then $u_n \longrightarrow u$ in E

Proof of Theorem 3.2. Let $\lambda \in (0, \lambda_*)$, where λ_* is given by (4.5). Then, it follows from Lemma 4.4 that on the boundary of the ball centred at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} \Phi_\lambda > 0. \tag{4.6}$$

Furthermore, from Lemma 4.2, there exists $\varphi \in E$ such that $\Phi_\lambda(t\varphi) < 0$ for $t > 0$ small enough. Using (4.4), we deduce that

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} \Phi_\lambda < 0. \tag{4.7}$$

Let choose $\epsilon > 0$. Then, we have

$$0 < \epsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda - \inf_{B_\rho(0)} \Phi_\lambda.$$

Using the above information, the functional $\Phi_\lambda : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$, is lower bounded on $\overline{B_\rho(0)}$ and $\Phi_\lambda \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Then by Ekeland's variational principle there exists $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \underline{c} \leq \Phi_\lambda(u_\epsilon) \leq \underline{c} + \epsilon, \\ \Phi_\lambda(u_\epsilon) < \Phi_\lambda(u) + \epsilon \cdot \parallel u - u_\epsilon \parallel_a, \quad u \neq u_\epsilon. \end{cases}$$

Since

$$\Phi_\lambda(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \epsilon \leq \inf_{B_\rho(0)} \Phi_\lambda + \epsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda,$$

we can infer that $u_\epsilon \in B_\rho(0)$.

Now, let define $I_\lambda : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$ by $I_\lambda(u) = \Phi_\lambda(u) + \epsilon \cdot \parallel u - u_\epsilon \parallel_a$. It is not difficult to see that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0,$$

for $t > 0$ small enough and any $v \in B_1(0)$. It yields from the above relation that

$$\frac{\Phi_\lambda(u_\epsilon + t \cdot v) - \Phi_\lambda(u_\epsilon)}{t} + \epsilon \cdot \parallel v \parallel_a \geq 0.$$

Letting $t \to 0$, we obtain

$$\langle d\Phi_\lambda(u_\epsilon), v \rangle + \epsilon \cdot \parallel v \parallel_a \geq 0$$

and this implies that $\parallel d\Phi_\lambda(u_\epsilon) \parallel_a \leq \epsilon$. Therefore, we deduce that there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$\Phi_\lambda(u_n) \longrightarrow \underline{c} \quad \text{and} \quad d\Phi_\lambda(u_n) \longrightarrow 0_{E^*}, \tag{4.8}$$

where $\underline{c}$ is given by (4.7). Hence, we have that $\{u_n\}$ is bounded in E. Thus, there exists a subsequence again denoted by $\{u_n\}$, and u in E such that, $\{u_n\}$ converges weakly to u in E. So, in view of Remark 3.3, we have

$$u_n \longrightarrow u, \quad \text{in } L^{r(x)}(\partial\Omega). \tag{4.9}$$

Furthermore, a direct calculation shows that

$$\begin{aligned} &\prec d\Psi_\lambda(u_n) - d\Psi_\lambda(u), u_n - u \succ \\ &= \prec d\Phi_\lambda(u_n) - d\Phi_\lambda(u), u_n - u \succ + \lambda \prec dJ(u_n) - dJ(u), u_n - u \succ . \end{aligned} \tag{4.10}$$

On the other hand, it is clear that

$$\prec d\Phi_\lambda(u_n) - d\Phi_\lambda(u), u_n - u \succ = \prec d\Phi_\lambda(u_n), u_n - u \succ - \prec d\Phi_\lambda(u), u_n - u \succ \longrightarrow 0. \tag{4.11}$$

Now, from hypothesis (**H**) and (4.9) we get

$$\begin{aligned} \prec dJ(u_n) - dJ(u), u_n - u \succ &= \int_{\partial\Omega} V(x) \left(|u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \right) (u_n - u) d\sigma \\ &\leq \int_{\partial\Omega} |V(x)| |u_n|^{q(x)-1} |u_n - u| d\sigma \\ &\quad + \int_{\partial\Omega} |V(x)| |u|^{q(x)-1} |u_n - u| d\sigma. \end{aligned}$$

Since

$$\begin{aligned} &\int_{\partial\Omega} |V(x)| |u_n|^{q(x)-1} |u_n - u| d\sigma \\ &\leq |V(x)|_{L^{s(x)}(\partial\Omega)} \left| |u_n|^{q(x)-1} \right|_{L^{q(x)s'(x)}(\partial\Omega)} |u_n - u|_{L^{r(x)}(\partial\Omega)} \longrightarrow 0 \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\int_{\partial\Omega} |V(x)| |u|^{q(x)-1} |u_n - u| d\sigma \\ &\leq |V(x)|_{L^{s(x)}(\partial\Omega)} \left| |u|^{q(x)-1} \right|_{L^{q(x)s'(x)}(\partial\Omega)} |u_n - u|_{L^{r(x)}(\partial\Omega)} \longrightarrow 0, \end{aligned} \tag{4.13}$$

combining (4.10), (4.12) and (4.13), we obtain

$$\prec d\Psi_\lambda(u_n) - d\Psi_\lambda(u), u_n - u \succ \longrightarrow 0.$$

Since $d\Psi$ is of (S_+) type, then $u_n \longrightarrow u$ in E. Moreover, $\Phi_\lambda \in C^1(E, \mathbb{R})$, then we conclude that

$$d\Phi_\lambda(u_n) \to d\Phi_\lambda(u) \quad \text{as} \quad n \to \infty. \tag{4.14}$$

Relations (4.8) and (4.14) shows that $d\Phi_\lambda(u) = 0$ and thus u is a weak solution for problem $(\mathbf{P}_\lambda)$. Moreover, by relation (4.8) it follows that $\Phi_\lambda(u) < 0$ and thus, u is a nontrivial weak solution for $(\mathbf{P}_\lambda)$.

Since $\Phi_\lambda(|u|) = \Phi_\lambda(u)$, then problem $(\mathbf{P}_\lambda)$ has a nonnegative one.

The proof of Theorem 3.2 is complete. □

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Received: January 6, 2017.
Revised: January 19, 2017.
Accepted: January 28, 2017.