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On the one-dimensional mean-field games with congestion

Abstract In this work, we consider a one-dimensional forward-forward model of Mean-field Games with congestion. We establish a connection between such models and conservation laws. Next, we show the existence of non trivial convex entropies. Finally, we investigate the existence of solutions in the parabolic case and derived some estimates thanks to the existence of such convex entropies.

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1. Introduction. Mean-Field Games (MFGs) provide a framework for the study of a class of competitive games involving a large number of agents. In these games, the agents are assumed indistinguishable and interact in a mean-field manner so that each agent has a negligible effect on the outcome of the games. Since the seminal work by Lasry and Lions (2006a, 2006b) in Mathematics and by Huang, Caines and Malhame in Engineering in 2006, the theory has propagated to other fields of research including crowd dynamics, economics, finance, biology ans social science (see [3, 22]). From a mathematical perspective, MFGs, developed as limit of non cooperative games, are captured by two important classes of PDEs: the Hamilton-Jacobi-Bellman equations viewed as backward equations and the Fokker Planck equations seen as forward equations. The Hamilton-Jacobi-Bellman equations express the optimality of a value function in relation to a control problem. On the other hand, the Fokker-Planck equations model the dynamics of the density of the agents governed by a velocity field derived from the Hamiltonian of the system. The coupling of these highly non-linear equations has since generated a lot of interest and has been studied under various conditions and from various perspectives.

The existence and uniqueness of classical solutions for MFGs were investigated in [14, 15, 18, 20, 24]. There, the authors exploited structural forms of the specific equations to obtain existence and uniqueness of solutions in the presence of viscosity as well as in the absence of the viscosity. On the other hand, in [4, 5, 24, 26] the authors developed a concept of weak solutions. Furthermore, stationary problems were considered in [12, 13, 17]. In these cases, the monotonicity method has played an important role in solving the equations. In [10, 11] and in [8], the authors even constructed explicit solutions to some Mean Field-Games problems. The articles [9, 16, 21] studied MFGs designed to model congestion problems which incorporate the difficulty for agents to move freely in areas with high density.

The one dimensional congestion problem can be stated as follows:

$$\begin{cases} -u_t + m^{\alpha} H\left(\frac{u_x}{m^{\alpha}}\right) = \varepsilon u_{xx} + g(m) \\ m_t - \left(H'\left(\frac{u_x}{m^{\alpha}}\right) m\right)_x = \varepsilon m_{xx}. \end{cases} \tag{1}$$

This system is supplemented by a terminal -initial conditions.

$$\begin{cases} u(x,T) = u_T(x) \\ m(x,0) = m_0(x). \end{cases} \tag{2}$$

In the system (1), $H : \mathbb{R} \longrightarrow \mathbb{R}$ denotes the Hamiltonian of the systems. $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is the coupling between the Hamiton-Jacobi equation and the Fokker-Planck equations. α is the congestions parameter. For simplicity, we consider the problem on a domain where the spatial variable x belongs to the torus which we identify with $[0,1]$ while the temporal variable t belongs to the interval $[0,T]$ where $T > 0$. The unknowns in the problems (1) and (2) are u and m. u represents the value function for an optimal control problem that determines the Hamilton-Jacobi problem ; m stands for a density of probability that describes the mean-field.

Mean-Field Games have been considered in the context of numerical methods [1]. In order to study the associated stationary problem from a numerical point of view, it has been conjectured that the forward-forward model as time goes to infinity will offer a good approximation to stationary states. Indeed, the forward-forward models were introduced by [2] and [6] as a variant of the standard Mean-Field Games originally with the hope that the long time convergence of such models will yield solution to the stationary MFGs. The forward-forward models take the following form:

$$\begin{cases} u_t + m^{\alpha} H\left(\frac{u_x}{m^{\alpha}}\right) = \varepsilon u_{xx} + g(m) \\ m_t - \left(H'\left(\frac{u_x}{m^{\alpha}}\right) m\right)_x = \varepsilon m_{xx} \end{cases} \tag{3}$$

with the *initial-initial conditions*:

$$\begin{cases} u(x,0) = u_0(x) \\ m(x,0) = m_0(x). \end{cases} \tag{4}$$

Various techniques have been developed to study the forward-forward MFGs. In [19], the authors used the entropy method to study the convergence of the one-dimensional forward-forward MFGs. In [9], the entropy method was adapted to study of the forward-forward congestion problem with no coupling for the quadratic Hamiltonian.

In this paper we investigate (3) for a coupling function g which is nonzero. More precisely, we will focus on the case where g is a non-increasing power function of m. The strategy implemented here to deal with the forward-forward problem is based on the relationship between the Hamilton-Jacobi equation and the conservation laws in dimension one. The key result is the identification of non trivial convex entropies.

The paper is organized in the following way. In section 1, we develop the preliminary results. In section 2, we show how these MFGs can be transformed into systems of one-dimensional conservation laws and we study important features of these conservation laws. In section 4, we consider parabolic forward forward MFGs and show short time existence of solution such MFGs equations. Besides, we exploit the existence of convex entropies to obtain some estimates.

2. Preliminaries. In this section, we recall some definitions and a few well-known results in the study of systems of conservation laws. For more extensive explanation, we refer the readers to [7, 25]. We first recall the general form for the equations of conservation laws:

$$\mathbf{w}_t + \partial_x \mathbf{G}(\mathbf{w}) = 0, \quad x \in \mathbb{R},\ t > 0,\ \mathbf{w}(t,x) \in V, \tag{5}$$

Here, V is an open set in $\mathbb{R}^2$, $\mathbf{G} : V \longrightarrow \mathbb{R}$ is the flux-function which is assumed differentiable on V. The function $\mathbf{w} = \mathbf{w}(t,x) : \mathbb{R}_+ \times \mathbb{R} \longrightarrow V$ is the unknown in (5). We say that (η, q) is an *entropy/entropy-flux* for (5) if

$$D\eta(w) D\mathbf{G}(w) = Dq(w), \tag{6}$$

for every $w \in V$. For such entropy-entropy flux, we have the following identity:

$$\eta(\mathbf{w})_t + (q(\mathbf{w}))_x = 0 \tag{7}$$

for any smooth solution $\mathbf{w}$ of (5). In the context of conservation laws the concept of hyperbolicity is important. We say that (5) is *hyperbolic* if the Jacobian matrix $DG(w)$ at each point $w \in V$ has two real eigenvalues $\lambda_i(w)$, $i = 1, 2$. We say that (5) is *strictly hyperbolic* if $\lambda_1(w) \neq \lambda_2(w)$ for each $w \in V$.

3. Systems of conservation laws and first-order, forward-forward MFGs. In this section, we consider the forward-forward MFGs with congestion and a quadratic Hamiltonian. We further assume that the coupling g is a power function:

$$\begin{cases} u_t + \frac{(c+u_x)^2}{2m^\alpha} = km^\alpha, \\ m_t - \left(\frac{c+u_x}{m^{\alpha-1}}\right)_x = 0, \end{cases} \tag{8}$$

with $\alpha > 0$ and $c \in \Re$. Next, let us assume that the solutions of (8) are smooth enough. We differentiate the first equation with respect to the spatial variable x and set $v = c + u_x$. We are thus led to the following system of conservation laws:

$$\begin{cases} v_t + \left(\frac{v^2}{2m^\alpha} - km^\alpha\right)_x = 0, \\ m_t - \left(\frac{v}{m^{\alpha-1}}\right)_x = 0. \end{cases} \tag{9}$$

3.1. Hyperbolicity. Now, we show that (9) is a hyperbolic, system of conservation laws. The flux function associated with the conservation laws in (9) is given by

$$G(v,m) = \left(\frac{v^2}{2m^\alpha} - km^\alpha, -\frac{v}{m^{\alpha-1}}\right) \tag{10}$$

for any $(v,m) \in \mathbb{R}^2$ with $m > 0$. To study the hyperbolicity of the equation in (9), we compute the Jacobian of G and get

$$DG(v,m) = \begin{pmatrix} m^{-\alpha}v & -\alpha km^{-1+\alpha} - \frac{1}{2}\alpha m^{-1-\alpha}v^2 \\ -m^{1-\alpha} & -(1-\alpha)m^{-\alpha}v \end{pmatrix}. \tag{11}$$

We define $\mathcal{B}$ as the set

$$\mathcal{B} = \left\{(v,m) \in \mathbb{R}^2 : v > 0,\ m > 0\right\}.$$

Proposition 3.1 *The system (9) is strictly hyperbolic on the set $\mathcal{B}$. More precisely, (11) has eigenvalues*

$$\lambda_1 = \frac{1}{2m^\alpha}\left(\alpha v - \sqrt{4\alpha km^{2\alpha} + (4-2\alpha+\alpha^2)v^2}\right) \tag{12}$$

and

$$\lambda_2 = \frac{1}{2m^\alpha}\left(\alpha v - \sqrt{4\alpha km^{2\alpha} + (4-2\alpha+\alpha^2)v^2}\right) \tag{13}$$

with corresponding eigenvectors

$$\mathbf{r}_1 = \left((\alpha-2)\,v + \sqrt{4k\alpha\, m^{2\alpha} + (4-2\alpha+\alpha^2)\,v^2},\ 2m\right) \tag{14}$$

and

$$\mathbf{r}_2 = \left((\alpha-2)\,v - \sqrt{4k\alpha\, m^{2\alpha} + (4-2\alpha+\alpha^2)\,v^2},\ 2m\right). \tag{15}$$

Proof The characteristic equation of $DF(v,m)$ is given by

$$X^2 - \alpha m^{-\alpha} v X + \alpha k + 1/2(-2+\alpha) m^{-2\alpha} v^2 = 0$$

Simple computations show that DF has eigenvalues given by (12) and (13), and that these eigenvalues are distinct on $\mathcal{B}$. Thus, (9) is a strictly hyperbolic system of conservation laws on the set $\mathcal{B}$. Next, we find the right eigenvectors corresponding to λ_1 and λ_2. Accordingly, we determine $\mathbf{r}_i$, $i = 1,2$, such that

$$DG^T \mathbf{r}_i = \lambda_i \mathbf{r}_i.$$

Again, straightforward computations ensure that $\mathbf{r}_1$, $\mathbf{r}_2$ can be chosen as in (14) and (15). ∎

3.2. Existence of convex entropies. First, we construct entropies for (9). We recall that (η, q) is an entropy/entropy-flux pair for (9) if

$$D\eta DG = Dq. \tag{16}$$

In the case of problem (9), we recall that the flux function is given by (10). A direct computation shows that η solves (16) if and only if it satisfies a certain second order partial differential equation. This is the object of the next lemma.

Lemma 3.2 *The entropies η of the system of conservation laws (9) satisfy the following equation:*

$$\frac{\alpha(2km^{2\alpha} + v^2)}{2m^{\alpha+1}} \eta_{vv} - \frac{1}{m^{\alpha-1}} \eta_{mm} + \frac{(2-\alpha)v}{m^\alpha} \eta_{vm} = 0. \tag{17}$$

Proof Let (η, q) be an entropy/ entropy-flux for (9). We note that

$$\eta(v,m)_t + q(v,m)_x = \partial_v \eta v_t + \partial_m \eta m_t + \partial_v q v_x + \partial_m q m_x.$$

By using the system (9), we obtain that

$$\begin{aligned} \eta(v,m)_t + q(v,m)_x &= \partial_v \eta \left(km^\alpha - \frac{v^2}{2m^\alpha} \right)_x + \partial_m \eta \left(\frac{v}{m^{\alpha-1}} \right)_x + \partial_v q v_x + \partial_m q m_x \\ &= \partial_v \eta \left(k\alpha m^{\alpha-1} m_x - \frac{v v_x}{m^\alpha} + \frac{\alpha v^2}{2m^{\alpha+1}} m_x \right) \\ &\quad + \partial_m \eta \left(\frac{v_x}{m^{\alpha-1}} + \frac{(1-\alpha)v}{m^\alpha} m_x \right) + \partial_v q v_x + \partial_m q m_x \end{aligned}$$

By collecting the terms in v_x and m_x, we obtain that

$$\begin{aligned} \eta(v,m)_t + q(v,m)_x = &\left(k\partial_v \eta \alpha m^{\alpha-1} + \frac{\alpha v^2}{2m^{\alpha+1}} \partial_v \eta + \frac{(1-\alpha)v}{m^\alpha} \partial_m \eta + \partial_m q \right) m_x \\ &+ \left(\frac{\partial_m \eta}{m^{\alpha-1}} - \frac{v}{m^\alpha} \partial_v \eta + \partial_v q \right) v_x \end{aligned} \tag{18}$$

As (η, q) is an *entropy/entropy-flux* for (5) we have that $\eta(v,m)_t + q(v,m)_x = 0$. In light of (18), this equation implies that

$$\partial_v q = -\left(-\partial_v \eta \frac{v}{m^\alpha} + \frac{\partial_m \eta}{m^{\alpha-1}}\right)$$

and

$$\partial_m q = -\left(k\alpha m^{\alpha-1}\partial_v \eta + \partial_v \eta \frac{\alpha v^2}{2m^{\alpha+1}} + \frac{(1-\alpha)v}{m^\alpha}.\partial_m \eta\right).$$

It follows that

$$\partial_m \left(-\partial_v \eta \frac{v}{m^\alpha} + \frac{\partial_m \eta}{m^{\alpha-1}}\right) = \partial_v \left(k\alpha m^{\alpha-1}\partial_v \eta + \partial_v \eta \frac{\alpha v^2}{2m^{\alpha+1}} + \frac{(1-\alpha)v}{m^\alpha}\partial_m \eta\right).$$

Thus,

$$\begin{aligned} \frac{\alpha v}{m^{\alpha+1}}\partial_v \eta - \partial_{vm}\eta \frac{v}{m^\alpha} \quad &+ \frac{(1-\alpha)}{m^\alpha}\partial_m \eta + \frac{1}{m^{\alpha-1}}\partial^2_{mm}\eta \\ &= \left(k\alpha m^{\alpha-1} + \frac{\alpha v^2}{2m^{\alpha+1}}\right)\partial^2_{vv}\eta \\ &\quad + \frac{\alpha v}{m^{\alpha+1}}\partial_v \eta + \frac{(1-\alpha)v}{m^\alpha}\partial^2_{mv}\eta + \frac{(1-\alpha)}{m^\alpha}\partial_m \eta. \end{aligned}$$

Therefore,

$$\left(k\alpha m^{\alpha-1} + \frac{\alpha v^2}{2m^{\alpha+1}}\right)\partial^2_{vv}\eta \qquad + \frac{(2-\alpha)v}{m^\alpha}\partial_{mv}\eta - \frac{1}{m^{\alpha-1}}\partial^2_{mm}\eta = 0.$$

In the next proposition, we investigate the existence of entropies and determine conditions under which these entropies are convex.

PROPOSITION 3.3 *The equation (17) has solutions of the form :*

$$\eta(v,m) = Q(m) + \frac{P(v)}{m^\beta}, \tag{19}$$

where $P(v) = av^2$ and $Q''(m) = \frac{2ak\alpha}{m^{\beta-2\alpha+2}}$ for any $a > 0$ and β such that $\beta^2 + (5-2\alpha)\beta + \alpha = 0$.

PROOF Assume η has the following form:

$$\eta(m,v) = Q(m) + \frac{P(v)}{m^\beta}.$$

We have that

$$\partial_m \eta = Q'(m) - \beta \frac{P(v)}{m^{\beta+1}} \quad \text{and} \quad \partial_v \eta = \frac{P'(v)}{m^\beta}.$$

We further compute the second partial derivatives

$$\partial_{mm}\eta = Q''(m) + \beta(\beta+1)\frac{P(v)}{m^{\beta+2}} \quad \partial_{vv}\eta = \frac{P''(v)}{m^\beta} \quad \text{and} \quad \partial_{vm}\eta = -\beta\frac{P'(v)}{m^{\beta+1}}.$$

$$\frac{\alpha(2km^{2\alpha}+v^2)}{2m^{\alpha+1}}\left(\frac{P''(v)}{m^\beta}\right) - \frac{1}{m^{\alpha-1}}\left(Q''(m) + \beta(\beta+1)\frac{P(v)}{m^{\beta+2}}\right) + \frac{(2-\alpha)v}{m^\alpha}\left(-\beta\frac{P'(v)}{m^{\beta+1}}\right) = 0.$$

That is,

$$\alpha(2km^{2\alpha}+v^2)\left(\frac{P''(v)}{m^\beta}\right) - 2m^2\left(Q''(m) + \beta(\beta+1)\frac{P(v)}{m^{\beta+2}}\right) + 2m(2-\alpha)v\left(-\beta\frac{P'(v)}{m^{\beta+1}}\right) = 0. \tag{20}$$

We rearrange the (20) to obtain

$$-2m^2Q''(m) - 2\beta(\beta+1)\frac{P(v)}{m^\beta} - 2v\beta(2-\alpha)\frac{P'(v)}{m^\beta} + \alpha\left(2km^{2\alpha}+v^2\right)\frac{P''(v)}{m^\beta} = 0. \tag{21}$$

The equation (21) can be rewritten as

$$2m^{2+\beta}Q''(m) + 2\beta(\beta+1)P(v) + 2(2-\alpha)\beta vP'(v) - \alpha\left(2km^{2\alpha}+v^2\right)P''(v) = 0. \tag{22}$$

Assume P is of the form $P(v) = av^2$. Then, (22) becomes

$$2m^{\beta+2}Q''(m) + 2\beta(\beta+1)(av^2) + 2(2-\alpha)\beta v(2av) - 2\alpha\left(2km^{2\alpha}+v^2\right)a = 0.$$

In other words,

$$2m^{\beta+2}Q''(m) - 4a\alpha km^{2\alpha} + [2a\beta(\beta+1) + 2a(4-2\alpha)\beta - 2a\alpha]v^2 = 0.$$

That is,

$$2m^{\beta+2}Q''(m) - 4a\alpha km^{2\alpha} + 2a[\beta^2 + (5-2\alpha)\beta - \alpha]v^2 = 0.$$

As $\alpha > 0$, we have that $(5-2\alpha)^2 + 4\alpha > 0$ and so, we choose β such that $\beta^2 + (5-2\alpha)\beta + \alpha = 0$. Then, $Q''(m) = \frac{2ak\alpha}{m^{\beta-2\alpha+2}}$. ■

In the next proposition, we investigate the convexity of the entropies exhibited in the previous proposition. As seen later, the convexity of the entropies provides useful information on the viscosity problems.

Proposition 3.4 *There exists* $\epsilon_0 > 0$ *such that whenever* $0 < \alpha < \epsilon_0$, *the functions defined by*

$$\eta(v,m) = Q(m) + \frac{av^2}{m^\beta}, \qquad a > 0, \tag{23}$$

where $Q''(m) = \frac{2ak\alpha}{m^{\beta-2\alpha+2}}$ *with* $\beta^2 + (5-2\alpha)\beta - \alpha = 0$, *are strictly convex.*

Proof We compute the Hessian of the function η:

$$D^2\eta(m,v) = \begin{pmatrix} Q''(m) + \dfrac{a\beta(\beta+1)v^2}{m^{\beta+2}} & -\dfrac{2a\beta v}{m^{\beta+1}} \\ -\dfrac{2a\beta v}{m^{\beta+1}} & \dfrac{2a}{m^\beta} \end{pmatrix}. \tag{24}$$

The determinant of the Hessian of $D^2\eta(m,v)$ is given by

$$\begin{aligned} \det\left(D^2\eta(m,v)\right) &= \frac{2a}{m^\beta}\left(Q''(m) + \frac{a\beta(\beta+1)v^2}{m^{\beta+2}}\right) - \left(-\frac{2a\beta v}{m^{\beta+1}}\right)^2 \\ &= \frac{2a}{m^\beta}Q''(m) + \frac{2a^2\beta(\beta+1)v^2}{m^{2\beta+2}} - \frac{4a^2\beta^2v^2}{m^{2\beta+2}} \\ &= \frac{2a}{m^\beta}Q''(m) + \frac{2a^2\beta^2v^2 + 2a^2\beta v^2}{m^{2\beta+2}} - \frac{4a^2\beta^2v^2}{m^{2\beta+2}} \\ &= \frac{2a}{m^\beta}Q''(m) + \frac{2a^2\beta(1-\beta)v^2}{m^{2\beta+2}}. \end{aligned} \tag{25}$$

We note that

$$\beta_1 := \beta_1(\alpha) = \frac{-5 + 2\alpha + \sqrt{25 - 16\alpha + 4\alpha^2}}{2}$$

satisfies $\beta^2 + (5-2\alpha)\beta - \alpha = 0$. Moreover, $\beta_1 > 0$ and $\beta_1(\alpha)$ converges to zero when α goes to zero. As a consequence, there exists $\epsilon_0 > 0$ such that $\beta_1(\alpha)\left(1 - \beta_1(\alpha)\right) > 0$ whenever $0 < \alpha < \epsilon_0$. In light of (25), it results that $\det\left(D^2\eta(m,v)\right) > 0$ for all $0 < \alpha < \epsilon_0$. As $Q''(m) + \dfrac{a\beta(\beta+1)v^2}{m^{\beta+2}} > 0$, we conclude that the Hessian of η is positive definite. Consequently, η is strictly convex. ■

4. Parabolic forward-forward with congestion. The parabolic version of the forward-forward MFG with congestion can be expressed in similar way to the non parabolic problem described above and as follows:

$$\begin{cases} v_t + \left(\frac{v^2}{2m^\alpha} + km^\alpha\right)_x = \varepsilon v_{xx}, \\ m_t - \left(\frac{v}{m^{\alpha-1}}\right)_x = \varepsilon m_{xx}, \end{cases} \tag{26}$$

with $\varepsilon > 0$ and initial conditions

$$\begin{cases} v(x,0) = v_0(x) \\ m(x,0) = m_0(x). \end{cases} \tag{27}$$

We assume that (v_0, m_0) is smooth and takes values in a compact subset, $\mathcal{C} \subset \mathcal{B}$. We recall that

$$\mathcal{B} = \left\{(v,m) \in \mathbb{R}^2 : v > 0,\ m > 0\right\}.$$

In the previous paragraph, we have shown the existence on convex entropies for (26) in the absence of viscosity. In this framework, standard theorems in conservation laws theory ensures the existence of unique classical solutions $(v^\varepsilon, m^\varepsilon)$ for (26)-(27) on $T_t \times [0, T_\infty)$ for some $0 < T_\infty \leq \infty$.

Remark 4.1 For smooth initial conditions, the short time existence of solutions in (26) implies the short time existence of a solution in the corresponding MFGs with congestion:

$$\begin{cases} u_t + \frac{(c+u_x)^2}{2m^\alpha} = km^\alpha + \varepsilon u_{xx}, \\ m_t - \left(\frac{c+u_x}{m^{\alpha-1}}\right)_x = \varepsilon m_{xx}. \end{cases} \tag{28}$$

The existence of long time solutions in (26) requires that the density m stays positive for all time. The following proposition provides a bound on the solutions of (26) for a class of entropies and is the first step in the attempt to establish such results.

Proposition 4.2 *Let $(v^\varepsilon, m^\varepsilon)$ be the classical solution for (26). Then, for any entropy η, we have*

$$\frac{d}{dt}\int_{\mathbb{T}} \eta(v^\varepsilon, m^\varepsilon)dx = -\varepsilon \int_{\mathbb{T}} (v_x^\varepsilon, m_x^\varepsilon)^T D^2\eta(v^\varepsilon, m^\varepsilon)(v_x^\varepsilon, m_x^\varepsilon)dx. \tag{29}$$

As a consequence, for convex entropies as in (19), we have

$$\int_{\mathbb{T}} Q(m^\varepsilon) + a\frac{(v^\varepsilon)^2}{(m^\varepsilon)^\beta}dx \leq \int_{\mathbb{T}} Q(m_0^\varepsilon) + a\frac{(v_0^\varepsilon)^2}{(m_0^\varepsilon)^\beta}dx =: c_0. \tag{30}$$

Here, Q, a and β are defined in proposition 3.4.

Proof The equality is straightforward using (26). Assume η is a convex entropy as in lemma 3.4. Then, $D^2\eta$ is positive definite so that the right hand side of (29) is non positive. It follows that $t \longrightarrow \int_{\mathbb{T}} \eta(v^\varepsilon, m^\varepsilon)dx$ is decreasing. As a result, $\int_{\mathbb{T}} \eta(v^\varepsilon, m^\varepsilon)dx \leq \int_{\mathbb{T}} \eta(v_0^\varepsilon, m_0^\varepsilon)dx$ which yields (30). ■

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O modelach gier uśrednionego oddziaływania z zarządzaniem.

Marc Sedjro

Abstract Praca poświęcona jest jednowymiarowym modelom gier z uśrednionym oddziaływaniem w zarządzaniu. Takie badania mają na celu analizę podejmowania strategicznych decyzji przez czynniki mało oddziaływające w bardzo dużych populacjach. Ustalany jest związek między takimi modelami a prawami zachowania. W wyniku tych badań pokazano istnienie nietrywialnych entropii wypukłych. W końcowej części badane jest istnienie rozwiązań w przypadku parabolicznym i wyprowadzane są pewne oszacowania z istnienia takich wypukłych entropii.

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