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UNIVERSAL EQUATIONS FOR PYTHAGOREAN AND SAUTER-TYPE FORMULAS OF MEAN VALUE CALCULATION AND CLASSIFICATION OF THE EXTENDED PYTHAGOREAN MEANS

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Abstract: A general equation, combining the Pythagorean and Sauter-type formulas, which can be used for calculation of a mean of a set of values is given and discussed in the paper. A classification of means, forming the Extended Pythagorean Means family, including power (arithmetic, geometric, quadratic, cubic, etc.), inverted power (harmonic, inverted quadratic, inverted cubic, etc.), and Sauter-type (volume-surface, volume-length, etc.) is presented. The calculated means, depending on their type, are different. Formulas for calculating mean values for a set of values treated individually, in groups and as a fraction (weighted mean) are also given.

Keywords: *average value, mean size, mean diameter, weighted mean*

1. INTRODUCTION

In very often there is a need for calculation of the mean value, especially when one deals with poly-dispersed in property particles, air bubbles and droplets. In such a situation, there is a question which formula should be chosen for the mean value calculation. The possible number of formulas of mean size is great (theoretically infinite) (Bullen 1990; 2003). However, it can be significantly narrowed to the so-called Pythagorean and Sauter-type means, which are simple, universal, meaningful and used

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in practice. The main Pythagorean means are geometric, arithmetic and harmonic (Bullen 2003) while the Sauter-type means (Sauter 1926; Pacek et al. 1998) form another broad family of means. In this paper the Pythagorean and Sauter-type means were combined into one mathematical formula called the Extended Pythagorean Means and the generated means were classified. The created general formula can be used for generating specific equations for determination of mean values of different properties of particles, bubbles and droplets including size and for instance mean specific rate of a separation process.

2. PYTHAGOREAN FAMILY OF MEANS

The generalized mean, also known as the power or Hölder mean (Stolarsky 1996), is an abstraction of the Pythagorean quadratic, arithmetic, geometric and harmonic means. It can be used for a set of n positive values d_i. The general mathematical formula for the Pythagorean means is (Wikipedia 2017):

$$d_x = \left(\frac{1}{n} \sum_{i=1}^{n} d_i^x \right)^{\frac{1}{x}}. \tag{1}$$

where d_x is the mean value. The values are numbered i = 1, 2, 3, ..., n., and x is a characteristic number for a specific type of mean. Equation 1 can be used for generating different means, including geometric, harmonic and arithmetic. To do that, an appropriate value of x, taken from Table 1, should be used.

Table 1. Values of x needed for calculation of specific Pythagorean means based on Eq. (1)

Mean	x
Maximum of d_i	$\rightarrow \infty$
Quadratic	2
Arithmetic	1
Geometric	$\rightarrow 0$
Harmonic	–1
Minimum of d_i	$\rightarrow -\infty$

There is another general formula for the Pythagorean means called quasi-arithmetic, Kolmogorov, or generalized *f*-mean (Kolmogorov 1930; 1991). Its mathematical form is (Anonimus 2017; Aczel and Dhombres 1989):

$$d_x = f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(d_i) \right). \tag{2}$$

To use Eq. (2) one needs to know the $f(d)$ and f^{-1} terms. The $f(d)$ term is specific for a given mean and can be taken from Table 2, while f^{-1} is much more complex because f must be injective in order for the inverse function f^{-1} to exist and is defined over an interval, $(f(d_1) + f(d_2))/2$, which lies within the domain of f^{-1}. Function f is a strictly monotonic, thus the f-mean is neither larger than the largest number of the tuple nor smaller than the smallest number in d (Wikipedia 2017).

Table 2. Values of $f(d)$ needed for calculation of specific Pythagorean means based on Eq. (2)

Mean	$f(d)$
Arithmetic	d
Harmonic	$1/d$
Power	d^x
Geometric	$\ln d$

3. SAUTER-TYPE FAMILY OF MEANS

Sauter mean, also called volumetric-surface mean, in commonly used in chemical engineering and mineral processing, especially for calculation the mean diameter for poly-dispersed spherical objects (Sauter 1928). The Sauter means calculations turn a poly-dispersed spherical objects system into equivalent with different number of objects but with identical, as original system, objects volume and surface (Kowalczuk and Drzymala 2017). There are many Sauter-based means for which the general formula (Pacek et al. 1998) is:

$$d_{xy} = \left(\frac{\sum_{i=1}^{n} d_i^x}{\sum_{i=1}^{n} d_i^y} \right)^{1/(x-y)} . \tag{3}$$

This formula, which is in fact consists of mean of two Pythagorean means,

$$d_x = \left(\frac{1}{n} \sum_{i=1}^{n} d_i^x \right)^{\frac{1}{x}} \quad \text{and} \quad d_y = \left(\frac{1}{n} \sum_{i=1}^{n} d_i^y \right)^{\frac{1}{y}} , \tag{4}$$

needs adjusting the power term to $1/(x-y)$, to have the unit of the final value the same as the original two means. For either x or y equal to 0 the term $\sum_{i=1}^{n}$ in Eq. (3) is equal to n. Equation (3) is a general formula for calculation of the Sauter-type means d_{xy} (Pacek et al. 1998), while x and y are constants which are characteristic for a given type of mean. Similarly to Eq. (1), d_i stands for a given value. The number of values is n and they are numbered i = 1, 2, 3, ..., n. Equation (3) is valid for as a set of numbers treated in calculations individually (no grouping of numbers or no weighted averaging).

The original Sauter mean is obtained when x in Eq. (3) is replaced with 3 and y with 2:

$$d_{32} = \left(\frac{\sum_{i=1}^{n} d_i^3}{\sum_{i=1}^{n} d_i^2} \right) \tag{5}$$

and is called d_{32}.

4. EXTENDED PYTHAGOREAN FAMILY OF MEANS

It appears that the Sauter-type equation (Eq. (3)) also covers the Pythagorean means, because by changing the x and y values, both Pythagorean and Sauter-type means be generated. Only geometric means is slightly different, because either it requires replacement of d_i with ln d_i, and consequently d_{xy} with ln d_{xy} or for strictly geometric mean $x \to 0$ while in practice $x = 0.01$ provides adequate accuracy of geometric mean determination. In the case of calculating geometric mean using logarithm, the mean is also expressed in a logarithmic form. It should be noticed that in this case not only decimal or natural, but any other logarithms can be used.

The formula given in Eq. (3), which can be called Extended Pythagorean Means (EPM) equation, provides variety of means which type depends on the chosen pair of x and y values. The x and y values provide a base for classification of means. To keep the classification logical, the names of some well-known means were changed, so they are slightly different from those used in literature. For instance arithmetic mean in our system is a member of the power means family with power of one, while with the higher power means are called quadratic, cubic, quartic, and so on. The harmonic mean usually has the power equal to minus one, while in our system there are different harmonic means. Therefore, in analogy to the power family means, the whole harmonic family will be called the inverse power mean. For power of one (–1), it will be called harmonic, while for higher powers, inverse quadratic, inverse volumetric, and so on. The specific means, resulting from the general formula given in Eq. (3) for the Extended Pythagorean Means, are presented in Table 3.

Table 3. Classification of Extended Pythagorean Means based on x and y values used for generation of specific means from the universal formulas given in this Table and Eq. (3)

Extended Pythagorean Means

$$d_{xy} = \left(\frac{\sum_{i=1}^{n} d_i^x}{\sum_{i=1}^{n} d_i^y} \right)^{1/(x-y)}, \quad d_{xy} = \left(\frac{\sum_{i=1}^{j} N_i d_i^x}{\sum_{i=1}^{j} N_i d_i^y} \right)^{1/(x-y)}, \quad d_{xy} = \left(\frac{\sum_{i=1}^{j} g_i d_i^x}{\sum_{i=1}^{j} g_i d_i^y} \right)^{1/(x-y)}$$

(when either x or y equal to 0 then $\sum_{i=1}^{n} = n$, $\sum_{i=1}^{j} N_i = n$, $\sum_{i=1}^{j} g_i = 1$)

Mean d_{xy} name	d_i	x	y	Mean symbol (d_{xy})*	Mean value (example)
Power	d_i				
– arithmetic		1	0	d_{10}	5.00
		0	1	d_{01}	5.00
– quadratic		2	0	d_{20} (or d_{02})	5.12
– cubic		3	0	d_{30} (or d_{03})	5.22
– ------		x	0	d_{x0} (or d_{0y})	
– geometric		$x \to 0$	0	$d_{x \to 0,0}$	4.88**
or	$\log d_i$, $\log d_{xy}$	1	0	$d_{\log, 10}$	4.88
– other					
Inverted power	d_i				
– harmonic		–1	0	d_{-10}	4.75
		0	–1	d_{0-1}	4.75
– inverted quadratic		–2	0	d_{-20} (or d_{0-2})	4.62
– inverted cubic		–3	0	d_{-30} (or d_{0-3})	4.48
– other		$-x$	0	d_{-x0} (or d_{0-y})	
Sauter-type	d_i				
– Sauter (volume-surface)		3	2	d_{32}	5.45
		2	3	d_{23}	5.45
– surface-length		2	1	d_{21}	5.23
– Sokolowski (1995, 1996)		0	–0.5	$d_{0-0.5}$(or $d_{-0.50}$)	4.82
– other				d_{xy}	

* for strictly geometric mean $x \to 0$, in practice $x = 0.01$ provides adequate accuracy,
** d_{xy} matrix is symmetric. Each mean d_{xy} has its equivalent: for instance $d_{10} = d_{01}$.

5. GROUPED AND WEIGHTED MEANS

Very often, the same or very similar values of a measured parameter is obtained many times. Then, instead of calculating the mean by adding the values one by one, starting with i = 1 and ending with n, and calculating the mean using equation:

$$d_{xy} = \left(\sum_{i=1}^{n} d_i^x \Big/ \sum_{i=1}^{n} d_i^y \right)^{\frac{1}{x-y}}, \text{ when either } x \text{ or } y \text{ is equal to 0 then } \sum_{i=1}^{n} = n \quad (6)$$

(for the geometric mean d_i can be replaced with ln d_i and obtained d_{xy} is in the log d_{xy} form) the same or similar values can be grouped and the mean calculated with the equation:

$$d_{xy} = \left(\sum_{i=1}^{j} N_i d_i^x \Big/ \sum_{i=1}^{j} N_i d_i^y \right)^{\frac{1}{x-y}} \left(\text{when either x or y is equal to 0 then } \sum_{i=1}^{j} N_i = n \right) \quad (7)$$

Thus, the mean is not calculated for individual values i = 1, 2, 3, ..., n but for groups j = 1, 2, 3, ... Now the number of groups j is smaller than the number of values n. The group can be also called fraction or class.

There is a third way of calculating the mean called weighted mean. Having groups of values we can make use of information how heavily each group contribute to the mean (contribution to the whole considered system). Thus, instead using i = 1, 2, 3, ..., n as in Eq. (3), now the weight (contribution) of each fraction g_i, based on mass, volume, surface, number, etc., is utilized, keeping in mind that the total weight of all fractions is equal to 1 ($\Sigma g_i = 1$). The number of fraction is j and i = 1, 2, 3, ..., j. For this case

$$d_{xy} = \left(\sum_{i=1}^{j} g_i d_i^x \Big/ \sum_{i=1}^{j} g_i d_i^y \right)^{\frac{1}{x-y}} \text{ when either } x \text{ or } y \text{ is equal to 0 then} \sum_{i=1}^{j} g_i = 1. \quad (8)$$

Calculation of a mean by taking into account individual objects (Eqs. (3) or (6)), objects groups into fraction (Eq. (7)), or weighing approach, requires different mathematical forms of the mean formula while the results of calculations are identical.

6. TYPICAL CALCULATIONS

To make calculation of the means using different formulas familiar, a set of values is consider as an example. Let us assume that there were measurements of size of 12 particles (n = 12) (Table 4).

Table 4. Data for calculation of the mean size of a set of particles using Eq. (6) valid for individual particles

$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$	$i = 10$	$i = 11$	$i = 12$
4 mm	5 mm	6 mm	5 mm	7 mm	4 mm	3 mm	5 mm	6 mm	5 mm	6 mm	4 mm

As shown in Fig. 1, the distribution of particles size was normal (Gaussian). The mean size, calculated using formula $d_{xy} = \left(\sum_{i=1}^{n} d_i^x \bigg/ \sum_{i=1}^{n} d_i^x \right)^{\frac{1}{x-y}}$, for $y = 0$ and $x = 1$, that is by means of the arithmetic mean, d_{10} was equal to 5. The values of the other means are given in Table 3.

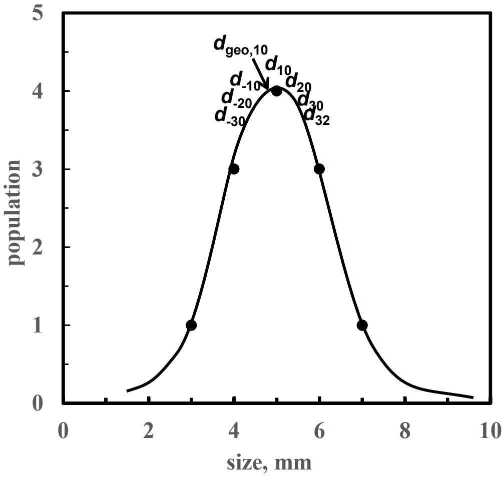

Fig. 1. Distribution of measured sizes and calculated means based on the Extended Pythagorean Means formula (Eqs. (6)–(8))

Since the particle size of 5 mm occurs 4 times, particles of size 4 and 6 mm 3 times each, the particles can be grouped into 5 classes (fractions) ($j = 5$) (Table 5) and the formula $d_{xy} = \left(\sum_{i=1}^{j} N_i d_i^x \bigg/ \sum_{i=1}^{j} N_i d_i^y \right)^{\frac{1}{x-y}}$ (Eq. (7)) can be used for calculations.

Table 5. Data for calculation of the mean size of the considered set of sized using Eq. (7) for grouped particles

$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
3 mm, $N = 1$	4 mm, $N = 3$	5 mm, $N = 4$	6 mm, $N = 3$	7 mm, $N = 1$

The arithmetic mean d_{10} for the grouped particles for the considered here example is the same as the arithmetic mean calculated for individual particles (Eq. (7)) and is equal to 5 mm.

The same arithmetic mean d_{10} = 5 mm is obtained by using formula $d_{xy} = \left(\sum_{i=1}^{j} g_i d_i^x \Big/ \sum_{i=1}^{j} g_i d_i^y \right)^{\frac{1}{x-y}}$ (Eq. (8), Table 6). The weights of each size fraction was calculated with the formula $g_i = N/n$, which for example for the 1st size fraction is equal to $g_1 = 1/12 = 0.083(3)$.

Table 6. Data for calculation of the mean size of the considered set of sizes using Eq. (8) valid for the weighted mean

$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
3 mm, $g = 0.083(3)$	4 mm, $g = 0.250$	5 mm, $g = 0.33(3)$	6 mm, $g = 0.250$	7 mm, $g = 0.083(3)$

Other means for the considered set of data are given in Table 3 and in a graphical form in Fig. 1.

7. CONCLUSIONS

A general equation for the Pythagorean means can be combined with the Sauter-type mean formula providing more general and universal one which can be used for generation and classification of many useful means, inclusion geometric, harmonic, arithmetic, Sauter, etc. The calculated means, depending on their type, are different. The same means are obtained for a given type of formula and a set of values when treated individually, or in groups as well as fractions with appropriate weight for weighted mean calculations.

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