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DROPPING LIQUID TIME-DEPENDENT FLOW IN A PIPE OF LENGTHWISE LINEARLY VARYING DIAMETER

Introduction

The objective of this paper is to discuss briefly unsteady flow of incompressible liquid in a rigid pipe the diameter of which is changed lengthwise by linear law. Within these conditions a differential equation was set up. To integrate the former boundary conditions were defined for speedup period of liquid flow. Instantaneous averaged velocity values for a number of cross-sections were plotted, stabilization rate and period were determined.

The problem is particularly difficult when the flow is slow and walls of the pipe and the liquid are deformable. A set of equations characterizing interaction of pressure and rate change have been established for these conditions. A slowing non-steady flow in a narrowing conical pipe was studied. Fluid medium and walls of the pipe are considered as elastic materials. The hydraulic shock phenomena in case of instantaneous closure of a valve mounted at the end of the conical pipe connected to the reservoir has been examined. Initial and boundary conditions for the problem solution have been set up. In accordance with these conditions obtained differential equations have been integrated using the method of separable variables. Solutions of equations were sought for in the form of infinite series wherein values of participating factors have been obtained from respective boundary conditions.

This solution enables to obtain the measure of pressure rise at the end of a conical pipe in case the valve is suddenly closed. The suggested technique enables to solve similar in nature problems in case of different boundary conditions.

The article studies unsteady flow of incompressible liquid in a rigid pipe of a lengthwise linearly varying diameter. Boundary conditions in acceleration starting time were defined to integrate a differential equation of flow derived for such conditions.

Instantaneous, sectional, and averaged velocity diagrams have been plotted, stabilization velocity and time determined. The problem is particularly a very complicated one for slow flow when the pipe wall and the fluid are deformable [1-5].

For these conditions a set of equations describing relation between pressure and velocity change was derived. To this end a section dx is separated on the pipe and a differential equation of its motion is formed (Fig. l).

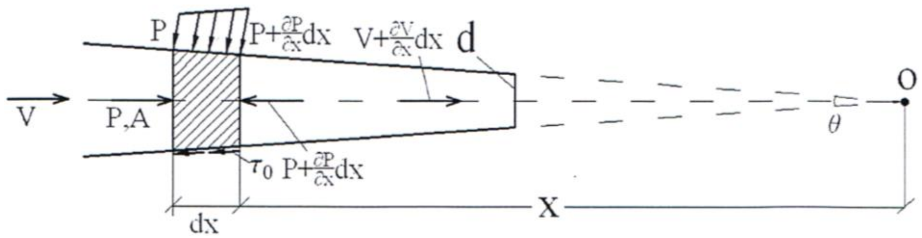

Fig. 1. Separated section of the pipe

$$\Delta M \frac{dV}{dt} = \Delta F + \Delta T \quad (1)$$

where ΔM is the mass of the separated section

$$\Delta M = \pi\, r_x^2 dx\rho \quad (2)$$

ΔF is a pressure force along flow

$$\Delta F = PA_x - \left(PA_x + \frac{\partial (PA_x)}{\partial x} dx \right) + \left(P + \frac{1}{2}\frac{\partial P}{\partial x} dx \right) \cdot 2\pi r_x\, dx \cdot \sin\frac{\theta}{2} \quad (3)$$

ΔT is a friction force in reverse direction

$$\Delta T = -2 r_x dx \cdot \tau_0 \cos\frac{\theta}{2} \quad (4)$$

τ_0 is a friction stress developed at the fixed pipe wall.

Substituting ΔM, ΔF, and ΔT in Eq. (1) we have

$$\rho \frac{dV}{dt} = -\frac{\partial P}{\partial x} + \frac{2\tau_0}{r_x} \cos\frac{\theta}{2} \quad (5)$$

The second equation is obtained from one-dimensional flow continuity condition [3, 5]

$$\frac{\partial (\rho A V)}{\partial x} + \frac{\partial (\rho A)}{\partial t} = 0 \quad (6)$$

After opening parentheses, we get

$$\rho A \frac{\partial V}{\partial x} + \rho V \frac{\partial A}{\partial x} + AV \frac{\partial \rho}{\partial x} + A\frac{\partial \rho}{\partial t} + \rho \frac{\partial A}{\partial t} = 0 \quad (7)$$

Now members of the above equation are calculated

a) $\frac{\partial \rho}{\partial x} = 0$, since the fluid density along the path remains constant

b) $\frac{\partial \rho}{\partial x} = \frac{\partial (\pi r^2)}{\partial x} = 2\pi r \frac{\partial r}{\partial x} = 2\frac{A}{r} tg\frac{\theta}{2} = \frac{2A}{x}$ (8)

c) $A\frac{\partial \rho}{\partial t} = A\frac{\rho}{K}\frac{\partial P}{\partial t}$ - according to the fluid elasticity (9)

d) $\rho\frac{\partial A}{\partial t} = \rho\frac{\partial (\pi r)}{\partial t} = 2\rho\pi r\frac{\partial r}{\partial t}$.

According to Hook's law $\frac{\partial r}{\partial t} = \frac{r}{E}\frac{\partial \sigma}{\partial t}$. Since $\sigma = P\frac{2r}{2\delta}$, therefore

$$\frac{\partial \sigma}{\partial t} = \frac{\partial P}{\partial t}\frac{r}{\delta} \text{ and } \frac{\partial r}{\partial t} = \frac{r\cdot r}{\delta\cdot E}\frac{\partial P}{\partial t},$$

$$\rho\frac{\partial A}{\partial t} = 2\rho\pi r\frac{r\cdot r}{\delta\cdot E}\frac{\partial P}{\partial t} = 2\rho A\frac{r}{\delta E}\frac{\partial P}{\partial t} \tag{10}$$

Substituting Eq. (8), (9), and (10) in Eq. (7), we have

$$\frac{\partial V}{\partial x} + 2V\frac{tg\frac{\theta}{2}}{r} + \frac{1}{\rho a^2}\frac{\partial P}{\partial t} = 0 \tag{11}$$

Where $a = \sqrt{\frac{\frac{K}{\upsilon}}{1+\frac{Kr}{\delta E}}}$ is shock wave propagation velocity at the section of r radius.

Since the pipe's diameter along its length is variable, then the shock wave propagation velocity also is variable [1, 2].

Eqs. (5) and (11) for liquid medium and the pipe's wall material elasticity conditions form a set of real fluid unsteady flow equations in a pipe of a lengthwise linearly diameter.

$$\begin{cases} \rho\frac{\partial V}{\partial t} = -\frac{\partial P}{\partial x} + \frac{2\tau_0}{r}\cdot\cos\frac{\theta}{2} \\ \frac{\partial V}{\partial x} + 2V\frac{tg\frac{\theta}{2}}{r} + \frac{1}{\rho a^2}\frac{\partial P}{\partial t} \end{cases} \tag{12}$$

When we deal with an ideal liquid, then between layers of the liquid and fixed walls of the pipe no friction stress is developed and as a result Eq. (12) takes more simple expression

$$\begin{cases} \rho \dfrac{\partial V}{\partial t} = -\dfrac{\partial P}{\partial x} \\ \dfrac{\partial V}{\partial x} + \dfrac{2V}{x} + \dfrac{1}{\rho a^2}\dfrac{\partial P}{\partial t} = 0 \end{cases} \tag{13}$$

where $r = x \cdot tg\dfrac{\theta}{2}$.

To determine hydrodynamic parameters $V(x, t)$, $P(x, t)$ of the cross section in case of unsteady flow, initial and boundary conditions are added to the set of equations (13). Let us now study the shock phenomena in a lengthwise narrowing conical pipe of length l at the initial point of which the pressure (that is in the basin) is constant and equal to P_0, and in $t = 0$ moment the valve installed at the end point is instantaneously closed. In such conditions the initial and boundary conditions of the set equations (13) are

$$V(l, t) = 0, \quad P(0, t) = \mathrm{P}_0 \quad (0 \le x \le l) \tag{14}$$

$$V(x,0) = V_0 \frac{r_0^2}{r^2}, \quad P(x, 0) = 0 \quad (0 \le t \le \infty) \tag{15}$$

Eliminating $P(x, t)$ from Eq. (13), we get [6]

$$\frac{1}{a^2}V_u = V_{xx} + \frac{2}{x}V_x - \frac{2}{x^2}V \tag{16}$$

Conditions (14) and (15) are transformed to the following conditions:

$$V(l, t) = 0, \quad V(0, t) = 0 \quad (0 \le x \le l) \tag{17}$$

$$V(x,0) = V_0 \frac{r_0^2}{r^2}, \quad \mathrm{V_t(x, 0) = 0} \quad (0 \le \mathrm{t} \le \infty) \tag{18}$$

Solution of Eq. (18) is sought as $V(x, t) = T(t)\,X(x)$. In this case instead of Eq. (16) we will have two independent equations relative to x and t variables.

$$T_{tt} + \lambda 2 a 2 T = 0 \tag{19}$$

$$X_{xx} + \frac{2}{x}X_x + (\lambda^2 - \frac{2}{x^2})X = 0 \tag{20}$$

Where λ are sought for arbitrary constant which subject to be determined.

Solutions of Eq. (20) are linearly independent from each functions $x^{-\frac{1}{2}}J_{\frac{3}{2}}(\lambda.x)$ and $x^{-\frac{1}{2}}N_{\frac{3}{2}}(\lambda.x)$. $N_{\frac{3}{2}}(\lambda.x)$ function have a singularity at $x = 0$ point. Solution of the following equation is their linear combination.

$$X(x) = \sum_{k=0}^{\infty} x^{-\frac{1}{2}}[A_k J_{\frac{3}{2}}(\lambda_k x) + B_k N_{\frac{3}{2}}(\lambda_k x)] \tag{21}$$

Where J and N are Bessel and Neumann functions, respectively [6].

$V(x, t)$ at $x = 0$ point is finite, therefore $B_k = 0$ and we have

$$X(x) = \sum_{k=0}^{\infty} x^{-\frac{1}{2}} A_k J_{\frac{3}{2}}(\lambda_k x) \tag{22}$$

From the boundary conditions we obtain $X(0) = 0$ and $X(l) = 0$. The first of these conditions always satisfies, as for the second one we get $X(l) = \sum_{k=0}^{\infty} l^{-\frac{1}{2}} A_k J_{\frac{3}{2}}(\lambda_k x) = 0$, from which it follows that $J_{\frac{3}{2}}(\lambda_k l) = 0$. This equation has infinite number roots of $\lambda_k = \frac{\gamma_k}{l}$, where γ_k are real roots of $J_{\frac{3}{2}}(\gamma_k) = 0$ equation. Eigenfunctions of the problem are $X_k(x) = x^{-\frac{1}{2}} J_{\frac{3}{2}}(\frac{\gamma_k}{l} x)$. It is known that $J_{\frac{3}{2}}(\lambda_k x) = 0$ functions are orthogonal of $\rho(x) = x$ weight, that is

$$\int_0^l xJ(\lambda_k x)J(\lambda_n x)dx = \begin{cases} 0, \lambda_k \neq \lambda_n \\ \left\| J_{\frac{3}{2}}(\lambda_n x) \right\|^2, \lambda_k = \lambda_n \end{cases} \tag{23}$$

Calculating the value of $\left\| J_{\frac{3}{2}}(\lambda_n x) \right\|^2$ expression, we get

$$\left\| J_{\frac{3}{2}}(\frac{\gamma_n}{l} x) \right\|^2 = \int_0^l xJ_{\frac{3}{2}}(\frac{\gamma_n x}{l})dx = \frac{l^2}{2}\{[J'_{\frac{3}{2}}(\gamma_n)]^2 + (1 - \frac{9}{4\gamma_n^2})J_{\frac{3}{2}}^2(\gamma_n)\} = \frac{l^2}{2}[J'_{\frac{3}{2}}(\gamma_n)]^2$$

as far as $J_{\frac{3}{2}}(\gamma_k) = 0$.

Solving of Eq. (19) in the form of infinite, we have [6]

$$T(t)=\sum_{k=0}^{\infty}(A_k Cosa\lambda_k t + B_k Sina\lambda_k t) \tag{24}$$

Using Eq. (22) and (24) the general solution of the problem is presented

$$V(x,t)=\sum_{k=0}^{\infty}(A_k Cosa\lambda_k t + B_k Sina\lambda_k t)x^{-\frac{1}{2}}J_{\frac{3}{2}}(\lambda_k x) \tag{25}$$

From the initial condition (18) it follows that $B_k = 0$ and A_k is obtained from the below equation

$$V_0\frac{r_0^2}{r^2}=\sum_{k=0}^{\infty}A_k x^{-\frac{1}{2}}J_{\frac{3}{2}}(\lambda_k x), \qquad 0<x<l \tag{26}$$

In order to determine A_k coefficients two parts of Eq. (26) are multiplied by $x^{\frac{3}{2}}J_{\frac{3}{2}}(\lambda_n x)$ function and integrated from 0 to l interval

$$V_0\frac{r_0^2}{tg^2\frac{\theta}{2}}\int_0^l x^{-\frac{1}{2}}J_{\frac{3}{2}}(\lambda_n x)dx=\sum_{k=0}^{\infty}A_n\int_0^l xJ_{\frac{3}{2}}(\lambda_k x)J_{\frac{3}{2}}(\lambda_n x)dx$$

Using the condition of orthogonality (23) of functions J, we have

$$V_0\frac{r_0^2}{tg^2\frac{\theta}{2}}\int_0^l x^{-\frac{1}{2}}J_{\frac{3}{2}}(\lambda_n x)dx=A_n\left\|J_{\frac{3}{2}}\left(\frac{\gamma_n}{l}x\right)\right\|^2$$

from which

$$A_n=\frac{V_0 r_0^2}{tg^2\frac{\theta}{2}\left\|J_{\frac{3}{2}}(\lambda_n x)\right\|^2}\int_0^l x^{-\frac{1}{2}}J_{\frac{3}{2}}(\lambda_n x)dx \tag{27}$$

Substituting values of B_k and A_k factors in Eq. (25) the value of $V(x, t)$ function is obtained

$$V(x,t)=\sum_{k=0}^{\infty}A_k x^{-\frac{1}{2}}J_{\frac{3}{2}}(\lambda_k x)Cosa\lambda_k t \tag{28}$$

To determine $P(x, t)$ the first equation of the set (13) is integrated from 0 to x

$$P(x,t) = P_0 + a\rho\sum_{k=0}^{\infty}\lambda_k A_k Sina\lambda_k t\int_0^x \xi^{-\frac{1}{2}} J_{\frac{3}{2}}(\lambda_k \xi)d\xi =$$

$$= P_0 + a\rho\sum_{k=0}^{\infty}\lambda_k A_k Sina\lambda_k t[\frac{2}{\sqrt{2\pi\lambda_k}} - \frac{J_{\frac{1}{2}}(\lambda_k x)}{\lambda_k \sqrt{x}}]$$

Taking into consideration that $\lambda_k = \frac{\gamma_k}{l}$, from the last equation we obtain the value of $P(x, t)$ function at $x = l$ section.

$$P(l,t) = P_0 + a\rho\sum_{k=0}^{\infty}\frac{A_k}{\sqrt{l}} Sina\frac{\gamma_k}{l}t[\sqrt{\frac{2\gamma_k l}{\pi}} - J_{\frac{1}{2}}(\gamma_k)] \quad (29)$$

Conclusion

The obtained set of equations of unsteady flow running in a pipe of lengthwise linearly varying diameter enables to obtain changing values of pressure and velocity, formed, in the framework of the given problem, for the initial and boundary conditions. Particularly, in case of instantaneous shutting of the valve installed at the terminal point of the pipe, initial and boundary conditions have been set up. For these conditions a set of the unsteady flow differential equations were integrated whereby regularities of the pressure and velocity variation along the entire length of the pipe, in particular in the last section, have been found.

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Abstract

The objective of this paper is to discuss briefly unsteady flow of incompressible liquid in a rigid pipe the diameter of which is changed lengthwise by linear law. Within these conditions a differential equation was set up. To integrate the former boundary conditions were defined for speedup period of liquid flow. Instantaneous averaged velocity values for a number of cross-sections have been plotted, stabilization rate and period have been determined. The problem is particularly difficult when the flow is slow and walls of the pipe and the liquid are deformable. A set of equations characterizing interaction of pressure and rate change have been established for these conditions. A slowing non--steady flow in a narrowing conical pipe was studied. Fluid medium and walls of the pipe are considered as elastic materials. The hydraulic shock phenomena in case of instantaneous closure of a valve mounted at the end of the conical pipe connected to the reservoir has been examined. Initial and boundary conditions for the problem solution have been set up. In accordance with these conditions obtained differential equations have been integrated using the method of separable variables. Solutions of equations were sought for in the form of infinite series wherein values of participating factors have been obtained from respective boundary conditions. This solution enables to obtain the measure of pressure rise at the end of a conical pipe in case the valve is suddenly closed. The suggested technique enables to solve similar in nature problems in case of different boundary conditions.

Przepływ spadającej cieczy zależny od czasu w rurze o podłużnie liniowo zmiennej średnicy

Streszczenie

Celem niniejszego artykułu jest zwięzłe omówienie nierównomiernego przepływu nieściśliwej cieczy w sztywnej rurze, której średnica jest podłużnie, liniowo zmienna. Przy tak określonych warunkach zostało wyprowadzone równanie różniczkowe. W celu zintegrowania dotychczasowych warunków brzegowych zostało określone przyspieszenie okresu przepływu cieczy. Uśrednione wartości prędkości chwilowych dla wielu przekrojów zostały przedstawione na wykresach. Ponadto określono współczynnik i okres stabilizacji. Problem ten jest szczególnie trudny, gdy przepływ jest powolny i ścianki rury są odkształcalne. Został określony układ równań charakteryzujących oddziaływanie ciśnienia i zmiany współczynników dla przyjętych warunków. Zbadano spowolnienie nierównomiernego przepływu w rurze zwężonej stożkowo. Nośnik płynu i ściany rury rozpatrywano jako materiały sprężyste. Przeanalizowano zjawisko uderzenia hydraulicznego w przypadku chwilowego zamknięcia zaworu, zamontowanego na końcu rury stożkowej, połączonej ze zbiornikiem. Zostały ustalone początkowe

i brzegowe warunki rozwiązania problemu. Zgodnie z tymi warunkami otrzymane równania różniczkowe połączono z wykorzystaniem metody zmiennych rozłącznych. Rozwiązania równań były poszukiwane w postaci nieskończonej serii, w której wartości czynników uczestniczących zostały uzyskane z odpowiednich warunków brzegowych. To rozwiązanie pozwala na uzyskanie wartości wzrostu ciśnienia na końcu rury stożkowej w przypadku nagłego zamknięcia zaworu. Sugerowana technika pozwala rozwiązać podobne problemy w przypadku różnych warunków brzegowych.