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A NOVEL HYBRID ITERATIVE METHOD FOR APPLIED MATHEMATICAL MODELS WITH TIME-EFFICIENCY

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Abstract. Non-linear phenomena appear in many fields of engineering and science. Research on numerical methods is continually extending with the improvement of the latest computing tools. In today's computational field, one requires maximum achievement in a minimum amount of time. Therefore, there is a need to modify the Newton-type method to achieve higher-order convergence to solve non-linear equations. While the modified methods are expected to be higher-order convergent, the minor computational information and the maximum time efficiency are also important factors. We propose a three-step hybrid iterative method having a non-linear nature. Per iteration, the method requires three function evaluations and three first-order derivatives. The method is theoretically proven to be tenth-order convergent. The mathematical results of the proposed strategy to solve models from fluid dynamics, electric field, and real gases demonstrated better performance. In light of error analysis, computational productivity, and CPU times, the proposed method's performance is compared to the famous Newton and a recently proposed tenth-order method.

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1. Introduction

The modeling of systems in various disciplines is performed in the form of non-linear equations of the type $f(x) = 0$. It is not always possible to obtain the exact solutions of such equations, and therefore primarily numerical methods are used. It is essential to be higher-order accurate and cost-efficient for a numerical method by using fewer evaluations and iterations. In simple words, the method must be efficient from the computational viewpoint.

Investigating effective methods for solving nonlinear problems is one of the most pressing issues in engineering and science. We often solve nonlinear equations by using the existing mathematical techniques such as the Newton Raphson method (NRM), which is one of the most effective methods with quadratic convergence and having two function evaluations per iteration [1]. However, the method has a pitfall related to a failure of the first-order derivative of $f(x) = 0$ at either initial guess or any other approximate value of the required solution during the iteration process. Because of its quadratic convergence, the Newton approach has received much attention among the nonlinear solvers [2]. As it is quoted by Traub and Kung [3], an ideal iterative solver should have its order equal to its number of function evaluations. In this regard, NRM is considered to be an optimal iterative process. While the objective of the ongoing investigation is to develop a method alternative to the Newton technique, we came to know that the Traub and Kung [3] conjecture is only a guess of numerous research analyses, which is negligible in progress for the development of a combination of numerical solvers [4]. The idea of efficiency index ($\zeta = p^{1/\omega}$) is likewise significant in rating the numerical techniques for the afore-mentioned purpose. The efficiency index considers the function evaluations per iteration (ω) and order of convergence (p) of a technique. Thus, order of convergence is essential for comparing two numerical solvers, but other factors, including efficiency index, number of function evaluations, asymptotic error constant, rate of convergence, speed, and stability, are also essential. The NRM through Taylor's expansion in [1], the Homotopy Perturbation technique [5], the variational iterative technique [6], the Adomian decomposition and the quadrature rules [7] are some of the strategies that are used to find approximate solutions of nonlinear models emerging from several research studies.

In [8], the development of an iterative technique was sped up by joining two distinct methods with orders of q_1 and q_2, respectively, to acquire a procedure with order $q_1 q_2$. While this methodology employs additional cost per iteration, the resulting strategy is always guaranteed to be faster than some lower-order techniques. Inspired by this methodology, we will put forward a tenth-order convergent method by merging the second-order Newton's technique (NRM) with an efficient Newton--type method of fifth-order convergence found in [8]. Compared to some available methodologies, the proposed procedure implies better performance. For example, Jaiswal and Choubey [9] presented a three-step method of eighth-order with five function evaluations. Similarly, Liu et al. in [10] developed a three-step iterative scheme of eighth-order of convergence with five function evaluations, and Cordero et al. in [11] also showed a four-step technique of order eight with five function evaluations.

Furthermore, it may also be noted that the proposed three-step hybrid method is equally applicable for solving systems of nonlinear equations, as has been discussed in several recently published of the type [12–14]. The solution of the systems requires an initial guess vector for the method to be useful since the method herein is a local method. However, the method's convergence is also guaranteed for the systems as it is in the above-cited papers.

This present research work aims to propose an efficient three-step hybrid method for dealing with nonlinear equations. The proposed strategy begins with Newton's step, used by various scholars [12–15]. The proposed three-step technique is formulated to achieve tenth-order convergence with only six function evaluations per iteration. After discretization of some differential equations [16–23], one needs to solve the resultant equations with numerical techniques such as the one proposed herein. Other researchers may continue working in this interesting field by their contributions regarding enhancement of the convergence order and the reduction of the computational effort required for the simulation of several physical models given in terms of nonlinear equations.

The fundamental component of the proposed technique is that it requires less CPU time than other commonly used techniques having similar order of convergence. As a result, the proposed technique is expected to outperform, match with other techniques, and sometimes beat other techniques. The convergence analysis of the proposed method for single variable models has also been carried out, and the asymptotic error constant is determined.

2. Materials and methods

Single-variable nonlinear equations can be commonly represented as $f(x) = 0$, where x is the required solution and $f(x)$ may appear in the polynomial or transcendental structure on the left-hand side of the equation. Most of the time, it is impossible to solve a nonlinear equation for x directly. In such situations, numerical techniques come to our rescue by generating the convergent solution of nonlinear equations. In this section, we will specifically discuss some currently existing techniques. Let us consider the famous Newton technique that can be found in several studies, for example, in [2]. The Newton Raphson Method (NRM) of quadratic convergence while using two function evaluations $f(x_n)$ and $f'(x_n)$ is given below:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, ..., \tag{1}$$

where initial approximation x_0 has been used for getting solution x_1, while the successive approximations will be started with an underlying guess at $x = x_0$. In [1], Abro & Shaikh introduced a three-step iterative technique (P6) of sixth-order convergence with five function evaluations. The technique is shown below:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, z_n = y_n - \frac{f(y_n)}{f'(y_n)}, x_{n+1} = y_n - \frac{f(y_n) + f(z_n)}{f'(y_n)}. \tag{2}$$

In [12], Waseem et al. proposed a new four-step fifth-order iterative method (WM). This method is derivative based method and denoted by WM as shown below:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, z_n = y_n - \frac{f(y_n)}{f'(x_n)}, w_n = z_n - \frac{f(z_n)}{f'(x_n)}, x_{n+1} = w_n - \frac{f(w_n)}{f'(x_n)}. \tag{3}$$

In 2007, Noor and Noor, in [15], suggested a two-step iterative method (AD) with fifth-order convergence, where the first-step is taken to be the well-known Halley method with third order convergence. Their method is described below:

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, x_{n+1} = x_n - \frac{2[f(x_n)+g(y_n)]f'(x_n)}{2f'^2(x_n) - [f(x_n)+g(y_n)]f''(x_n)}. \tag{4}$$

In [4] Shah et al., in 2016, preferred a pair of sixth-order convergence, denoted by SA1 and SA2. These two techniques utilize three function evaluations and two first-order derivatives per iteration. The following conditions depict the calculations for SA1 and SA2:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, z_n = y_n - \frac{f(y_n)}{f'(y_n)}, x_{n+1} = z_n - \frac{f(z_n)}{f'(y_n) - f(y_n)}. \tag{5}$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, z_n = y_n - \frac{f(y_n)}{f'(y_n)}, x_{n+1} = z_n - \frac{f(z_n)f(y_n)}{f'(y_n)f(x_n) - 2f(z_n)}. \tag{6}$$

In [9] Jaiswal and Choubey, in 2013, proposed a new three-step iterative scheme (NEO) for solving non-linear equations. With five function evaluations, the scheme approaches the eighth-order convergence as given below:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, z_n = y_n - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)}\left(\frac{f(y_n)}{f'(x_n)}\right), x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{7}$$

In [10], the authors discussed an eighth-order convergent technique (O81) for solving a nonlinear models. The method having first derivative in each step, as illustrated below:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, z_n = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)}\left(\frac{f(y_n)}{f'(x_n)}\right), \\ x_{n+1} &= z_n - \left[\left(\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}\right)^2 + \frac{f(z_n)}{f(y_n) - 5f(z_n)} + \frac{4f(z_n)}{f(x_n) - 7f(z_n)}\right]\left(\frac{f(z_n)}{f'(x_n)}\right). \end{aligned} \tag{8}$$

In [11] Cordero et al., in 2021, proposed a method that consists of a four-step iterative method (O82) with the eighth-order of convergence. The efficiency index of this scheme is about 1.5157, which is better than NRM and some other existing methods. The steps involved in the method are described below:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, z_n = x_n - \frac{f(x_n)}{8f'(x_n)} - \frac{3f(x_n)}{8f'(y_n)}, w_n = x_n - \frac{6f(x_n)}{f'(x_n) + f'(y_n) + 4f'(z_n)}, \\ x_{n+1} &= w_n - \frac{f'(x_n) + f'(y_n) - f'(z_n)}{2f'(y_n) - f'(z_n)}\left(\frac{f(w_n)}{f'(x_n)}\right). \end{aligned} \tag{9}$$

Recently, in [13], Tassaddiqa et al., in 2021, proposed a method with tenth-order of convergence (ASA) having three steps with six evaluations. The efficiency index of this technique is around 1.4678, which is better than several existing methods. The computational scheme of the method is as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, z_n = y_n - \frac{f(y_n)}{f'(y_n)}, x_{n+1} = z_n - \frac{f'(z_n)+3f'(y_n)}{5f'(z_n)-f'(y_n)}\left(\frac{f(z_n)}{f'(y_n)}\right). \quad (10)$$

3. Proposed hybrid iterative method

Motivated by some recent findings [1, 13], we propose a new strategy with the use of a classical Newton step while merging it with the fifth-order convergent and efficient Newton-type method [8] as shown below:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, x_{n+1} = y_n - \frac{5f'^2(x_n)+3f'^2(y_n)}{f'^2(x_n)+7f'^2(y_n)}\left(\frac{f(y_n)}{f'(x_n)}\right). \quad (11)$$

If the techniques to be mixed are not wisely selected, it causes some additional function evaluations. Nevertheless, quick convergence is guaranteed with an additional computational cost. Therefore, inspired by the methodology of [1,13], we are proposing a tenth-order strategy by merging the fifth-order Newton-type method with the traditional second-order Newton technique, but it guarantees a faster order of convergence than these two techniques themselves. The proposed hybrid three-step technique is represented below:

$$\begin{aligned} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad z_n = y_n - \frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} = z_n - \frac{5f'^2(y_n)+3f'^2(z_n)}{f'^2(y_n)+7f'^2(z_n)}\left(\frac{f(z_n)}{f'(y_n)}\right). \end{aligned} \quad (12)$$

The proposed strategy given above in (12) is denoted by the word ”Proposed” during the numerical simulations. It will be shown to be a good and efficient choice among several numerical solvers used for the said purpose.

3.1. Order of convergence

This section explains the derivation of the asymptotic error term, and consequently the order of convergence for the proposed nonlinear hybrid iterative method given in (12).

Theorem 1. Assume that $\kappa \in \mathbb{S}$ is the exact simple root of a differentiable function $f\colon \mathbb{S} \subset \mathbb{R} \to \mathbb{R}$ on an open interval so that the three-step iterative technique N10, i.e., Eq. (12), then exhibits tenth-order convergence, and the resulting error term is:

$$\varepsilon_{i+1} = -\frac{\left(\frac{d^2}{d\kappa^2} f(\kappa)\right)^7 \left(4\left(\frac{d}{d\kappa} f(\kappa)\right)\frac{d^3}{d\kappa^3} f(\kappa) - 21\left(\frac{d^2}{d\kappa^2} f(\kappa)\right)^2\right)}{3072\left(\frac{d}{d\kappa} f(\kappa)\right)^9}\varepsilon_i^{10} + \mathscr{O}(\varepsilon_i^{11}). \tag{13}$$

where $\varepsilon_i = x_i - \kappa$.

Proof. Suppose κ is the root of $f(x_i)$, where x_i is the i-th iteration nearly to the root by N10 and $\varepsilon_i = x_i - \kappa$ is the error term after i-th iteration. Utilizing Taylor's series for $f(x_i)$ about κ, we have

$$f(x_i) = \left(\frac{d}{d\kappa} f(\kappa)\right)\varepsilon_i + \frac{\left(\frac{d^2}{d\kappa^2} f(\kappa)\right)\varepsilon_i{}^2}{2} + \frac{\left(\frac{d^3}{d\kappa^3} f(\kappa)\right)\varepsilon_i{}^3}{6}. \tag{14}$$

By Taylor's series for $\dfrac{1}{f'(x_i)}$ about κ, we obtained:

$$\frac{1}{f'(x_i)} = \left(\frac{d}{d\kappa} f(\kappa)\right)^{-1} - \frac{\left(\frac{d^2}{d\kappa^2} f(\kappa)\right)\varepsilon_i}{\left(\frac{d}{d\kappa} f(\kappa)\right)^2} + \frac{\varepsilon_i{}^2}{\frac{d}{d\kappa} f(\kappa)}\left(-\frac{\frac{d^3}{d\kappa^3} f(\kappa)}{2\frac{d}{d\kappa} f(\kappa)} + \frac{\left(\frac{d^2}{d\kappa^2} f(\kappa)\right)^2}{\left(\frac{d}{d\kappa} f(\kappa)\right)^2}\right). \tag{15}$$

Multiplying (14) and (15) and putting the result in the first step of (12), we get:

$$\sigma_i = \frac{\left(2\left(\frac{d^3}{d\kappa^3} f(\kappa)\right)\frac{d}{d\kappa} f(\kappa) - 3\left(\frac{d^2}{d\kappa^2} f(\kappa)\right)^2\right)\varepsilon_i{}^3}{6\left(\frac{d}{d\kappa} f(\kappa)\right)^2} + \frac{\varepsilon_i{}^2\frac{d^2}{d\kappa^2} f(\kappa)}{2\frac{d}{d\kappa} f(\kappa)}. \tag{16}$$

By Taylor's series for $f(y_i)$ about κ, we obtained:

$$f(y_i) = \left(\frac{d}{d\kappa} f(\kappa)\right)\sigma_i + \frac{\left(\frac{d^2}{d\kappa^2} f(\kappa)\right)\sigma_i{}^2}{2} + \frac{\left(\frac{d^3}{d\kappa^3} f(\kappa)\right)\sigma_i{}^3}{6}. \tag{17}$$

By Taylor's series for $\dfrac{1}{f'(y_i)}$ about κ, we obtained:

$$\frac{1}{f'(y_i)} = \left(\frac{d}{d\kappa} f(\kappa)\right)^{-1} - \frac{\left(\frac{d^2}{d\kappa^2} f(\kappa)\right)\sigma_i}{\left(\frac{d}{d\kappa} f(\kappa)\right)^2} + \frac{\sigma_i{}^2}{\frac{d}{d\kappa} f(\kappa)}\left(-\frac{\frac{d^3}{d\kappa^3} f(\kappa)}{2\frac{d}{d\kappa} f(\kappa)} + \frac{\left(\frac{d^2}{d\kappa^2} f(\kappa)\right)^2}{\left(\frac{d}{d\kappa} f(\kappa)\right)^2}\right). \tag{18}$$

Multiplying (17) and (18) and putting the result in the second step of (12), we get:

$$\eta_i = \frac{\left(2\left(\frac{d^3}{d\kappa^3}f(\kappa)\right)\frac{d}{d\kappa}f(\kappa) - 3\left(\frac{d^2}{d\kappa^2}f(\kappa)\right)^2\right)\sigma_i{}^3}{6\left(\frac{d}{d\kappa}f(\kappa)\right)^2} + \frac{\sigma_i{}^2\frac{d^2}{d\kappa^2}f(\kappa)}{2\frac{d}{d\kappa}f(\kappa)}. \tag{19}$$

By Taylor's series for $f(z_i)$ about κ, we obtained:

$$f(z_i) = \left(\frac{d}{d\kappa}f(\kappa)\right)\eta_i + \frac{\left(\frac{d^2}{d\kappa^2}f(\kappa)\right)\eta_i{}^2}{2} + \frac{\left(\frac{d^3}{d\kappa^3}f(\kappa)\right)\eta_i{}^3}{6}. \tag{20}$$

By Taylor's series for $f'(z_i)$ about κ, we obtained:

$$f'(z_i) = \frac{d}{d\kappa}f(\kappa) + \left(\frac{d^2}{d\kappa^2}f(\kappa)\right)\eta_i + \frac{\left(\frac{d^3}{d\kappa^3}f(\kappa)\right)\eta_i{}^2}{2} + \frac{\left(\frac{d^4}{d\kappa^4}f(\kappa)\right)\eta_i{}^3}{6}. \tag{21}$$

By Taylor's series for $f'(y_i)$ about κ, we obtained:

$$f'(y_i) = \frac{d}{d\kappa}f(\kappa) + \left(\frac{d^2}{d\kappa^2}f(\kappa)\right)\sigma_i + \frac{\left(\frac{d^3}{d\kappa^3}f(\kappa)\right)\sigma_i{}^2}{2} + \frac{\left(\frac{d^4}{d\kappa^4}f(\kappa)\right)\sigma_i{}^3}{6}. \tag{22}$$

Finally, we substitute all required values in third step of (12) to get the following:

$$\varepsilon_{i+1} = -\frac{\left(\frac{d^2}{d\kappa^2}f(\kappa)\right)^7\left(4\left(\frac{d}{d\kappa}f(\kappa)\right)\frac{d^3}{d\kappa^3}f(\kappa) - 21\left(\frac{d^2}{d\kappa^2}f(\kappa)\right)^2\right)}{3072\left(\frac{d}{d\kappa}f(\kappa)\right)^9}\varepsilon_i^{10} + \mathscr{O}(\varepsilon_i^{11}). \tag{23}$$

The above equation demonstrates that the proposed hybrid iterative method is tenth-order convergent.

4. Comparative analysis

One of the most fundamental approaches to examine the strength of a numerical technique for solving $f(x) = 0$ is to compare its order of convergence p, efficiency index ζ, and the number of function evaluations per iteration ω to the existing techniques. We have examined these parameters for the proposed novel hybrid iterative method with some existing methods as shown in the following Table 1.

Table 1. Comparison of some iterative methods based on order, the efficiency index, and the function evaluations per iteration

Method	p	ζ	ω
Proposed	10	1.4678	6
Newton	2	1.4142	2
ASA	10	1.4678	6

5. Mathematical analyses

To compare the proposed tenth-order hybrid iterative method (12) with other techniques, we consider various scalar nonlinear models from existing literature. Every calculation is performed in MAPLE 2020, installed on an Intel® Core™ i5 hp PC with 8 GB of RAM and a processing speed of 2.6 GHz. Critical analysis of models showing many real-world phenomena that contain fixed parameters is necessary for higher precision because they oscillate between exceptionally high range (in 10^{70} or significantly higher) and sometimes short range (in 10^{-30} or even lower). The numerical techniques should be highly accurate when models are nonlinear. Higher-order methods are required to obtain better results in a reasonable amount of time. The same situations have also been presented by [1, 4], where higher-order and accurate methods are required to deal with several physical and natural phenomena. The use and recommendation of higher-order nonlinear methods for scalar equations are also encouraged in previous and current articles. For example, the eighth-order method in [9–11], and fifteenth-order method in [14], have been devised. While solving nonlinear models, we have employed three methods known as the proposed method given in (12), the well-known Newton's method given in Eq. (1), and a recently devised ASA method given above in Eq. (10). The comparison is based on some parameters, including the number of iterations (i), the number of function evaluations ω, the computational cost COC $= i \times \omega$, the absolute error at the last iteration $\delta = |x_n - \kappa|$, and the CPU time (in seconds).

In the case of nonlinear case study problems, the final absolute error has been calculated using:

$$\delta_{i+1} = |x_{i+1} - x_i|. \tag{24}$$

The pre-determined tolerance is maintained as: $|\delta| < 10^{-100}$ with 50 maximum number of iterations. The computational cost (COC) of a technique is the result of the quantity of iterations (i) needed to satisfy the indicated tolerance, and number of function evaluations (ω) utilized per iteration. It is calculated as:

$$COC = i \times \omega. \tag{25}$$

5.1. Open-channel flow system

In standard and ecological design, it remains a challenge to connect the water stream with factors influencing the flow inside open channels such as trenches, seepage ditches, drains, and sewers. A stream rate is the volume of a stream passing a specific point through space during a specified period. However, another difficult circumstance arises when the viable channel becomes clogged. In this situation, the Manning's condition becomes an important factor for the water stream in an unlocked channel flowing under steady stream positions:

$$f(h) = \frac{\sqrt{m}}{n} Wh \left[\frac{Wh}{W+2h}\right]^{\frac{2}{3}} - Q. \tag{26}$$

The depth of water h in the channel has been assessed while expecting the remainder of the boundaries as $Q = 14.15$ m^3/s, $W = 4.572$ m, $n = 0.017$ and $m = 0.0015$. The underlying supposition for the initial estimate is $h_0 = 18.5$ m. The mathematical outcomes under various methods are displayed in Table 2. It can be observed from this Table that the proposed hybrid iterative method is computationally inexpensive while producing the same accurate result (x_i^*) as produced by existing approaches.

Table 2. Mathematical outputs for the model (26) with the initial guess $h_0 = 18.5$ m

Method	i	ω	COC	δ	$f(x_i^*)$	x_i^*	time
Proposed	4	6	24	3.2321e-577	0.0000e+00	1.4651	6.3e-02
Newton	9	2	18	2.9458e-169	1.1803e-337	1.4651	6.3e-02
ASA	4	6	24	1.9480e-591	0.0000e+00	1.4651	1.57e-01

5.2. Conversion in a chemical reactor

Consider the following nonlinear model:

$$f(x) = \frac{x}{1.0-x} - 5.0\ln\left(0.4\frac{1.0-x}{0.4-0.5x}\right) + 4.45977, \tag{27}$$

where x stands for the fractional ($x \in (0,1)$) conversion of species in a chemical reactor. The numerical simulations produced some mathematical outcomes that are represented in Table 3 for the simulations of the above chemical reactor problem while assuming the initial guess to be 0.76. Once again, the proposed hybrid iterative method is found to have smallest utilization of the machine time.

Table 3. Mathematical outputs for the problem 2 with the initial guess x_0 = 0.76

Method	i	ω	COC	δ	$f(x_i^*)$	x_i^*	time
Proposed	3	6	18	5.7523e-136	4.9071e-1340	7.574e-01	3.1e-02
Newton	8	2	16	5.8129e-179	4.2738e-354	7.574e-01	1.09e-01
ASA	3	6	18	4.2199e-144	4.0372e-1422	7.574e-01	4.7e-02

5.3. Volume from Van der Waals equation

The van der Waals nonlinear condition is a well-known scientific model for deciding the contrast between great and genuine gases, and it is as follows:

$$f(V) = PV^3 - V^2(RT + hP)n + n^2kV - hkn^3. \tag{28}$$

The temperature of gas is $R \approx 0.0820578$. The accompanying polynomial with third-degree can be gotten under $k = 16$, $h = 0.1243$, $n = 1.29$, $P = 37$ atm, and $T = 380^{\circ}$C. For $V = 3.25$, the mathematical results are represented in Table 4.

Table 4. Mathematical outputs for the problem 3 with the initial guess $V_0 = 3.25$

Method	i	ω	COC	δ	$f(x_i^*)$	x_i^*	time
Proposed	4	6	24	1.4302e-151	4.1257e-1506	1.9708	0.00
Newton	11	2	22	4.8618e-188	3.3383e-373	1.9708	1.6e-02
ASA	4	6	24	4.5979e-181	1.8698e-1801	1.9708	1.09e-01

Sometimes, it may happen that at the cost of a little more computational time, the errors will be drastically reduced and thus the accuracy is guaranteed to be improved.

6. Conclusion

A new tenth-order convergent technique is proposed by merging second-order Newton Raphson and an efficient Newton-type method with fifth-order convergence. The amalgamation gives us a tenth-order convergent technique with six function evaluations required per iteration. Using Taylor's expansion, the asymptotic error and order of convergence of the proposed hybrid method have been derived theoretically and confirmed via computational order of convergence. The efficiency index is calculated at about 1.4678. In addition, a comprehensive comparison of the proposed method with other existing techniques on various non-linear models taken from the literature has been carried out. The new tenth-order technique takes less CPU time than other well-known methods. In the current era, many scholars developed such methods that are initiated by classical Newton's method with good performance. Our main concern, in the present research work, is to improve the performance of the Newton's method by suitably blending it with the existing two-step methods. In future studies, we plan to consider the Homotopy methods, as done in the research works in [24, 25] and most of the references cited therein for the possible improvement of the present approach regarding semilocal convergence analysis.

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