Control and Cybernetics

vol. **46** (2017) No. 1

Fuzzy similarity measure based on fuzzy sets*

by

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Abstract: This paper extends the notion of fuzzy similarity measure between fuzzy sets. A definite class of fuzzy similarity measures between fuzzy sets is also introduced. Using some theorems and examples, it is shown that the introduced extended similarity measures satisfy many common and desired properties, based on the common axiomatic definitions, introduced for fuzzy similarity measures. Some illustrative and practical examples from the areas of pattern recognition and approximate reasoning systems are provided in order to present the possible applications of the proposed fuzzy similarity measures.

Keywords: approximate reasoning system, fuzzy similarity measure, ranking method, pattern recognition

1. Introduction

Since Zadeh (1965) introduced the notion of fuzzy sets, which can be used to represent the uncertainty through imprecise numbers, the fuzzy set theory has been successfully applied in various fields of scientific endeavour as a suitable tool for handling vague information, such as decision making, approximate reasoning, logic programming, fuzzy risk analysis, pattern recognition, and cluster analysis (see, for instance, Hejazi et al., 2011; Mondal et al., 2006; Turksen and Zhong, 1988; Yang and Shih, 2001). An important class of these methods measures the degree of similarity between fuzzy sets. Many researchers have proposed various similarity measures between fuzzy sets during the last decades. For instance, several similarity measures between fuzzy sets were reviewed and compared by Zwick et al. (1987) based on both geometric and set-theoretic premises. Pappis and Karacapilidis also (1993) introduced some similarity measures between fuzzy sets. After that, many researchers proposed more similarity measures between fuzzy sets (Balopoulos et al., 2007; Beg and Ashraf, 2009; Bouchon-Meunier et al., 2010; Bosteels and Kerre, 2007; Fan and Xie, 1999; Kehagis and Konstantinidou, 2003; Wang et al., 1995; Xuechang, 1992; Yang et al., 2005;

*Submitted: March 2015; Accepted: April 2017

Zeng et al., 2006). However, the literature, concerning this kind of measure can be divided into two main streams: 1) some papers introducing axiomatic definitions (see, for instance, Chachi and Taheri, 2013; Jenhani et al., 2010; Santini and Jain, 1999) 2) a great number of works introducing or reviewing some parametric families of measures (see, for instance, Chen et al., 1995; DeBaets et al., 2009, 2005, 2001; Guha and Chakraborty, 2010; Hejazi et al., 2011).

As it can be seen, the above methods present crisp similarity measures for fuzzy sets. However, since the sets themselves are not known exactly, it is logical that the similarity measure between them should also be treated as a non-exact value. Therefore, a similarity measure for fuzzy sets, supposed to represent the similarity between two uncertain sets, may also constitute an uncertain set. Based on this reasoning, using axiomatic definitions, some new classes of fuzzy similarity measures between fuzzy sets are introduced. Then, the main properties of the proposed fuzzy similarity measures are investigated. It is shown that these fuzzy similarity measures satisfy the main axiomatic conditions expected from a similarity measure in the fuzzy environment. For practical reasons, the proposed ranking method and similarity measures are illustrated using some application examples.

This paper is organized as follows. The next section reviews some concepts concerning fuzzy sets. In Section 3, some new definitions and relevant properties with respect to (crisp) similarity measures between fuzzy sets are proposed and discussed. Then, some fuzzy similarity measures between fuzzy sets are introduced and some basic properties of the proposed fuzzy similarity measures are also subject to investigation. Several illustrative and practical examples from the areas of decision making and approximate reasoning systems are presented and analyzed in Section 4. Finally, concluding remarks are forwarded in Section 5.

2. Preliminaries

A fuzzy set (Zadeh, 1965) of $\mathbb{X}$ is a mapping $\widetilde{A} : \mathbb{X} \to [0,1]$, which assigns to each $x \in \mathbb{X}$ a degree of membership $0 \leq \widetilde{A}(x) \leq 1$. The class of all fuzzy sets is denoted by $\mathcal{H}(\mathbb{X})$. $\widetilde{A}$ is called a normal fuzzy set if there exists a unique real number x^* such that $\widetilde{A}(x^*) = 1$. For each $\alpha \in (0,1]$, the subset $\{x \in \mathbb{X} \mid \widetilde{A}(x) \geq \alpha\}$ is called the level set or α-cut of $\widetilde{A}$ and is denoted by $\widetilde{A}[\alpha]$. The lower and upper bounds of $\widetilde{A}[\alpha]$ are denoted by $\widetilde{A}^L_\alpha = \inf\{x \in \mathbb{X} \mid \widetilde{A}(x) \geq \alpha\}$ and $\widetilde{A}^U_\alpha = \sup\{x \in \mathbb{X} \mid \widetilde{A}(x) \geq \alpha\}$, respectively. The set $\widetilde{A}[0] = \{x \in \mathbb{X} | \widetilde{A}(x) \neq 0\}$ is called the support of $\widetilde{A}$. The following operators on fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ are used in this paper (Dubois and Prade, 1978, 1980).

1. $\widetilde{A} = \widetilde{B}$ if and only if $\widetilde{A}(x) = \widetilde{B}(x)$, for all $x \in \mathbb{R}$, which is equivalent to $\widetilde{A}[\alpha] = \widetilde{B}[\alpha]$ for every $\alpha \in [0,1]$.
2. $\widetilde{A} \neq \widetilde{B}$ if and only if $\widetilde{A}(x) \neq \widetilde{B}(x)$, for some $x \in \mathbb{R}$, which is equivalent to $\widetilde{A}[\alpha_0] = \widetilde{B}[\alpha_0]$ for some $\alpha_0 \in [0,1]$.

3. $\widetilde{A} \subseteq \widetilde{B}$ if and only if $\widetilde{A}(x) \leq \widetilde{B}(x)$, for all $x \in \mathbb{R}$, which is equivalent to $\widetilde{A}[\alpha] \subseteq \widetilde{B}[\alpha]$ for every $\alpha \in [0,1]$.
4. $\widetilde{A}^c$ is the complement of fuzzy number $\widetilde{A}$ with the membership function $\widetilde{A}^c(x) = 1 - \widetilde{A}(x)$, for all $x \in \mathbb{R}$.
5. $\widetilde{A} \cup \widetilde{B}$ is the union of fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ with the membership function $(\widetilde{A} \cup \widetilde{B})(x) = \max\{\widetilde{A}(x), \widetilde{B}(x)\}$, for all $x \in \mathbb{R}$.
6. $\widetilde{A} \cap \widetilde{B}$ is the intersection of fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ with the membership function $(\widetilde{A} \cap \widetilde{B})(x) = \min\{\widetilde{A}(x), \widetilde{B}(x)\}$, for all $x \in \mathbb{R}$.

In this paper, a fuzzy set $\widetilde{A}$ of $\mathbb{X} = \mathbb{R}$ is called a fuzzy number when it satisfies the two following conditions:

1) $\widetilde{A}$ is normal.
2) For each $\alpha \in [0,1]$, the set $\widetilde{A}[\alpha]$ is a compact interval.

We denote by $\mathbb{F}(\mathbb{R})$ the set of all fuzzy real numbers. A fuzzy number $\widetilde{A} \in \mathbb{F}(\mathbb{R})$ is called positive, symbolized by $\widetilde{A} > 0$, if $\widetilde{A}[0] \subseteq (0, \infty)$. A fuzzy number $\widetilde{A} \in \mathbb{F}(\mathbb{R})$ is also called negative, symbolized by $\widetilde{A} < 0$, if $\widetilde{A}[0] \subseteq (-\infty, 0)$. However, in this paper, we focus on a special case of fuzzy numbers, which is called the triangular fuzzy number, **TFN**, denoted by $\widetilde{A} = (a^l, a, a^r)_T$, with the following membership function:

$$\widetilde{A}(x) = \begin{cases} \frac{x-a^l}{a-a^l} & a^l \leq x < a, \\ \frac{a^r-x}{a^r-a} & a \leq x < a^r. \end{cases}$$

Here, we recall the Secant Approximation for the multiplication of two **TFN**s (for more details, see Hanss, 2005, p. 60).

- If $\widetilde{A} = (a^l, a, a^r)_T > 0$ and $\widetilde{B} = (b^l, b, b^r)_T > 0$ then:

  $$\widetilde{A} \otimes \widetilde{B} \simeq (a^l b^l, ab, a^r b^r)_T,$$

- If $\widetilde{A} = (a^l, a, a^r)_T > 0$ and $\widetilde{B} = (b^l, b, b^r)_T < 0$ then:

  $$\widetilde{A} \otimes \widetilde{B} \simeq (c^l, c, c^r)_T,$$

  where

  $$\begin{aligned} c^l &= ab - (a(b - b^l) - b(a^r - a) + (a - a^l)(b - b^l)), \\ c &= ab, \\ c^r &= ab + (a(b^r - b) - b(a - a^l) - (b^r - b)(a - a^l)). \end{aligned}$$

In this paper, we are going to extend the concept of similarity measure between two fuzzy sets as a **TFN**. To generalize the properties of a fuzzy similarity measure, therefore, we need to apply a proper ranking method of **TFN**s. During the last decades, researchers interested in ranking of fuzzy sets proposed different ranking methods, derived on the basis of various principles (for instance, see Akyar et al., 2012; Allahviranlo and Saneifard, 2012; Boulmakoul et al., 2013; Nasseri et al., 2012; Wang et al., 2009; Wu, 2005). However, a common preference criterion for comparing fuzzy numbers was introduced by Nakamura (1986).

Definition 1 *For two fuzzy numbers* $\widetilde{A}$ *and* $\widetilde{B} \in \mathbb{F}(\mathbb{R})$, *let:*

$$\Delta_{\widetilde{A}\widetilde{B}} = \int_{\{\alpha:\widetilde{A}^L_\alpha \geq \widetilde{B}^L_\alpha\}} (\widetilde{A}^L_\alpha - \widetilde{B}^L_\alpha) d\alpha + \int_{\{\alpha:\widetilde{A}^U_\alpha \geq \widetilde{B}^U_\alpha\}} (\widetilde{A}^U_\alpha - \widetilde{B}^U_\alpha) d\alpha. \tag{1}$$

Then the degree of "$\widetilde{A}$ *is greater or equal to* $\widetilde{B}$*", is defined by:*

$$D(\widetilde{A} \succeq \widetilde{B}) = \begin{cases} 0.5 & \widetilde{A} = \widetilde{B}, \\ \frac{\Delta_{\widetilde{A}\widetilde{B}}}{\Delta_{\widetilde{A}\widetilde{B}} + \Delta_{\widetilde{B}\widetilde{A}}} & \widetilde{A} \neq \widetilde{B}. \end{cases} \tag{2}$$

Conform to this definition, for given fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, the preference criterion D measures the degree to which "$\widetilde{A}$ is greater than or equal to $\widetilde{B}$".

Definition 2 *For two fuzzy numbers* $\widetilde{A}$ *and* $\widetilde{B}$, *we say that* $\widetilde{A}$ *is greater than or equal to* $\widetilde{B}$, *denoting by* $\widetilde{A} \succeq_D \widetilde{B}$, *if* $D(\widetilde{A} \succeq \widetilde{B}) \geq 0.5$.

The preference criterion D has the following properties (for more details, see Yang et al., 2005).

Proposition 1 *Let* $\widetilde{A}$, $\widetilde{B}$ *and* $\widetilde{C}$ *be three fuzzy numbers. Then*

1) *D is reciprocal, i.e.,* $D(\widetilde{A} \succeq \widetilde{B}) = 1 - D(\widetilde{B} \succeq \widetilde{A})$.
2) *D is reflexive, i.e.,* $\widetilde{A} \succeq_D \widetilde{A}$.
3) *D is transitive, i.e.,* $\widetilde{A} \succeq_D \widetilde{B}$ *and* $\widetilde{B} \succeq_D \widetilde{C}$ *imply* $\widetilde{A} \succeq_D \widetilde{C}$.
4) $D(\widetilde{A} \succeq \widetilde{B}) = 1$ *if and only if* $\widetilde{B}^L_\alpha \leq \widetilde{A}^L_\alpha$ *and* $\widetilde{B}^U_\alpha \leq \widetilde{A}^U_\alpha$ *for all* $\alpha \in [0, 1]$ *and* $\widetilde{B}^L_{\alpha_0} < \widetilde{A}^L_{\alpha_0}$ *or* $\widetilde{B}^U_{\alpha_0} < \widetilde{A}^U_{\alpha_0}$ *for some* $\alpha_0 \in [0, 1]$.

We will use D in the following sections, to indicate the maximum fuzzy similarity measure between fuzzy numbers due to its reasonable properties.

Remark 1 *Let* $\widetilde{A} = (a^l, a, a^r)_T$ *and* $\widetilde{B} = (b^l, b, b^r)_T$ *be two* ***TFN****s. If* $a^l \geq b^l$, $a \geq b$ *and* $a^r \geq b^r$, *then it is easy to verify that* $\widetilde{A} \succeq_D \widetilde{B}$.

Remark 2 *Using the preference degree D, we can sort n fuzzy numbers* $\widetilde{A}_i$, *by using the following natural procedure: sort* $\{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\}$ *by calculating* $\binom{n}{2}$ *preference degrees to obtain the set in the form* $\{\widetilde{A}_{k_1}, \widetilde{A}_{k_2}, \ldots, \widetilde{A}_{k_n}\}$, *so that for any* $i < j$, $D(\widetilde{A}_j \succeq \widetilde{A}_i) \geq 0.5$. *The feasibility of the sorting is guaranteed by Proposition 1. Based on the sorting, therefore,* $\widetilde{A}_{k_n}$ *is the most preferred choice (which is denoted by* $\widetilde{\max}_{i=1}^n\{\widetilde{A}_i\}$*),* $\widetilde{A}_{k_{(n-1)}}$ *is the second, etc.*

3. Fuzzy similarity measure

In this section, we are going to extend a similarity measure between two fuzzy sets as a **TFN**. For this purpose, some new constructions for (crisp) similarity measures for fuzzy sets are proposed and it is shown that the proposed similarity measures satisfy the properties of the axiomatic definition provided by Couso et al. (2013). Then, based on the proposed (crisp) similarity measures, some class of fuzzy similarity measures for fuzzy sets is introduced and the properties of these measures are discussed by means of several theorems in fuzzy environment.

Definition 3 *For two normal fuzzy sets* $\widetilde{A}$ *and* $\widetilde{B}$*, a mapping* $S : \mathcal{H}(\mathbb{X}) \otimes \mathcal{H}(\mathbb{X}) \to \mathbb{R}$ *is called a similarity measure if it satisfies the following properties:*
1. $S(\widetilde{A}, \widetilde{B}) \in [0, 1]$.
2. $S(\widetilde{A}, \widetilde{B}) = S(\widetilde{B}, \widetilde{A})$.
3. $S(\widetilde{A}, \widetilde{B}) = 1$ *if and only if* $\widetilde{A} = \widetilde{B}$.
4. *If* $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$*, then* $S(\widetilde{A}, \widetilde{C}) \leq \min\{S(\widetilde{A}, \widetilde{B}), S(\widetilde{B}, \widetilde{C})\}$.

Example 1 *Let* $S_1^{a,b} : \mathbb{F}(\mathbb{R}) \times \mathbb{F}(\mathbb{R}) \to [0, 1]$ *be defined by:*

$$S_1^{a,b}(\widetilde{A}, \widetilde{B}) = \int_0^1 \frac{b|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|}{a|\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha]| + b|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|} d\alpha, \;\; a, b > 0, \tag{3}$$

for all $\widetilde{A}, \widetilde{B} \in \mathbb{F}(\mathbb{R})$ *where* $A\Delta B$ *denotes the symmetric difference of two ordinary sets* A *and* B *and* $|A|$ *denotes the length of* A*. Then,* S_1^{ab} *is a similarity measure. Moreover,* $S_1^{a,b}$ *satisfies the following properties for any* $a, b > 0$*:*
(I) If $\widetilde{A} \cap \widetilde{B} = \emptyset$ *then* $S_1^{ab}(\widetilde{A}, \widetilde{B}) = 0$*, otherwise* $S_1^{ab}(\widetilde{A}, \widetilde{B}) > 0$*,*
(II) $S_1^{ab}(\widetilde{A} \cap \widetilde{B}, \widetilde{A} \cup \widetilde{B}) = S_1^{ab}(\widetilde{A}, \widetilde{B})$*,*
(III) $S_1^{ab}(\widetilde{A} \cup \widetilde{C}, \widetilde{B} \cup \widetilde{C}) \geq S_1^{ab}(\widetilde{A}, \widetilde{B})$*.*
To check the above assertions, first assume that $\widetilde{A} = \widetilde{B}$*. Then it is readily seen that*

$$\frac{b|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|}{a|\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha]| + b|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|} = 1$$

for all $\alpha \in [0, 1]$ *and therefore* $S_1^{a,b}(\widetilde{A}, \widetilde{B}) = \int_0^1 1 d\alpha = 1$*. In reverse, assume that* $S_1^{a,b}(\widetilde{A}, \widetilde{B}) = 1$ *or*

$$\int_1^0 (1 - \frac{b|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|}{a|\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha]| + b|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|}) d\alpha = 0.$$

Since

$$\frac{b|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|}{a|\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha]| + b|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|} \leq 1$$

for any $\alpha \in [0, 1]$*, it follows that:*

$$1 - \frac{b|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|}{a|\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha]| + b|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|} = 0,$$

which leads to the conclusion that $|\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha]| = 0$*, that is* $\widetilde{A} = \widetilde{B}$*. Now, assume that* $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$*. It follows that*

$$S_2^a(\widetilde{A}, \widetilde{C}) = \int_0^1 \frac{b|\widetilde{A}[\alpha]|}{a|\widetilde{C}[\alpha]| + (b - a)|\widetilde{A}[\alpha]|} d\alpha.$$

Since $f(x) = \frac{bx}{ac+(b-a)x}, c \geq 0$ *is an increasing function of* $x > 0$ *and* $|\widetilde{A}[\alpha]| \leq |\widetilde{B}[\alpha]|$ *for every* $\alpha \in [0, 1]$*, we have* $S_2^a(\widetilde{A}, \widetilde{C}) \leq S_2^a(\widetilde{A}, \widetilde{B})$ *and similarly* $S_2^a(\widetilde{A}, \widetilde{C})$

$\leq S_2^a(\widetilde{B},\widetilde{C})$. *Therefore,* $S_1^{a,b}$ *is a similarity measure from Definition 3. If* $\widetilde{A}\cap\widetilde{B}=\emptyset$ *then it follows that* $|\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha]|=0$ *for all* $\alpha\in[0,1]$ *and therefore* $S_1^{a,b}(\widetilde{A},\widetilde{B})=0$. *In addition, if* $\widetilde{A}\cap\widetilde{B}\neq\emptyset$ *then it can be observed that*

$$\frac{b|\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha]|}{a|\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha]|+b|\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha]|}>0,$$

for all $\alpha\in[0,1]$ *and any* $a,b>0$. *Hence,* $S_1^{ab}(\widetilde{A},\widetilde{B})>0$, *which leads to the conclusion (I). The assertion (II) immediately follows, since* $(\widetilde{A}[\alpha]\cup\widetilde{B}[\alpha])\cap(\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha])=\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha]$ *and* $(\widetilde{A}[\alpha]\cup\widetilde{B}[\alpha])\triangle(\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha])=\widetilde{A}[\alpha]\triangle\widetilde{B}[\alpha]$. *To prove the assertion (III), let*

$$g^{a,b}(\widetilde{A}[\alpha],\widetilde{B}[\alpha])=\frac{a|\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha]|+b|\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha]|}{b|\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha]|}=\frac{a}{b}\frac{|\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha]|}{|\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha]|}+1.$$

Consequently, we get

$$g^{a,b}(\widetilde{A}[\alpha]\cup\widetilde{C}[\alpha],\widetilde{B}[\alpha]\cup\widetilde{C}[\alpha])=\frac{a}{b}\frac{|(\widetilde{A}[\alpha]\cup\widetilde{C}[\alpha])\Delta(\widetilde{B}[\alpha]\cup\widetilde{C}[\alpha])|}{|(\widetilde{A}[\alpha]\cup\widetilde{C}[\alpha])\cap(\widetilde{B}[\alpha]\cup\widetilde{C}[\alpha])|}+1.$$

Using the relations $(\widetilde{A}[\alpha]\cup\widetilde{C}[\alpha])\Delta(\widetilde{B}[\alpha]\cup\widetilde{C}[\alpha])\subseteq\widetilde{A}[\alpha]\triangle\widetilde{B}[\alpha]$ *and* $\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha]\subseteq(\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha])\cup\widetilde{C}[\alpha]$, *it follows that* $g^{a,b}(\widetilde{A}[\alpha]\cup\widetilde{C}[\alpha],\widetilde{B}[\alpha]\cup\widetilde{C}[\alpha])\leq g^{a,b}(\widetilde{A}[\alpha],\widetilde{B}[\alpha])$ *and therefore*

$$S_1^{ab}(\widetilde{A}\cup\widetilde{C},\widetilde{B}\cup\widetilde{C})=\int_0^1\frac{1}{g^{a,b}(\widetilde{A}[\alpha]\cup\widetilde{C}[\alpha],\widetilde{B}[\alpha]\cup\widetilde{C}[\alpha])}d\alpha\geq\int_0^1\frac{1}{g^{a,b}(\widetilde{A}[\alpha],\widetilde{B}[\alpha])}d\alpha=S_1^{ab}(\widetilde{A},\widetilde{B}).$$

Consider some special cases of $S_1^{a,b}$. *Assuming* $a=b=1$, *we get:*

$$S_1^{1,1}(\widetilde{A},\widetilde{B})=\int_0^1\frac{|\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha]|}{|\widetilde{A}[\alpha]\cup\widetilde{B}[\alpha]|}d\alpha. \tag{4}$$

In addition, let $a=1$, $b=2$, *then we get:*

$$S_1^{1,2}(\widetilde{A},\widetilde{B})=\int_0^1\frac{2|\widetilde{A}[\alpha]\cap\widetilde{B}[\alpha]|}{|\widetilde{A}[\alpha]|+|\widetilde{B}[\alpha]|}d\alpha. \tag{5}$$

EXAMPLE 2 *For two* $\widetilde{A},\widetilde{B}\in\mathcal{H}(\mathbb{X})$, *define* $S_2^a:\mathcal{H}(\mathbb{X})\times\mathcal{H}(\mathbb{X})\to[0,1]$ *as follows:*

$$S_2^a(\widetilde{A},\widetilde{B})=\inf_{\alpha\in(0,1]}\frac{|(\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha])^c|}{a|\widetilde{A}[\alpha]\Delta\widetilde{B}[\alpha]|+|(\widetilde{A}[\alpha]\triangle\widetilde{B}[\alpha])^c|},\quad a>0. \tag{6}$$

Then, S_2^a *is a similarity measure. It also satisfies the following properties:*

(I) $S_2^a(\widetilde{A} \cap \widetilde{B}, \widetilde{A} \cup \widetilde{B}) = S_2^a(\widetilde{A}, \widetilde{B})$,
(II) $S_2^a(\widetilde{A}^c, \widetilde{B}^c) = S_2^a(\widetilde{A}, \widetilde{B})$,
(III) $S_2^a(\widetilde{A} \cap \widetilde{C}, \widetilde{B} \cap \widetilde{C}) \geq S_2^a(\widetilde{A}, \widetilde{B})$,
(IV) $S_2^a(\widetilde{A} \cup \widetilde{C}, \widetilde{B} \cup \widetilde{C}) \geq S_2^a(\widetilde{A}, \widetilde{B})$.

To verify that S_2^a *is a similarity measure, assume that*

$$|\widetilde{A}[\alpha] \bigtriangleup \widetilde{B}[\alpha]| = |\widetilde{B}[\alpha] \bigtriangleup \widetilde{A}[\alpha]|,$$

for any $\alpha \in [0,1]$*. Therefore, it follows that* $S_2^a(\widetilde{A}, \widetilde{B}) = S_2^a(\widetilde{B}, \widetilde{A})$*. If* $\widetilde{A} = \widetilde{B}$ *then it is easily seen that* $S_2^a(\widetilde{A}, \widetilde{B}) = 1$*. Now assume that* $S_2^a(\widetilde{A}, \widetilde{B}) = 1$ *which leads to the conclusion that* $|\widetilde{A}[\alpha] \bigtriangleup \widetilde{B}[\alpha]| = 0$ *for every* $\alpha \in [0,1]$*, that is,* $\widetilde{A}[\alpha] = \widetilde{B}[\alpha]$ *for all* $\alpha \in [0,1]$ *or* $\widetilde{A} = \widetilde{B}$*. Assume that* $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$*. So, we have*

$$S_2^a(\widetilde{A}, \widetilde{C}) = \inf_{\alpha \in (0,1]} \frac{n - |\widetilde{C}[\alpha] - \widetilde{A}[\alpha]|}{n + (a-1)|\widetilde{C}[\alpha] - \widetilde{A}[\alpha]|}.$$

Since $f(x) = \frac{n-x}{n+(a-1)x}$ *is a decreasing function of* $x > 0$ *and* $|\widetilde{B}[\alpha] - \widetilde{A}[\alpha]| \leq |\widetilde{C}[\alpha] - \widetilde{A}[\alpha]|$ *for every* $\alpha \in [0,1]$*, it follows that* $S_2^a(\widetilde{A}, \widetilde{C}) \geq S_2^a(\widetilde{A}, \widetilde{B})$*. Similarly, we have* $S_2^a(\widetilde{A}, \widetilde{C}) \geq S_2^a(\widetilde{B}, \widetilde{C})$ *and therefore* S_2^a *is a similarity measure. To prove (I)-(IV), for all* $\alpha \in [0,1]$ *note that*

$$|(\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]) \bigtriangleup (\widetilde{A}[\alpha] \cup \widetilde{B}[\alpha])| = |\widetilde{A}[\alpha] \bigtriangleup \widetilde{B}[\alpha]|.$$

Therefore, $S_2^a(\widetilde{A} \cap \widetilde{B}, \widetilde{A} \cup \widetilde{B}) = S_2^a(\widetilde{A}, \widetilde{B})$ *and so (I) follows. (II) is immediately derived since* $(\widetilde{A}[\alpha])^c \bigtriangleup (\widetilde{B}[\alpha])^c = \widetilde{A}[\alpha] \bigtriangleup \widetilde{B}[\alpha]$*. The proofs of (III) and (IV) are easily verified by the following relations:*

$$(\widetilde{A}[\alpha] \cap \widetilde{C}[\alpha]) \triangle (\widetilde{B}[\alpha] \cap \widetilde{C}[\alpha]) \subseteq \widetilde{A}[\alpha] \bigtriangleup \widetilde{B}[\alpha],$$
$$(\widetilde{A}[\alpha] \cup \widetilde{C}[\alpha]) \triangle (\widetilde{B}[\alpha] \cup \widetilde{C}[\alpha]) \subseteq \widetilde{A}[\alpha] \bigtriangleup \widetilde{B}[\alpha] \text{ for all } \alpha \in [0,1],$$

and $S_2^a(\widetilde{A}[\alpha], \widetilde{B}[\alpha])$ *is a decreasing function of* $|\widetilde{A}[\alpha] \bigtriangleup \widetilde{B}[\alpha]|$*. Especially, in Eq.*(6)*, let* $a = 1$*. Therefore, we get:*

$$S_2^1(\widetilde{A}, \widetilde{B}) = 1 - \frac{1}{n} \sup_{\alpha \in (0,1]} |\widetilde{A}[\alpha] \Delta \widetilde{B}[\alpha]|. \tag{7}$$

Now, let $a = 2$*, then we obtain:*

$$S_2^2(\widetilde{A}, \widetilde{B}) = \inf_{\alpha \in (0,1]} \frac{n - |\widetilde{A}[\alpha] \Delta \widetilde{B}[\alpha]|}{n + |\widetilde{A}[\alpha] \Delta \widetilde{B}[\alpha]|}. \tag{8}$$

In addition, let $a = 3$*, then we have:*

$$S_2^3(\widetilde{A}, \widetilde{B}) = \inf_{\alpha \in (0,1]} \frac{n - |\widetilde{A}[\alpha] \Delta \widetilde{B}[\alpha]|}{n + 2|\widetilde{A}[\alpha] \Delta \widetilde{B}[\alpha]|}. \tag{9}$$

Remark 3 *It is worth to note that* $S_1^{a,b}$ *is increasing with respect to* b *and decreasing with respect to* a*. In addition,* S_2^a *is decreasing with respect to* a*.*

In the sequel, a concept of fuzzy similarity measure between two fuzzy sets is introduced.

Definition 4 *Let* $\mathbb{X}$ *be a universal set. Any mapping* $\widetilde{S} : \mathcal{H}(\mathbb{X})\otimes\mathcal{H}(\mathbb{X}) \to \mathcal{F}(\mathbb{R})$ *is said to be a fuzzy similarity measure (**FSM**) if for all normal fuzzy sets* $\widetilde{A}, \widetilde{B}$ *and* $\widetilde{C}$ *in* $\mathcal{H}(\mathbb{X})$*, it fulfills the following axioms:*

1. $\widetilde{S}(\widetilde{A}, \widetilde{B})[0] \subseteq [0, 1]$.
2. $\widetilde{S}(\widetilde{A}, \widetilde{B}) = \widetilde{S}(\widetilde{B}, \widetilde{A})$.
3. $\widetilde{S}(\widetilde{A}, \widetilde{B})(x) = I(x = 1)$ *if and only if* $\widetilde{A} = \widetilde{B}$.
4. *If* $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$ *then* $\widetilde{S}(\widetilde{A}, \widetilde{C}) \preceq_D \widetilde{S}(\widetilde{A}, \widetilde{B})$ *and* $\widetilde{S}(\widetilde{A}, \widetilde{C}) \preceq_D \widetilde{S}(\widetilde{B}, \widetilde{C})$.

Theorem 1 *Let* S_1*,* S_2 *and* S_3*:* $\mathcal{H}(\mathbb{X}) \otimes \mathcal{H}(\mathbb{X}) \to [0, 1]$ *be three similarity measures satisfying the following condition:*

$$S_1(\widetilde{A}, \widetilde{B}) \leq S_2(\widetilde{A}, \widetilde{B}) \leq S_3(\widetilde{A}, \widetilde{B}) \quad \textit{for all} \quad \widetilde{A}, \widetilde{B} \in \mathcal{H}(\mathbb{X}).$$

Then $\widetilde{S} = (S_1, S_2, S_3)_T$ *is an **FSM**.*

Proof. Since S_1, S_2 and S_3 are similarity measures, the assertions (1)-(3) are trivially satisfied. To prove (4), assume that $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$. Since S_1, S_2 and S_3 are similarity measures, it follows that:

$$S_i(\widetilde{A}, \widetilde{C}) \leq \min\{S_i(\widetilde{A}, \widetilde{B}), S_i(\widetilde{B}, \widetilde{C})\}, \quad \text{for} \quad i = 1, 2, 3, \tag{10}$$

which completes the proof by Remark 1. □

Now, by applying the similarity measure $S_1^{ab}(\widetilde{A}, \widetilde{B})$ defined in Eq. (3), we can construct an **FSM** as follows.

Theorem 2 *For any* $\widetilde{A}$*,* $\widetilde{B}$*, and* $\widetilde{C}$ *in* $\mathcal{F}(\mathbb{R})$*, let*

$$\widetilde{S}_1(\widetilde{A}, \widetilde{B}) = (S_1^{a_3b_1}(\widetilde{A}, \widetilde{B}), S_1^{a_2b_2}(\widetilde{A}, \widetilde{B}), S_1^{a_1b_3}(\widetilde{A}, \widetilde{B}))_T, \tag{11}$$

where $0 < a_1 < a_2 < a_3$*,* $0 < b_1 < b_2 < b_3$*, and* $S_1^{ab}(\widetilde{A}, \widetilde{B})$ *is defined in Eq.* (3)*. Then,* $\widetilde{S}_1(\widetilde{A}, \widetilde{B})$ *is an **FSM**. Moreover,* $\widetilde{S}_1$ *satisfies the following properties:*

I $\widetilde{S}_1(\widetilde{A} \cap \widetilde{B}, \widetilde{A} \cup \widetilde{B}) = \widetilde{S}_1(\widetilde{A}, \widetilde{B})$.

II *If* $\widetilde{A} \cap \widetilde{B} = \emptyset$ *then* $\widetilde{S}_1(\widetilde{A}, \widetilde{B})(x) = I(x = 0)$*, otherwise* $\widetilde{S}_1(\widetilde{A}, \widetilde{B})[0] \subseteq (0, 1]$.

III $\widetilde{S}_1(\widetilde{A} \cup \widetilde{C}, \widetilde{B} \cup \widetilde{C}) \succeq_D \widetilde{S}_1(\widetilde{A}, \widetilde{B})$.

Proof. It is easy to verify that the similarity measure S_1^{ab} is an increasing function of $b > 0$ and a decreasing function of $a > 0$. Therefore, if $0 < a_1 < a_2 < a_3$ and $0 < b_1 < b_2 < b_3$, then we have the following inequalities:

$$S_1^{a_3b_1}(\widetilde{A}, \widetilde{B}) \leq S_1^{a_2b_2}(\widetilde{A}, \widetilde{B}) \leq S_1^{a_1b_3}(\widetilde{A}, \widetilde{B}), \text{ for all } \widetilde{A}, \widetilde{B}, \widetilde{C} \in \mathcal{F}(\mathbb{R}). \tag{12}$$

From Theorem 1, $\widetilde{S}_1$ is then an **FSM**. The assertions (I)-(III) immediately follow from Theorem 1 and Remark 1. □

From Theorem 2, therefore, one can produce a large class of **FSM**s on a given universal set $\mathbb{X} = \{x_1, x_2, \ldots, x_k\}$.

Example 3 *Consider a universal set* $\mathbb{X} = \{x_1, x_2, \ldots, x_k\}$. *For all* $\widetilde{A}$, $\widetilde{B}$, *and* $\widetilde{C}$ *in* $\mathcal{H}(\mathbb{X})$, *let*

$$\widetilde{S}_2(\widetilde{A}, \widetilde{B}) = (S_2^{a_3}(\widetilde{A}, \widetilde{B}), S_2^{a_2}(\widetilde{A}, \widetilde{B}), S_2^{a_1}(\widetilde{A}, \widetilde{B}))_T, \tag{13}$$

where $0 < a_1 < a_2 < a_3$ *and* $S_2^a(\widetilde{A}, \widetilde{B})$ *is defined in Eq.* (6). *Since* S_1^a *is a decreasing function of* $a > 0$, *for given positive constants* $a_1 < a_2 < a_3$, *it follows that:*

$$S_2^{a_3}(\widetilde{A}, \widetilde{B}) \leq S_2^{a_2}(\widetilde{A}, \widetilde{B}) \leq S_2^{a_1}(\widetilde{A}, \widetilde{B}), \text{ for all } \widetilde{A}, \widetilde{B} \in \mathcal{H}(\mathbb{X}). \tag{14}$$

Therefore, $\widetilde{S}_2$ *is an* ***FSM*** *on the basis of Theorem 1. Moreover, it is easy to verify that* $\widetilde{S}_2$ *satisfies the following properties:*
(I) $\widetilde{S}_2(\widetilde{A} \cap \widetilde{B}, \widetilde{A} \cup \widetilde{B}) = \widetilde{S}_2(\widetilde{A}, \widetilde{B})$,
(II) $\widetilde{S}_2(\widetilde{A}^c, \widetilde{B}^c) = \widetilde{S}_2(\widetilde{A}, \widetilde{B})$,
(III) $\widetilde{S}_2(\widetilde{A} \cap \widetilde{C}, \widetilde{B} \cap \widetilde{C}) \succeq_D \widetilde{S}_2(\widetilde{A}, \widetilde{B})$,
(IV) $\widetilde{S}_2(\widetilde{A} \cup \widetilde{C}, \widetilde{B} \cup \widetilde{C}) \succeq_D \widetilde{S}_2(\widetilde{A}, \widetilde{B})$.

Remark 4 *There are several defuzzification techniques available to defuzzify a fuzzy similarity measure meant to obtain a (crisp) similarity measure, such as the centroid, center of maxima and mean of maxima methods, etc. (for instance, see Leekwijck and Kerre, 1999). However, one can apply the most common and appealing of all the defuzzification methods, introduced by Sugeno (1985), the so-called "center of gravity" method. For an* $\widetilde{A} \in \mathcal{F}(\mathbb{R})$, *the center of gravity of* $\widetilde{A}$ *is defined by* $C_{\widetilde{A}} = \frac{\int x\widetilde{A}(x)dx}{\int \widetilde{A}(x)dx}$.

Remark 5 *It should be mentioned that the proposed families of (crisp) similarity measures introduced in this paper, are simply exemplary and so the results of the present work will not be lost by changing this choice to the ones which fit the demands of the decision makers. For instance, one can apply the family of similarity measures introduced by Li et al. (2014) to construct a class of* ***FSMs*** *for fuzzy sets. However, the proposed similarity measures can also provide a flexible tool for both fuzzy sets and fuzzy numbers which allows the decision maker to construct a similarity measure with desired properties according to an axiomatic structure as it is done in this paper.*

4. Application examples

In this section, we present some practical examples to demonstrate the **FSM**s between fuzzy sets. For practical reasons, we would explain **FSM**s between fuzzy sets by some examples revelent to pattern recognition and approximate reasoning systems.

Table 1. The optimal choice based on **FSM** defined in Theorem 3 for Example 4

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$
$\widetilde{S}(\widetilde{\mathcal{O}}, \widetilde{A}_i)$	$(0.25, 0.33, 0.5)_T$	$(0.5, 0.6, 0.75)_T$	$(0.10, 0.14, 0.25)_T$	$I\{0\}$

4.1. Decision making example

Example 4 *The problem considered concerns four applicants for a position* $(\widetilde{A}_1, \widetilde{A}_2, \widetilde{A}_3, \widetilde{A}_4)$*, each of them being evaluated over four attributes, namely:*

1. *Experience in the specific job function, denoted as* x_1*,*
2. *Educational background, denoted as* x_2*,*
3. *Adaptability, denoted as* x_3*, and*
4. *Aptitude for teamwork, denoted as* x_4*.*

Suppose that we obtain the fuzzy decision matrix $\widetilde{\mathcal{D}}$ *as follows:*

$$\widetilde{\mathcal{D}} = \begin{matrix} \\ \widetilde{A}_1 \\ \widetilde{A}_2 \\ \widetilde{A}_3 \\ \widetilde{A}_4 \end{matrix} \begin{matrix} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{pmatrix} 0.4 & 1.0 & 0.9 & 0.7 \\ 0.4 & 0.6 & 1.0 & 0.8 \\ 0.5 & 1.0 & 0.8 & 0.6 \\ 1.0 & 0.9 & 0.8 & 0.7 \end{pmatrix} \end{matrix},$$

where $[\widetilde{\mathcal{D}}]_{ij} = \widetilde{A}_i(x_j)$*,* $i = 1, 2, 3, 4$ *and* $j = 1, \ldots, 4$*. The following fuzzy reference sequence* $\widetilde{\mathcal{O}}$ *is composed of the optimal membership values of the indicator over all the attributes*

$$\widetilde{\mathcal{O}} = optimal \begin{matrix} \begin{matrix} x_1 & x_2 & x_3 & x_4 \end{matrix} \\ \begin{pmatrix} 0.4 & 0.6 & 1.0 & 0.9 \end{pmatrix} \end{matrix}.$$

*We need to choose the best candidate for the position by calculating the **FSMs** between the reference sequence* $\widetilde{\mathcal{O}}$ *and the alternative sequences* $\widetilde{A}_i$*, i.e.* $\widetilde{S}_2(\widetilde{\mathcal{O}}, \widetilde{A}_i) = (S_2^3, S_2^2, S_2^1)_T$*,* $i = 1, 2, 3, 4$*, defined in Eq.* (13)*. The results are then summarized in Table 1. Based on the ranking method given in Remark 2, the following order is established among* $\widetilde{S}_2(\widetilde{A}_1, \widetilde{\mathcal{O}})$ *through* $\widetilde{S}_2(\widetilde{A}_4, \widetilde{\mathcal{O}})$*:*

$$\widetilde{S}_2(\widetilde{A}_4, \widetilde{\mathcal{O}}) \preceq_D \widetilde{S}_2(\widetilde{A}_3, \widetilde{\mathcal{O}}) \preceq_D \widetilde{S}_2(\widetilde{A}_1, \widetilde{\mathcal{O}}) \preceq_D \widetilde{S}_2(\widetilde{A}_2, \widetilde{\mathcal{O}}).$$

Therefore, the optimal alternative is $\widetilde{A}_2 = \arg \widetilde{\max}_{i=1}^4 \widetilde{S}(\widetilde{A}_i, \widetilde{\mathcal{O}}) = (0.5, 0.6, 0.75)_T$ *according to an extension of the "principle of maximum degree". Now, based on the center of gravity criterion, the defuzzified values of **FSMs*** $\widetilde{S}_2(\widetilde{A}_i, \widetilde{\mathcal{O}})$*,* $i = 1, 2, 3, 4$ *are given in Table 2. Therefore, the optimal alternative remains* $\widetilde{A}_2 = \arg \widetilde{\max}_{i=1}^4 C_{\widetilde{S}(\widetilde{A}_i, \widetilde{\mathcal{O}})} = 0.62$ *whenever the **FSMs** are defuzzified according to the center of gravity criterion.*

Table 2. Defuzzified values of **FSM**s in Example 4

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$
$C_{\widetilde{S}(\widetilde{\mathcal{O}},\widetilde{A}_i)}$	0.35	0.62	0.16	0

4.2. Approximate reasoning examples

In the following example, the application of the proposed **FSM**s in a forward approximate reasoning system is examined. The structure of a typical fuzzy if-then rule that uses the "and" fuzzy operator is demonstrated in the example statement below (see, for instance, Ross, 2005; Zimmermann, 2001):

$$R_i \ : \ \text{if } \widetilde{X}_1 \text{ is } \widetilde{A}_{1i}, \ \textit{and } \widetilde{X}_2 \text{ is } \widetilde{A}_{2i}, \ldots, \ \textit{and } \widetilde{X}_p \text{ is } \widetilde{A}_{pi}, \ \text{then } \widetilde{Y} \text{ is } \widetilde{B}_i.$$

In this scheme, R_i $(i = 1, \ldots, n)$ is the ith production rule, n is the number of rules, $\widetilde{X}_j$ $(j = 1, \ldots, p)$ are the fuzzy input (antecedent) variables, $\widetilde{Y}$ are the fuzzy output (consequent) variable, $\widetilde{A}_{ji}$'s are **TFN**s for the antecedent variables, and $\widetilde{B}_i$'s are **TFN**s for the consequent variable. Suppose that the antecedent statement is expressed as follows:

$$\textit{Antecedent} \ : \ \widetilde{X}_1 \text{ is } \widetilde{A}_1^*, \ \textit{and } \widetilde{X}_2 \text{ is } \widetilde{A}_2^*, \ldots, \ \textit{and } \widetilde{X}_p \text{ is } \widetilde{A}_p^*,$$

where $\widetilde{A}_j^*$'s $(j = 1, \ldots, p)$ are **TFN**s for the antecedent variables. Finally, the consequence of the approximate reasoning scheme is determined as:

$$\textit{Consequence} \ : \ \widetilde{Y} \text{ is } \widetilde{B}^*,$$

where $\widetilde{B}^*$ is a **TFN** for the consequent variable. Here, applying a method similar to that described by Chen et al. (1997) (see also Zhang and Yiu, 2013), we apply the following algorithm for the ith rule:

1. Compute $\widetilde{s}_{ji} = \widetilde{S}(\widetilde{A}_{ji}, \widetilde{A}_j^*)$, the **FSM**s for **TFN**s $\widetilde{A}_{ji}$, and $\widetilde{A}_j^*$,
2. Let $\widetilde{s}_i = \widetilde{\max}_{1\le j\le p}\widetilde{s}_{ji}$, and $\widetilde{B}_i^* = \widetilde{s}_i \otimes \widetilde{B}_i$,
3. The deduced consequence of rule R_i is "$\widetilde{Y}$ is $\widetilde{B}_i^*$".

Thus, the deduced consequence of the approximate reasoning scheme is "$\widetilde{Y}$ is $\widetilde{B}^*$" in which:

$$\widetilde{B}^* = \widetilde{B}_1^* \cup \widetilde{B}_2^* \cup \ldots \cup \widetilde{B}_n^*.$$

EXAMPLE 5 *Let us consider the following forward approximate reasoning scheme based on* ***TFNs***

$$R_1 \ : \ \textit{if } \widetilde{X}_1 \textit{ is } \widetilde{A}_{11}, \ \textit{and } \widetilde{X}_2 \textit{ is } \widetilde{A}_{21}, \ \textit{and } \widetilde{X}_3 \textit{ is } \widetilde{A}_{31}, \ \textit{then } \widetilde{Y} \textit{ is } \widetilde{B}_1,$$
$$R_2 \ : \ \textit{if } \widetilde{X}_1 \textit{ is } \widetilde{A}_{12}, \ \textit{and } \widetilde{X}_2 \textit{ is } \widetilde{A}_{22}, \ \textit{and } \widetilde{X}_3 \textit{ is } \widetilde{A}_{32}, \ \textit{then } \widetilde{Y} \textit{ is } \widetilde{B}_2,$$
$$R_3 \ : \ \textit{if } \widetilde{X}_1 \textit{ is } \widetilde{A}_{13}, \ \textit{and } \widetilde{X}_2 \textit{ is } \widetilde{A}_{23}, \ \textit{and } \widetilde{X}_3 \textit{ is } \widetilde{A}_{33}, \ \textit{then } \widetilde{Y} \textit{ is } \widetilde{B}_3,$$

*The **TFN**s used in this scheme are given in Table 3. Now, suppose that the antecedent statement is expressed as follows*

$$\text{Antecedent} \ : \ \widetilde{X}_1 \text{ is } \widetilde{A}_1^*, \text{ and } \widetilde{X}_2 \text{ is } \widetilde{A}_2^*, \text{ and } \widetilde{X}_3 \text{ is } \widetilde{A}_3^*,$$

where

$$\widetilde{A}_1^* = (0.31, 0.35, 0.38)_T, \quad \widetilde{A}_2^* = (5.1, 5.4, 5.6)_T, \quad \widetilde{A}_3^* = (15, 17, 20)_T.$$

We need to determine the consequence of the approximate reasoning scheme, which is

$$\text{Consequence} \ : \ \widetilde{Y} \text{ is } \widetilde{B}^*.$$

In this example, we apply $\widetilde{S}_1$*, introduced in Eq.* (11)*, with* $a_1 = 1$*,* $a_2 = 2$*,* $a_3 = 3$*, and* $b_1 = 1$*,* $b_2 = 2$*,* $b_3 = 3$*, that is:*

$$\widetilde{S}_1 = (S_1^{3,1}, S_1^{2,2}, S_1^{1,3})_T,$$

where

$$S_1^{3,1}(\widetilde{A}, \widetilde{B}) = \int_0^1 \frac{|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|}{3|\widetilde{A}[\alpha] \Delta \widetilde{B}[\alpha]| + |\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|} d\alpha,$$

$$S_1^{2,2}(\widetilde{A}, \widetilde{B}) = \int_0^1 \frac{|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|}{|\widetilde{A}[\alpha] \cup \widetilde{B}[\alpha]|} d\alpha,$$

$$S_1^{1,3}(\widetilde{A}, \widetilde{B}) = \int_0^1 \frac{3|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|}{|\widetilde{A}[\alpha] \Delta \widetilde{B}[\alpha]| + 3|\widetilde{A}[\alpha] \cap \widetilde{B}[\alpha]|} d\alpha.$$

*The **FSM**s are then evaluated as shown in Table 4. Thus,* $\widetilde{B}^* = \cup_{i=1}^3 \widetilde{B}_i^*$ *whose*

Table 3. The **TFN**s in the forward approximate reasoning of Example 5

i	$\widetilde{A}_{1i}$	$\widetilde{A}_{2i}$	$\widetilde{A}_{3i}$	$\widetilde{B}_i$
1	$(0.30, 0.35, 0.39)_T$	$(4.5, 5.1, 5.5)_T$	$(16, 19, 23)_T$	$(39, 42, 45)_T$
2	$(0.32, 0.36, 0.40)_T$	$(5.0, 5.4, 5.7)_T$	$(18, 21, 24)_T$	$(41, 45, 48)_T$
3	$(0.28, 0.34, 0.38)_T$	$(4.0, 4.8, 5.3)_T$	$(14, 18, 19)_T$	$(43, 47, 52)_T$

Table 4. **FSMs** $\widetilde{s}_{ji} = \widetilde{S}_1(\widetilde{A}_{ji}, \widetilde{A}_i^*)$ $(i = 1, 2, 3,\ j = 1, 2, 3)$ between **TFNs** in the forward approximate reasoning of Example 5

i	1	2	3
$\widetilde{s}_{1i}$	$(0.54, 0.78, 0.91)_T$	$I\{0\}$	$I\{0\}$
$\widetilde{s}_{2i}$	$I\{0\}$	$(0.45, 0.71, 0.88)_T$	$I\{0\}$
$\widetilde{s}_{3i}$	$I\{0\}$	$I\{0\}$	$(0.44, 0.76, 0.87)_T$
$\widetilde{s}_i = \widetilde{\max}_j \widetilde{s}_{ji}$	$(0.54, 0.78, 0.91)_T$	$(0.45, 0.71, 0.88)_T$	$(0.44, 0.76, 0.87)_T$
$\widetilde{B}_i^* = \widetilde{s}_i \otimes \widetilde{B}_i$	$(21.06, 32.76, 40.95)_T$	$(18.64, 32.13, 42.36)_T$	$(19.13, 33.18, 45.24)_T$

membership function is given by:

$$\mu_{\widetilde{B}^*}(x) = \begin{cases} 0 & x < 18.64, \\ \frac{x-18.64}{13.49} & 18.64 \le x < 32.13, \\ \frac{42.36-x}{10.23} & 32.13 \le x < 32.42, \\ \frac{x-21.06}{11.70} & 32.42 \le x < 32.76, \\ \frac{40.95-x}{8.19} & 32.76 \le x < 32.91, \\ \frac{x-19.13}{14.05} & 32.91 \le x < 33.18, \\ \frac{45.24-x}{12.06} & 33.18 \le x < 45.24, \\ 0 & x \ge 45.24. \end{cases}$$

5. Conclusion

This paper proposes a notion of fuzzy similarity measure for measuring the degree of similarity between fuzzy sets. For this purpose, some novel crisp similarity measures are first introduced and the main properties of the axiomatic definition for these crisp similarity measures are also verified. Subsequently, the main properties of the axiomatic definition for the proposed fuzzy similarity measures are investigated in the fuzzy environment. Applicability and effectiveness of the proposed fuzzy similarity measures are demonstrated with the results of examples relevant to pattern recognition and approximate reasoning systems. The main concerns of the examples provided in this paper are to indicate that these fuzzy similarity measures can provide a useful way for measuring the degree of similarity between fuzzy sets. The proposed fuzzy similarity measures can be also applied in many other fields, such as image processing, fuzzy neural networks, fuzzy reasoning, fuzzy risk analysis and fuzzy control.

On the other hand, over the past years, many studies have introduced the relevant notions for other kinds of constructs, such as, for instance, inclusion measures or entropy in the space of fuzzy sets (Farhadinia, 2013; Mao et al., 2013; Xuechang, 1992; Zeng et al., 2006; Zhang et al., 2009). In this regard, extending the concepts of fuzzy inclusion measures, or of fuzzy entropy, may well constitute some of the potential subjects for future studies.

Acknowledgements. I would like to thank the referees and the editor for their remarks and criticisms that have led to the improvement of presentation of this work.

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