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Transient analysis of transversely functionally graded Timoshenko beam (TFGTB) in conjunction with finite element method

In this work, transient and free vibration analyses are illustrated for a functionally graded Timoshenko beam (FGM) using finite element method. The governing equilibrium equations and boundary conditions (B-Cs) are derived according to the principle of Hamilton. The materials constituents of the FG beam that vary smoothly along the thickness of the beam (along beam thickness) are evaluated using the rule of mixture method. Power law index, slenderness ratio, modulus of elasticity ratio, and boundary conditions effect of the cantilever and simply supported beams on the dynamic response of the beam are studied. Moreover, the influence of mass distribution and continuous stiffness of the FGM beam are deeply investigated. Comparisons between the current free vibration results (fundamental frequency) and other available studies are performed to check the formulation of the current mathematical model. Good results have been obtained. A significant effect is noticed in the transient response of both simply supported and cantilever beams at the smaller values of the power index and the modulus elasticity ratio.

1. Introduction

Recently, several studies have been accomplished about functionally graded materials, which are considered as a novel type of composite and smart materials that promise a great future for mechanical and electrical engineering industries. The functionally graded material is a non-homogeneous material. It generally consists of two different types of materials and is manufactured in a specific way so that its

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properties vary continuously along different axis [1]. The mechanical and physical properties of the functionally graded, continuous composite materials are improved as they improve the required properties and diminish the undesirable ones [2]. The common type of the functionally graded material is the composite of ceramic and metal joined together to provide a novel product that has desirable strength and thermal resistances as compared to a metal (high strength properties-low thermal resistance) and ceramic (lower strength properties-higher thermal resistance) only [3]. Further, the continuous distribution of the properties helps to get best reliability of the FGM composite like an incongruity material properties problem due to sudden changes in the material properties. These unique properties make FGMs superior materials for many industries: aerospace, automobile, defense, and nuclear [4]. Therefore, the characteristics of FGMs and their static and dynamic response under different conditions have been considered as research projects. Hard work for understanding both vibration and pre-post buckling of the FGM materials have been indexed in several theoretical models. In addition, there have been great tendency for extrudinge the performance of FGM structures [5]. For instance, H.T. Thai et al. [6] studied static and free vibration of functionally graded beams with different higher-order shear deformation beam theories in conjunction with various boundary condition, and the equation of motion are derived by Hamilton's principle. M. Al-Shujairi et al. [7] investigated free vibration and buckling analysis of sandwich FG micro beam with elastic foundation based on the different higher order shear deformation beam theories. They used generalized differential quadrature method to evaluate the fundamental frequencies and buckling load. M. Şimşek et al, [8] presented bending and free vibration of FG sandwich beam subjected to successive moving harmonic forces based on Timoshenko beam theory. M. Aydogdu et al., [9] worked on free vibration of FG beam with simply supported boundary condition. The equation of motion is derived by the Hamilton's principle and the natural frequency can be obtained by Navier type solution method. M. Şimşek et al., [10] investigated the free and forced vibration of FG beam subjected to the moving harmonic loads based on Euler–Bernoulli beam theory with simply supported boundary condition. K. Pradhan et al., [11] studied free vibration of functionally graded beam with different boundary conditions based on Euler and Timoshenko beam theories in conjunction with Rayleigh-Ritz method. Y. Yang et al., [12] estimated free vibration of sandwich FG beam based on Timoshenko beam theory with a meshfree boundary-domain integral equation method. L. Ke et al., [13] studied buckling analysis and free vibration of a cracked functionally graded beam with Timoshenko beam theory in conjunction with different boundary conditions. M. Şimşek [14] illustrated the free vibration of functionally graded beam based on the first-order beam theory with various boundary conditions. M. Şimşek [15] investigated the forced and free vibration of bi-directional functionally graded beam subjected to moving load based on Timoshenko and Euler–Bernoulli beam theories. J. Ying et al., [16] presented static and free vibration of FG beam resting on elastic foundation with the two-dimensional theory of elasticity S. Kapuria et al., [17] studied the static and free vibration of FG beam based on the modified rule of mixtures with Timoshenko beam theory. J. Yang et al., [18] studied the free vibration and buckling analysis of cracked functionally graded beam based on Euler–Bernoulli beam theory. The fundamental frequencies and buckling loads are evaluated with various boundary conditions by analytical solutions. The transient behavior and free vibration of functionally graded beam (FGB) under the action of thermal loads based on the Reddy's higher order shear deformation beam theory are studied by A. Doroushi et al. [19]. Euler–Bernoulli beam theory and finite element method are used to study the transient and model analyses of axially functionally graded beam as estimated by M. Aubad et al. [20]. The properties of this material change continuously and smoothly in the axial direction (x-direction) based on the rule of mixture approach. Paper [21] presented the transient response and free vibration of functionally graded curved beam for various BCs using first order-shear deformation theory. Based on the Voigt's rule of mixture, the mechanical properties of FG curved beam vary smoothly through the thickness direction. Work [22] investigated the transient vibration of the electro elastic beam with clamed-free boundary conditions based on the Timoshenko beam theory in conjunction with finite element method. Article [23] studied transient vibration with nonlinear behavior of sandwich functionally graded curved shell with isotropic core using the third-order deformation theory. Mixture rule is used to express properties variations of the faces in the thickness direction. Galerkin method and Hamilton's approach are used for the derivation of the nonlinear differential equations of motion. Paper [24] studied the transient vibration of functionally graded plate based on the hybrid numerical method. Modulus of elastically and density changed in the plate transverse direction. T. Yokoyama [25] studied the free vibration and transient analysis of beams based on Timoshenko beam theory on elastic foundation with hinge-hinge boundary condition by finite element method. The governing equilibrium equations of the beams are mathematically formulated using Hamilton's principle. E. Amal [26] studied free vibration of FGM beam based on the Euler–Bernoulli beam theory. Virtual work is used to derive the equations of motion and related boundary conditions. The material constituents vary gradually in both x and z directions. S. Taeprasartsit [27] illustrated nonlinear free vibration analysis and bending deformation of functionally graded beam using Euler–Bernoulli beam theory and Karman nonlinearity. The material properties vary along the beam thickness and the rule of mixture is used for that purpose. N. Pradhan et al. [28] illustrated free vibration of functional graded beam using Timoshenko beam theory for the formulation and finite element method at various B-Cs. The material properties vary in z-direction according to the rule of mixture method. N. Fouda et al. [29] studied bending, buckling and free vibration of functionally graded beam with influence of porosity based on Euler–Bernoulli beams in conjunction with finite element method. Li-long et al. [30] obtained natural frequencies and mode shapes of FGMB using Timoshenko beam theory. The material constituents vary smoothly across the thickness. The governing equations and the corresponding B-Cs are formulated based on Hamilton's principle. Work [31] investigated the buckling and free vibration analysis of FG beam. The refined deformation theory is used for the mathematical modeling and solved numerically by finite element method.

According to the presented literature survey, it can be seen that, relatively very few investigations have studied dynamic (transient) analysis of Timoshenko FG beam in conjunction with finite element method where the properties change with beam thickness. In addition, more parametric studies need to be shown to estimate the most sensitive parameters that control the dynamic response of this beam. To the best of our knowledge, no similar work is reported. The current research is expected to contribute in understanding dynamic characteristics by bearing in mind the transient analysis of FGM Timoshenko beam, and the respective reader can easily recognize the parameters that play a crucial role in the dynamic response of functional graded material. In this study, transient analysis and modal analysis of functionally graded (FG) beam are illustrated based on Timoshenko beam theory (TBT) and finite element method with different B-Cs like (C-F and S-S). The properties vary smoothly in the beam transverse direction. Effect of several parameters: power index, BCs and modulus ratio on the frequencies and dynamic response are explained. Some of the results are compared to other works for mathematical modeling validation.

2. Theoretical concept

2.1. Material properties

The functionally graded beam is present in Fig. 1 with the length (L) in axial x-direction, the width (b) in the y-direction, the thickness (h) in the z-direction and the $O(x\ y\ z)$ is chosen at the center point of the FG beam.

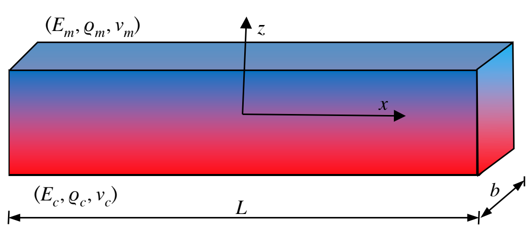

Fig. 1. A functionally graded (FG) beam

The functionally graded beam is made of the metallic (m) and the ceramic (c) components. The ceramic and metallic components are changing from the lower surface to the upper surface, as shown in Fig. 1. The volume fraction of the ceramic constitutive $V_c\,(z)$ of FG beam can be consider as

$$V_c(z) = \left(\frac{z}{h} + \frac{1}{2}\right)^n, \tag{1}$$

where (n) is an index parameter used to describe the material changing through the z-direction direction (thickness) of the FGM beam. Modulus of elasticity (E), Poisson's ratio (ν) and density (ρ) that vary smoothly through the z direction can be obtained according to the rule of mixture method. The relation between the volume fractions of the metallic and the ceramic constituents is as follows

$$V_m(z) + V_c(z) = 1. \tag{2}$$

The properties of material H can be obtained as follows

$$H = H_m V_m + H_c V_c , \tag{3}$$

where H_m and H_c are the properties of material of the metallic and ceramic constituents, respectively. The Young's modulus (E_{FGM}) and Poisson's ratio (ν_{FGM}), shear modulus (G_{FGM}) and mass density (ρ_{FGM}) can be obtained in the following equations

$$E_{\text{FGM}} = \left(\frac{z}{h} + \frac{1}{2}\right)^n (E_c - E_m) + E_m , \tag{4a}$$

$$\nu_{\text{FGM}} = \left(\frac{z}{h} + \frac{1}{2}\right)^n (\nu_c - \nu_m) + \nu_m , \tag{4b}$$

$$\rho_{\text{FGM}} = \left(\frac{z}{h} + \frac{1}{2}\right)^n (\rho_c - \rho_m) + \rho_m , \tag{4c}$$

$$G_{\text{FGM}} = \left(\frac{z}{h} + \frac{1}{2}\right)^n (G_c - G_m) + G_m . \tag{4d}$$

Fig. 2 shows the variation of ceramic constituent volume fraction along the beam thickness direction.

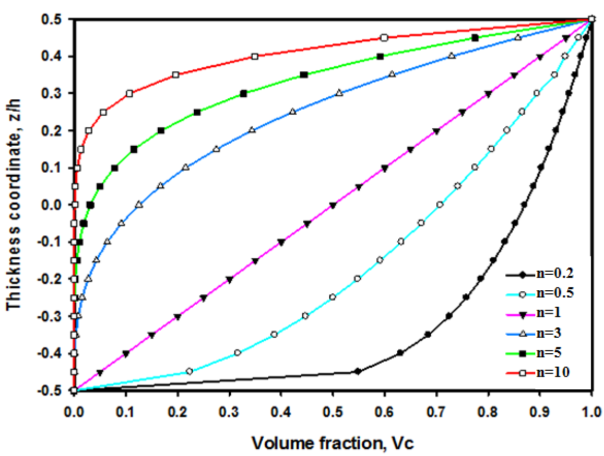

Fig. 2. Variation of the volume fraction of the ceramic constituent along the thickness

2.2. Equation of motion

Based on Timoshenko beam theory (TBT), the displacement fields are [32], as shown in Fig. 3

$$u(x,z) = z\theta(x), \tag{5a}$$

$$w(x,z) = w(x), \tag{5b}$$

where (w) is the transverse displacement in z direction.

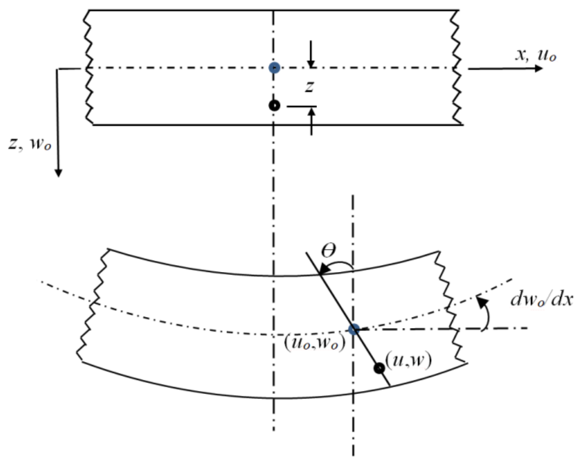

Fig. 3. Deformation of a Timoshenko beam

The formulation of the strain equations of TBT can be obtained as:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = z\,\frac{\partial \theta(x)}{\partial x}, \tag{6a}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \theta(x) + \frac{\partial w(x)}{\partial x}, \tag{6b}$$

where (ε_x) and (γ_{xz}) are the axial and shear strain, respectively.

Eqs. (6a) and (6b) can be written in a matrix form

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} -z & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} & \dfrac{\partial w}{\partial x} + \theta \end{Bmatrix}^T . \tag{6c}$$

The relation between the stress and strain of FG materials of beam obeys a Hooke's action. Therefore, the relation can be written as

$$\sigma = \begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix}. \tag{7}$$

Here E, G, σ_{xx} and τ_{xz} are the modulus, shear modulus of elasticity, stress and shear stress, respectively.

The equation of motion is derived by Hamilton's Principle, [32]

$$\delta\Pi = \int_{t_1}^{t_2} (\delta S_E - \delta K_E - \delta W)\, \mathrm{d}t = 0, \tag{8}$$

where δS_E, δK_E and δW are the strain, kinetic energy and the work of external forces.

The strain and kinetic energy are presented as:

$$S_E = \frac{1}{2} \int_0^L \int_A [\sigma]^T [\varepsilon] \mathrm{d}A \mathrm{d}x, \tag{9a}$$

$$K_E = \frac{1}{2} \int_0^L \int_A \rho \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right] \mathrm{d}A \mathrm{d}x, \tag{9b}$$

where A and L is the area and the length of the FG beam, respectively.

By substituting Eq. (6c) and Eg. (7) into Eq. (9a) we obtain the strain energy

$$S_E = \frac{1}{2} \int_0^L \begin{Bmatrix} \dfrac{\partial\theta}{\partial x} \\ \dfrac{\partial w}{\partial x} + \theta \end{Bmatrix}^T \begin{bmatrix} EI & 0 \\ 0 & kAG \end{bmatrix} \begin{Bmatrix} \dfrac{\partial\theta}{\partial x} \\ \dfrac{\partial w}{\partial x} + \theta \end{Bmatrix} \mathrm{d}x. \tag{10a}$$

Then, substituting Eqs. (5a) and (5b) into Eq. (9b) we obtain the kinetic energy

$$K_E = \frac{1}{2} \int_0^L \begin{Bmatrix} \dfrac{\partial w}{\partial t} \\ \dfrac{\partial\theta}{\partial t} \end{Bmatrix}^T \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w}{\partial t} \\ \dfrac{\partial\theta}{\partial t} \end{Bmatrix} \mathrm{d}x, \tag{10b}$$

where, A cross section area, I second moment of inertia and k shear correction factor of cross section.

The work of the external load can be given as [32]:

$$W = \int_0^L \begin{Bmatrix} w \\ \theta \end{Bmatrix}^T \begin{bmatrix} F_{\mathrm{DL}} \\ M_{moment} \end{bmatrix} \mathrm{d}x, \tag{11}$$

where F_{DL} and M_{moment}, are the distributed load and moment along the length of beam, while in this paper the external load, applied as a transient load in a short time, is discussed in the next section.

For functionally graded Timoshenko beam, the material with constant cross-section area (bh) has properties varying along thickness. The modulus of elasticity, rigidity and density from Eqs. (4a), (4c), (4d) will be substituted in equations of strain and kinetic energy to obtain another form of equation for functionally graded Timoshenko beam that can written as:

$$S_E = \frac{1}{2}\int_0^L \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial w}{\partial x} + \theta \end{Bmatrix}^T \begin{bmatrix} E_{\mathrm{FGM}} I & 0 \\ 0 & E_{\mathrm{FGM}} A \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial w}{\partial x} + \theta \end{Bmatrix} \mathrm{d}x, \tag{11a}$$

$$K_E = \frac{1}{2}\int_0^L \begin{Bmatrix} \dfrac{\partial w}{\partial t} \\ \dfrac{\partial \theta}{\partial t} \end{Bmatrix}^T \begin{bmatrix} \rho_{\mathrm{FGM}} A & 0 \\ 0 & \rho_{\mathrm{FGM}} I \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w}{\partial t} \\ \dfrac{\partial \theta}{\partial t} \end{Bmatrix} \mathrm{d}x. \tag{11b}$$

Eqs. (11a), (11b) and Eqs. (4a), (4c) (4d) are solved and can be written again after some mathematical manipulations as

$$S_E = \frac{1}{2}\int_{-L/2}^{L/2} \left[A_{xx}\left(\frac{\partial w}{\partial x}\right)^2 + 2B_{xx}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial \theta}{\partial x}\right) + k_s A_{xz}\left(\frac{\partial w}{\partial x} + \theta\right)^2 \right.$$
$$\left. + D_{xx}\left(\frac{\partial \theta}{\partial x}\right)^2 \right] \mathrm{d}x, \tag{12}$$

where

$$(A_{xx},\, B_{xx},\, D_{xx}) = \int_A E_{\mathrm{FGM}} \left(1,\, z,\, z^2\right) \mathrm{d}A, \tag{13a}$$

$$A_{xz} = k_s \int_A G_{\mathrm{FGM}}\, \mathrm{d}A, \tag{13b}$$

and the kinetic energy of the functionally graded beam can be given as,

$$K_E = \frac{1}{2}\int_{-1/2}^{1/2} \left[I_A\left(\frac{\partial w}{\partial t}\right)^2 + 2I_B\left(\frac{\partial w}{\partial t}\right)\left(\frac{\partial \theta}{\partial t}\right) + 2I_D\left(\frac{\partial \theta}{\partial t}\right)^2 \right] \mathrm{d}x, \tag{14}$$

where

$$(I_A,\, I_B,\, I_D) = \int_A \rho_{\mathrm{FGM}} \left(1,\, z,\, z^2\right) \mathrm{d}A. \tag{15}$$

2.3. Finite element formulation

The steady-state equilibrium equation of Timoshenko beam in terms of the displacement filed can be written as:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}+\theta\right) &= 0, \\ \frac{\partial^2\theta}{\partial x^2}-\frac{kGA}{EI}\left(\frac{\partial w}{\partial x}+\theta\right) &= 0. \end{aligned} \tag{16}$$

A specific polynomial is assumed to achieve the continuity of the two-node finite element, so that the shape function can be written as, [32]

$$\left\{\begin{matrix} w \\ \theta \end{matrix}\right\} = \begin{bmatrix} [N_w] \\ [N_\theta] \end{bmatrix} \{q\}, \tag{17}$$

$$[N_w]^T = \frac{1}{8(1+\phi_z)} \begin{bmatrix} 2(\zeta-1)\left(\zeta^2+\zeta-2-2\phi_z\right) \\ L\left(\zeta^2-1\right)(\zeta-1-\phi_z) \\ -2(\zeta+1)\left(\zeta^2-\zeta-2-2\phi_z\right) \\ L\left(\zeta^2-1\right)(\zeta+1+\phi_z) \end{bmatrix}, \tag{18a}$$

$$[N_\theta]^T = \frac{1}{4L(1+\phi_z)} \begin{bmatrix} 6\left(\zeta^2-1\right) \\ L\left(\zeta^2-1\right)(3\zeta+1-2\phi_z) \\ -6\left(\zeta^2-1\right) \\ L\left(\zeta^2+1\right)(3\zeta-1+2\phi_z) \end{bmatrix}, \tag{18b}$$

where (ϕ_z) is the ratio of the beam bending stiffness to the shear stiffness given by

$$\phi_z = \frac{12\int E_{\mathrm{FGM}}\, z^2\, \mathrm{d}z}{L^2 k \int G_{\mathrm{FGM}}\, \mathrm{d}z}$$

with

$$\phi_z = \frac{12 D_{bending}}{L^2 k\, D_{shearing}}, \quad D_{bending} = b\int_{-h/2}^{h/2} E_{\mathrm{FGM}}\, z^2\, \mathrm{d}z, \quad D_{shearing} = b\int_{-h/2}^{h/2} G_{\mathrm{FGM}}\, \mathrm{d}z,$$

and the array of nodal displacements and rotations is given

$$\{q\}^T = \{w_1,\ \theta_1,\ w_2,\ \theta_2\}^T. \tag{19}$$

It is interesting to note the strong dependency the shape functions have upon (θ), which is a ratio of the beam bending stiffness to the shear stiffness. The finite element representation of the equations of motion can be developed by substituting the energies and work equations into Hamilton's principle Eq. (8) and carrying out the integration over the beam length as, [33]

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{f\} = [N]^T f_0 , \tag{20}$$

where: $[M]$ is the mass matrix, $[C]$ is the damping matrix, $[K]$ is the stiffness matrix, $[N]$ is the shape function, f_0 is the value of the concentrated load (transient force), and $\{q\}$, $\{\dot{q}\}$ and $\{\ddot{q}\}$ are displacement, velocity and acceleration vectors, respectively.

Normally, the mass, stiffness and damping matrices are given as:

$$[K]^e = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}, \tag{21}$$

where

$$k_{11} = \frac{12D_b}{L^3(H+1)^2} + \frac{k\,D_s\,H^2}{L(H+1)^2}, \qquad k_{12} = k_{21} = \frac{6D_b}{L^2(H+1)^2} + \frac{k\,D_s\,H^2}{2(H+1)^2},$$

$$k_{13} = k_{31} = \frac{-12D_b}{L^3(H+1)^2} - \frac{k\,D_s\,H^2}{L(H+1)^2}, \qquad k_{14} = k_{41} = \frac{6D_b}{L^2(H+1)^2} - \frac{k\,D_s\,H^2}{2(H+1)^2},$$

$$k_{22} = \left(\frac{D_b}{L} + \frac{3D_s}{L(H+1)^2}\right) + \frac{kD_sH^2L}{4(H+1)^2}, \qquad k_{23} = k_{32} = \frac{-6D_b}{L^2(H+1)^2} - \frac{kD_sH^2}{2(H+1)^2},$$

$$k_{24} = k_{42} = \left(\frac{-D_b}{L} + \frac{3D_s}{L(H+1)^2}\right) + \frac{kD_sH^2L}{4(H+1)^2}, \qquad k_{33} = \frac{12D_b}{L^3(H+1)^2} + \frac{kD_sH^2}{L(H+1)^2},$$

$$k_{34} = k_{43} = \frac{-6D_b}{L^2(H+1)^2} - \frac{k\,D_s\,H^2}{2(H+1)^2}, \qquad k_{44} = \frac{D_b}{L} + \frac{3D_b}{L(H+1)^2} + \frac{k\,D_s\,H^2\,L}{4(H+1)^2},$$

$$[m_e] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}, \tag{22}$$

where

$$m_{11} = \frac{6I_{22}}{5L(H+1)^2} + \frac{I_{11}L\left(70H^2 + 147H + 78\right)}{210(H+1)^2},$$

$$m_{12} = m_{21} = -\frac{I_{22}(5H-1)}{10(H+1)^2} + \frac{I_{11}L^2\left(35H^2 + 77H + 44\right)}{840(H+1)^2},$$

$$m_{13} = m_{31} = -\frac{6I_{22}}{5L(H+1)^2} + \frac{I_{11}L^2\left(35H^2 + 63H + 27\right)}{210(H+1)^2},$$

$$
\begin{aligned}
m_{14} &= m_{41} = -\frac{I_{22}(5H-1)}{10(H+1)^2} - \frac{I_{11}L^2\left(35H^2+63H+26\right)}{840(H+1)^2},\\
m_{22} &= \frac{I_{11}L^3}{840(H+1)^2} + \frac{I_{11}L^3}{120} + \frac{I_{22}L\left(-5H^2+5H+1\right)}{30(H+1)^2},\\
m_{23} &= m_{32} = \frac{I_{22}(5H-1)}{10(H+1)^2} + \frac{I_{11}L^2\left(35H^2+63H+26\right)}{840(H+1)^2},\\
m_{24} &= m_{42} = \frac{I_{11}L^3}{840(H+1)^2} - \frac{I_{11}L^3}{120} - \frac{I_{22}L\left(-5H^2+5H+1\right)}{30(H+1)^2},\\
m_{33} &= \frac{6I_{22}}{5L(H+1)^2} + \frac{I_{11}L\left(70H^2+147H+78\right)}{210(H+1)^2},\\
m_{34} &= m_{43} = \frac{I_{22}(5H-1)}{10(H+1)^2} - \frac{I_{11}L^2\left(35H^2+77H+44\right)}{840(H+1)^2},\\
m_{22} &= \frac{I_{11}L^3}{120} + \frac{I_{11}L^3}{840(H+1)^2} + \frac{I_{22}L\left(10H^2+5H+4\right)}{30(H+1)^2},
\end{aligned}
$$

where L is element length

$$
\begin{aligned}
H &= \frac{12D_b}{k\,D_s\,L^2},\\
D_b &= \int\limits_A z^2E(z)\,\mathrm{d}A, \qquad D_s = \int\limits_A G(z)\,\mathrm{d}A,\\
I_{11} &= \int\limits_A \rho(z)\,\mathrm{d}A, \qquad I_{22} = \int\limits_A z^2\rho(z)\,\mathrm{d}A,\\
\mathrm{d}A &= b\,\mathrm{d}z
\end{aligned}
$$

while the damping matrix $[C]$ is Rayleigh-type damping [34] and can be represented as a linear combination of mass and stiffness matrices as:

$$
[C] = \alpha\,[M] + \beta\,[K], \tag{23}
$$

where α and β are proportionality factors which depend on the material types.

3. Free vibration analysis

Generally, the governing equation of the free vibration of the beam (no external load) can be expressed as [34],

$$
[M]\{\ddot{q}\} + [K]\{q\} = \{0\}. \tag{24}
$$

The general dynamic response can be generally given by a sinusoidal form as:

$$\{q_i\} = \{\varphi_i\} \sin \omega_i t; \quad i = 1, 2, \ldots, k, \tag{25}$$

where k, $\{\varphi_i\}$ and ω_i are the system total degrees of freedom, mode shape vector of $\left\{i^{\text{th}}\right\}$ vibration mode, and the corresponding natural frequency, respectively

$$\left([K] - \omega_i^2 [M]\right) \{\varphi_i\} = \{0\}. \tag{26}$$

4. Numerical results

4.1. Verification of the present results

The geometrical and mechanical property of the Timoshenko beam steel is regrouped in Table 1. The results of the approaches of exact, finite element method and spectral element analysis as Ref. [33] are regrouped in Table 2.

Table 1.
Properties of the beam [33]

L [m]	b [m]	h [m]	ν	E [MPa]	G [MPa]	ρ [kg/m^3]	k
1	0.05	0.15	0.305	$207 \cdot 10^9$	$79.3 \cdot 10^9$	$76.5 \cdot 10^3$	5/6

Table 2 illustrates the first five fundamental frequencies obtained using exact, FEM, and spectral element analyses for simply supported boundary condition. In general, a good agreement has been noticed between frequencies of the present study and the results from Ref [33].

Table 2.
First five frequencies using various methods (rad/s)

Mode	1	2	3	4	5
Ref. [33]	677.8829	2473.3231	4947.9316	7776.1021	10776.4430
Exact	677.8829	2473.3231	4947.9316	7776.1021	10776.4430
Present work	677.8900	2473.4021	4948.0000	7778.1021	10778.0065

4.2. Vibration modes

The beam is made of alumina (Al_2O_3) and steel metal. Their properties are presented in Table 3. Three mode shapes of the FGM Timoshenko beam for a simply supported and cantilever boundary conditions are illustrated in Fig. 4 and Fig. 5. The results are produced using finite element method for ($h/l = 0.15$) and 5/6 for both slenderness ration and cross sectional shape factor, respectively. It is noted that finite element method exactly predicts the mode shapes of the FGM beam. The first mode shapes for both cases are simply bending modes and become more complicated with increasing the mode number.

Table 3.
Material properties of the functionally graded material beam constituent Ref. [32]

Material	Property		
	Young modulus (E)	Density (ρ)	Poisson's ratio (ν)
Steel	210 GPa	7800 kg/m^3	0.3
Alumina (Al_2O_3)	390 GPa	3960 kg/m^3	0.3

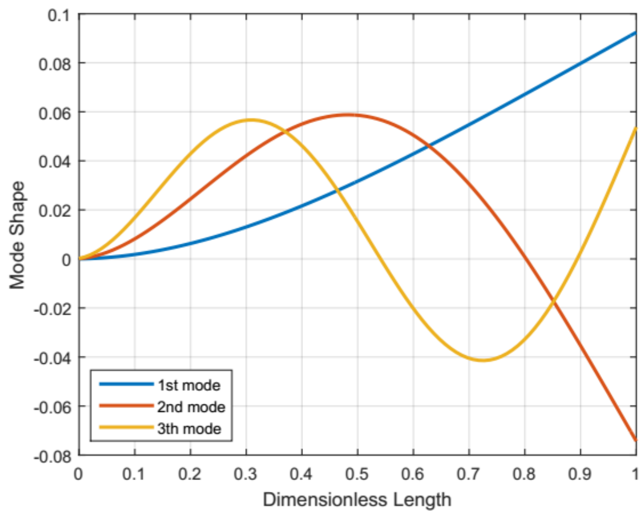

Fig. 4. Transversal vibration of the first three modes for S-S beam

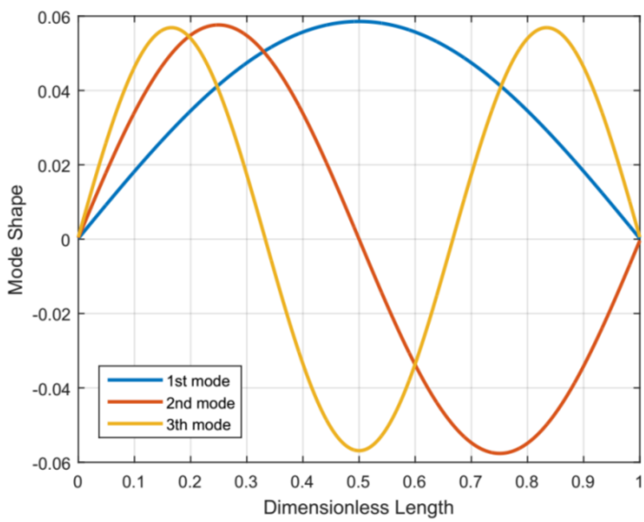

Fig. 5. Transversal vibration of the first three modes for C-F beam

4.3. Transient analysis

In this section, transient response of the FGM beam is illustrated for different values of gradient index (n) and slenderness ratio (L/h). For convenience, the effects will be presented in separate sections.

4.3.1. Effect of power law index

Figs. 6–9 present the response of a cantilever beam at various power indices for a step input force of 50 N. It is shown that the power index plays a critical function for the assessment of the transient response of the cantilever FGM beam. It is observed that the displacement response and the time required for steady state increased with

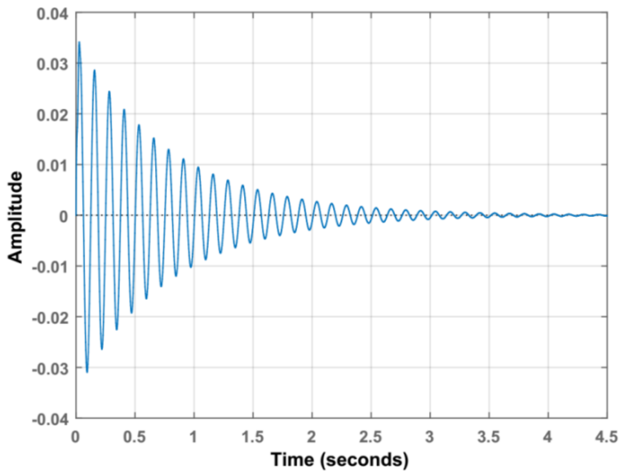

Fig. 6. Transient response of C-F beam with 50 N impulse force ($n = 0$)

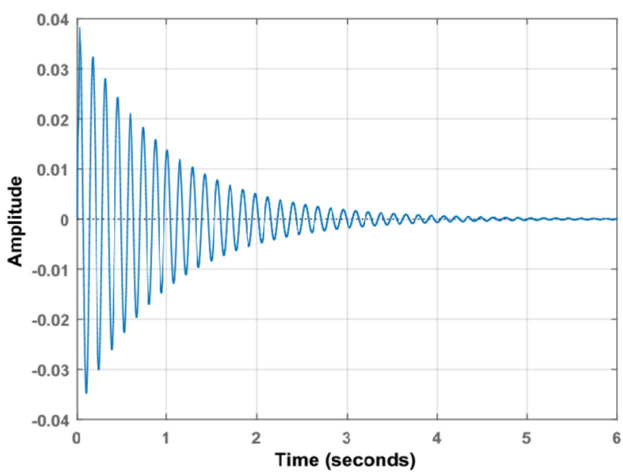

Fig. 7. Transient response of C-F beam with 50 N impulse force ($n = 0.5$)

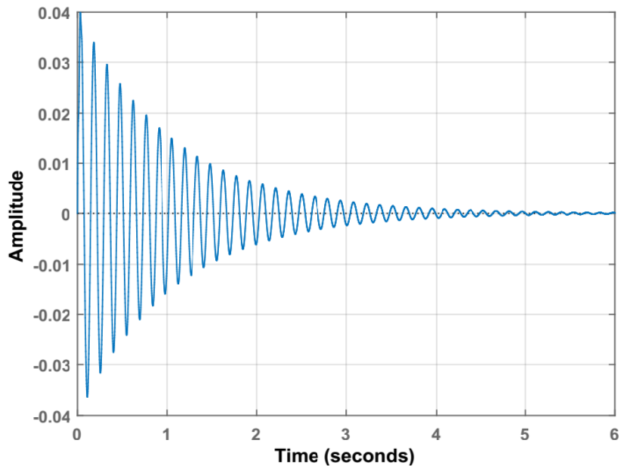

Fig. 8. Transient response of C-F beam with 50 N impulse force (n = 1)

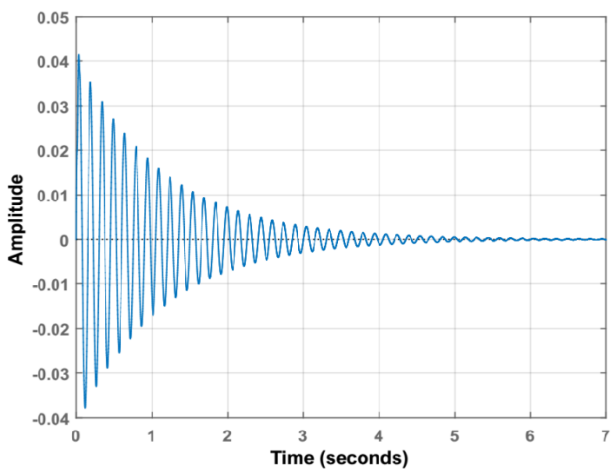

Fig. 9. Transient response of C-F beam with 50 N impulse force (n = 2)

the power law index (direct relationship). The smaller power law index, the smaller displacement and time for steady-state response. The reason of this effect is the dominance of Al_2O_3 properties for smaller values of the power index-over steel properties (the respected reader is referred to effective stiffness equation). Al_2O_3 domination results in a higher structural stiffness of the FGM beam and hence provides a stiffer beam and higher damped frequencies, as presented in Fig. 7.

Figs. 10–13 show the effect of power law index for both cantilever and simply supported beam. It is noticed obviously that transient response increases as the index increases due to the domination of the metal of the lower stiffness over the one of the material of higher metal. Furthermore, it is important to notice that the difference in the transient response with an increase in the power law index

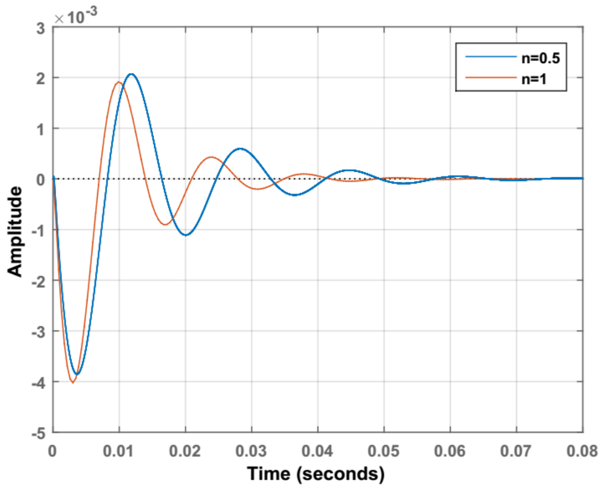

Fig. 10. Transient response of S-S beam for ($L/h = 0.06$ and $n = 0.5, 1$)

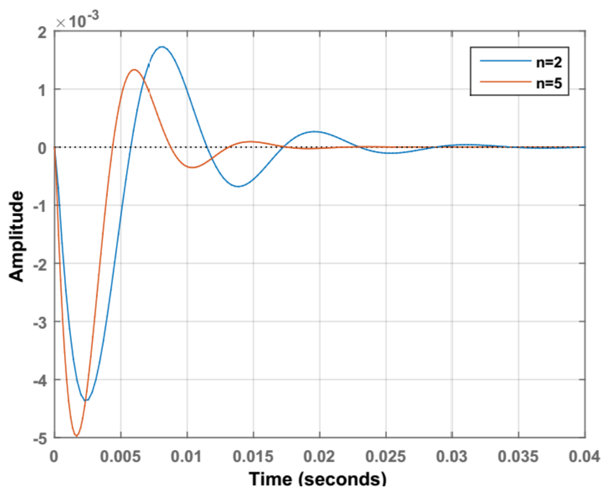

Fig. 11. Transient response of S-S beam for ($L/h = 0.06$ and $n = 2, 5$)

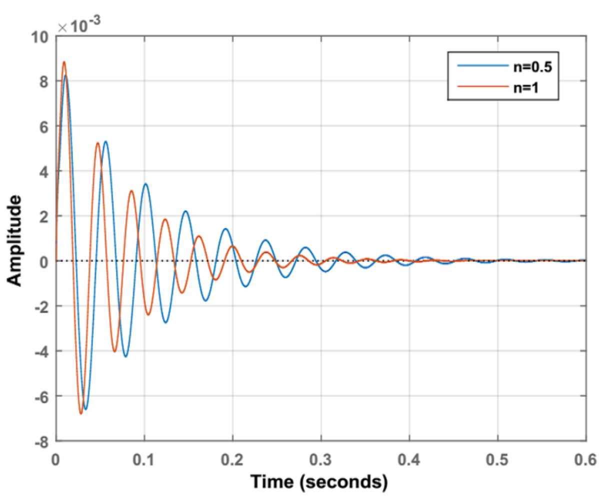

Fig. 12. Transient response of C-F beam for ($L/h = 0.06$ and $n = 0.5, 1$)

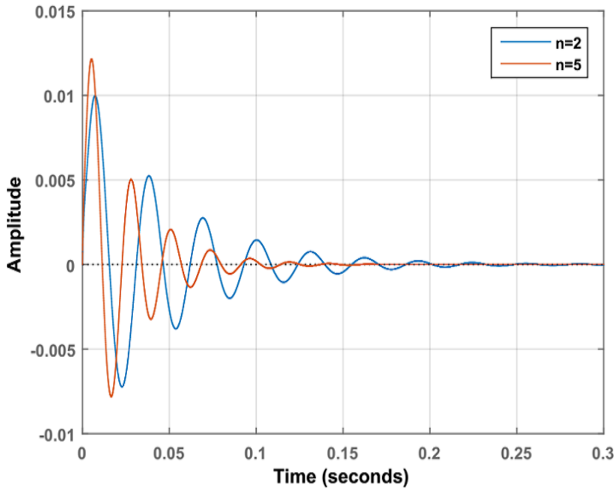

Fig. 13. Transient response of C-F beam for (L/h = 0.06 and n = 2, 5)

increases for both beam boundary conditions. However, the trend of percentage increase for the C-F beam is higher than that corresponding in S-S. In addition, the simply supported FGM beam shows more stable response than the cantilever FGM beam due to the beam constraint and the structural damping effect.

4.3.2. Effect of slenderness ratio

Figs. 14–17 present the effect of slenderness ratio on the transient analysis and steady-state time response. It is noted that peak displacement and the time required for the steady-state decreased with increasing the slenderness ratio due

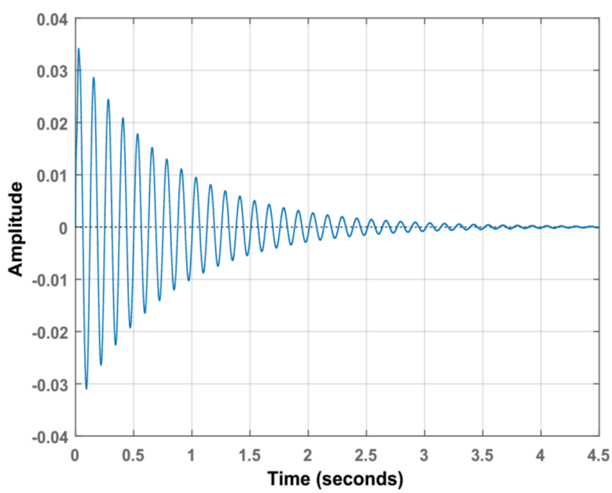

Fig. 14. Transient response of C-F beam for (L/h = 0.03) with (n = 0)

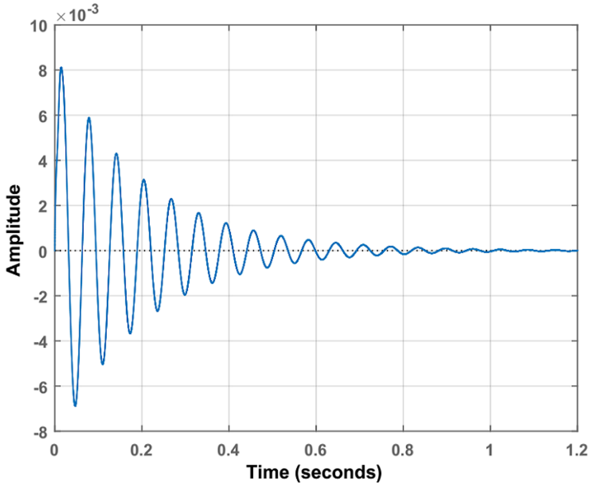

Fig. 15. Transient response of C-F beam for $(L/h = 0.06)$ with $(n = 0)$

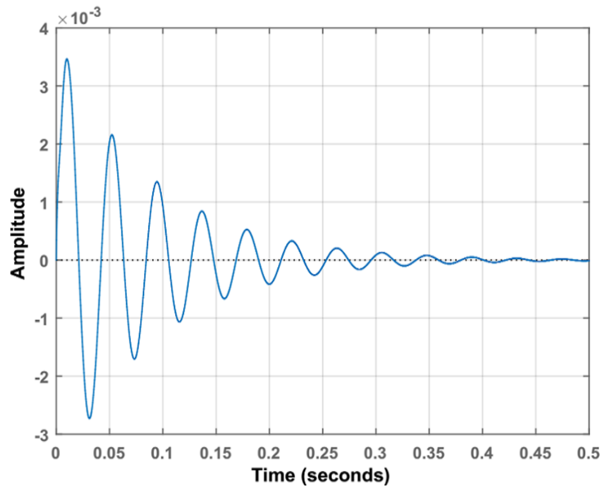

Fig. 16. Transient response of C-F beam for $(L/h = 0.09)$ with $(n = 0)$

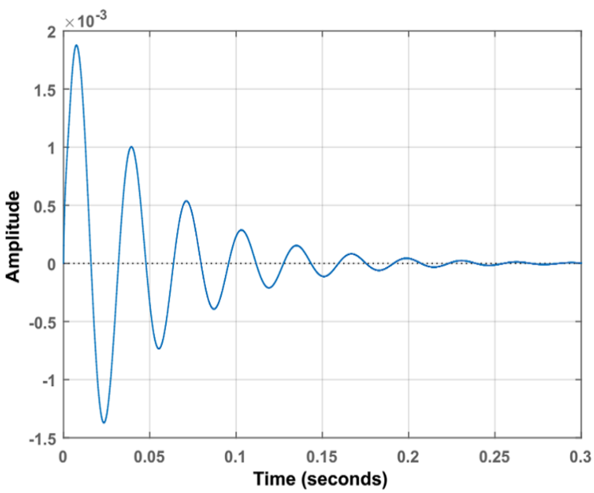

Fig. 17. Transient response of C-F beam for $(L/h = 0.12)$ with $(n = 0)$)

to the increase of the cross sectional area and the second moment of area of the FGM beam, thereby, increasing the stiffness of the beam and getting a stiffer FGM. In addition, the combined effect of increasing the stiffness and mass matrix, due to the increase of the cross sectional area, is the main reason for increasing the damped frequency and the steady state time.

For the simply supported FGM beam, lower transient response and steady state time is noticed, as shown in Figs. 18–21, due to the support conditions which make the FGM stiffer, thereby, minimum time response is noticed. While the increasing in the slenderness ratio is constant (0.03 for each step), the rate of the transient

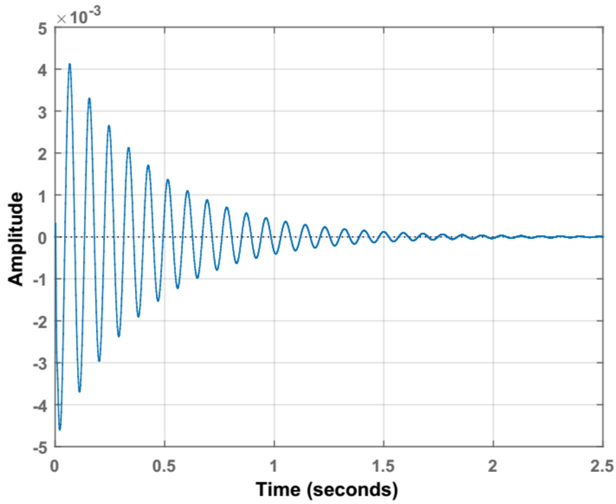

Fig. 18. Transient response of S-S beam for $(L/h = 0.03)$ with $(n = 0)$

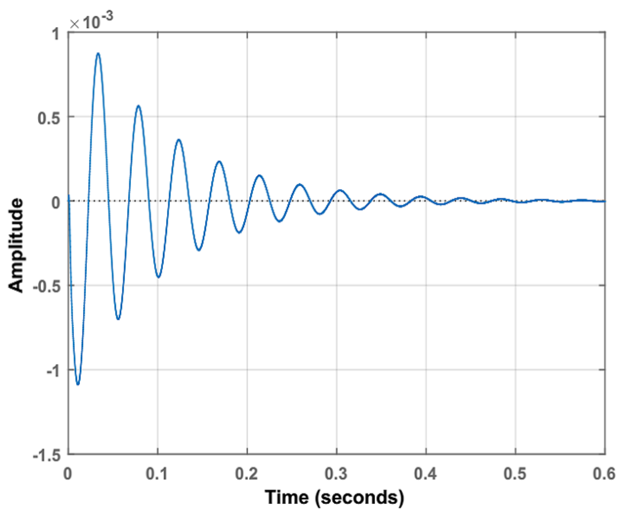

Fig. 19. Transient response of S-S beam for $(L/h = 0.06)$ with $(n = 0)$

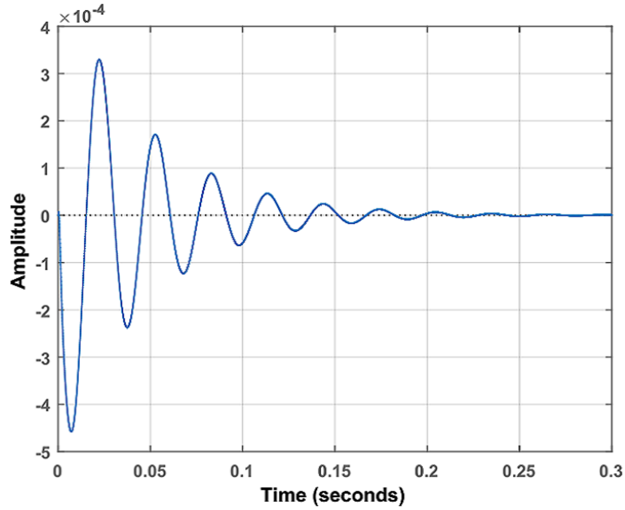

Fig. 20. Transient response of S-S beam for $(L/h = 0.09)$ with $(n = 0)$

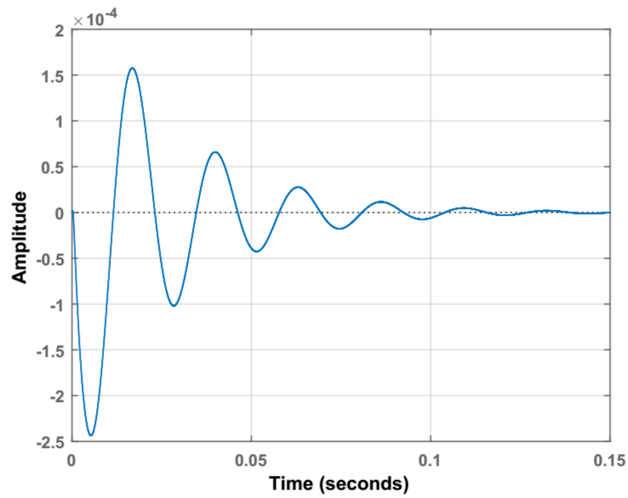

Fig. 21. Transient response of S-S beam for $(L/h = 0.12)$ with $(n = 0)$

response for both beams isn't constant. In other words, the rate of reduction in the response for the C-F beam is higher than that corresponding in the S-S beam, which means that slenderness effect is predominant for the C-F beam.

5. Conclusions

In this study, transient analysis of Timoshenko Transversely Functional Graded (TTFG) beam is studied and solved by finite element method along with Timoshenko beam theory. Classical mixture rule for description of the variation of the properties in the transverse direction of the beam is used. The effect of several parameters on the transient response of the beams is studied and the results are discussed. Some conclusions are drawn from the results, and the most important ones can be showed as:

1. The transient response of the TTFG beam is well predicted by the FE method for different B-Cs.
2. A significant effect is noticed in the transient response of both simply support and cantilever beams at smaller values of the gradient index.
3. Although the increase in the power law index and slenderness ratio is constant, step of 0.5 and 0.03, respectively, the percentage decrease and increase in the transient response, respectively, are not constant.
4. While power law index and slenderness ratio exert a noticeable effect on the peak response, the type of beam constraint plays the major impact in that response. The response of the C-F beam is more sensitive on variations of the above parameters.

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