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On a boundary value problem for the system of partial differential equations describing non-simple thermoelasticity

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Abstract. In this paper we study the Dirichlet problem for the system of equations describing non-simple thermoelasticity. Using the general theory of the elliptic problem we show that this problem is elliptic one.
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1. Introduction

The theory of non-simple elastic and thermoelastic materials was studied in various papers. R.A. Toupin [6] derived for the first time the equations of motions, constitutive equations and boundary conditions of the strain-gradient theory in general nonlinear form. On the basis of the conservation principle, R.D. Mindlin and N.N. Eshel [5] obtained the linear theory of elasticity in which the potential energy density depends not only on strain but also on the gradient of strain. G. Ahmadi and K. Firoozbakhsh [2] derived the strain-gradient theory of thermoelasticity based on Clausius-Duhem inequality. In paper [3] we derived the Cauchy problem for the non-stationary system of partial differential equations describing non-simple thermoelasticity (see also [4]).

In what follows, Ω is a bounded domain in 3-dimensional Euclidean space $\mathbb{R}^3$ with the smooth boundary $\partial\Omega$. In this paper we consider the following boundary value problem:

$$c_2^2 l_2^2 \triangle^2 u + (c_1^2 l_1^2 - c_2^2 l_2^2)\nabla \mathrm{div} \triangle u + \frac{m}{\rho}\nabla\theta - c_2^2\triangle u - (c_1^2 - c_2^2)\nabla \mathrm{div} u = f, \tag{1}$$

$$-\frac{k}{c}\triangle\theta = g \qquad \text{on} \quad \Omega,$$

$$u = u^0, \;\; \theta = \theta_0, \;\; \frac{\partial u}{\partial n} = u^1 \qquad \text{on} \quad \partial\Omega, \tag{2}$$

or briefly

$$P(D_x)V = F \qquad \text{on} \quad \Omega, \qquad \tilde{B}(D_x)V = G \qquad \text{on} \quad \partial\Omega.$$

where $x \in \mathbb{R}^3, D_x = (D_{x_1}, D_{x_2}, D_{x_3}), D_{x_j} = \frac{\partial}{\partial_{x_j}}, u : \bar{\Omega} \to \mathbb{R}^3$, $\frac{\partial}{\partial n}$ is the normal derivative, $\theta : \bar{\Omega} \to \mathbb{R}$,
u denotes the displacement, θ — the temperature disturbance, f is a given vector-valued function on Ω, θ is a given function on $\partial\Omega$ and u^0, u^1, θ_0 are given functions on $\partial\Omega$. $c_1, c_2, l_1, l_2, m, k, c, \rho$ are some constant physical parameters.

Theorem. *The boundary value problem (1-2) is elliptic.*

Proof. Let P_0 be the principal part of the operator P.

$$P_0(\xi) = \begin{bmatrix} c_2^2 l_2^2|\xi|^4 + \gamma|\xi|^2\xi_1^2 & \gamma|\xi|^2\xi_1\xi_2 & \gamma|\xi|^2\xi_1\xi_3 & 0 \\ \gamma|\xi|^2\xi_1\xi_2 & c_2^2 l_2^2|\xi|^4 + \gamma|\xi|^2\xi_2^2 & \gamma|\xi|^2\xi_2\xi_3 & 0 \\ \gamma|\xi|^2\xi_1\xi_3 & \gamma|\xi|^2\xi_2\xi_3 & c_2^2 l_2^2|\xi|^4 + \gamma|\xi|^2\xi_3^2 & 0 \\ 0 & 0 & 0 & K|\xi|^2 \end{bmatrix},$$

where $\gamma = c_1^2 l_1^2 - c_2^2 l_2^2, K = \frac{k}{c}$.

It is well known that the boundary problem (1-2) is elliptic if the pair $(P_0(D_x), \tilde{B})$ fulfils an algebraic complementing boundary condition (see [1]). In ours case the operator P_0 is elliptic in the sense of Agmon, Douglis, and Nirenberg. The boundary problem $P_0(D_x)V = F$ on Ω, $\tilde{B}(D_x)V_{|\partial\Omega} = G$ splits into two independent problems:

$$A(D_x)u = f \qquad \text{on} \quad \Omega,$$

$$u_{|\partial\Omega} = u^0, \frac{\partial u}{\partial n} = u^1 \qquad \text{on} \quad \partial\Omega, \quad \text{orbrifly } B(D_x)u_{\partial\Omega} = g,$$

and the Dirichlet problem for the Laplace operator.
The symbol of the operator $A(D_x)$ is

$$A(\xi) = c_2^2 l_2^2 |\xi|^4 I + \gamma|\xi|^2 \xi \otimes \xi,$$

where $I = id, \xi \otimes \xi = \{\xi_i \xi_j\}$. Of course this second problem is elliptic. Let $\xi = s + \lambda n$, where $s \neq 0$ is any tangent and n ($|n| = 1$) the normal to $\partial\Omega$, at x. We have

$$B(\xi) = \{B_{hj}\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix},$$

$$\det A(\xi) = \beta|\xi|^{12} = \beta(\lambda - i|s|)^6(\lambda + i|s|)^6, \qquad 0 \neq \beta = \text{const}.$$

Denote by M^+ the expression $(\lambda - i|s|)^6$. Note that $\bar{A}^t(\xi) = A(\xi) = \{A^{jk}\}$. Let us regard M^+ and the elements of the matrix BA as polynomials in the indeterminate λ.
According to the complementing boundary condition, the rows of latter matrix are required to be linearly independent modulo M^+, i.e.,

$$\sum_{h=1}^{6} c_h \sum_{j=1}^{3} B_{hj} A^{jk} \equiv 0 (\text{mod } M^+),$$

only if the constants c_h are all zero.
Reduce the polynomials (in λ) $\sum_{j=1}^{3} B_{hj} A^{jk}$ modulo M^+. We have

$$\sum_{j=1}^{3} B_{hj} A^{jk} \equiv \sum_{\beta=0}^{5} q_h^{k\beta} \lambda^\beta (\text{mod } M^+).$$

Then, construct the matrix $Q = \{q_h^{k\beta}\}$ having 6 rows: $h = 1, 2, ..., 6$, and 18 columns: $\beta = 0, ..., 5, k = 1, 2, 3$. Under the complementing boundary condition, the rank of Q will be 6.

$$BA = \begin{bmatrix} A \\ \lambda A \end{bmatrix},$$

Since
$\alpha|\xi|^4+\gamma|\xi|^2\xi_k^2=\alpha(s^2+\lambda^2)^2+\gamma(s^2+\lambda^2)(s_k+\lambda n_k)^2=(\alpha+\gamma n_k^2)\lambda^4+$
$+2\gamma s_k n_k\lambda^3+(2\alpha s^2+\gamma s^2 n_k^2+\gamma s_k^2)\lambda^2+2\gamma s^2 s_k n_k\lambda+\alpha s^4+\gamma s_k^2 s^2$,
where $\alpha=c_2^2 l_2^2$,
and $\gamma(s^2+\lambda^2)(s_i+\lambda n_i)(s_j+\lambda n_j)=\gamma n_i n_j\lambda^4+\gamma(s_i n_j+n_i s_j)\lambda^3+$
$+\gamma(s^2 n_i n_j+s_i s_j)\lambda^2+\gamma s^2(s_i n_j+n_i s_j)\lambda+\gamma s_i s_j s^2$.
The matrix Q has the following form: $Q=[A_1,...,A_6]$, where A_i $(i=$ $=1,...,6)$ are the matrices as follows:

$$A_1=\begin{bmatrix} 0 & \alpha+\gamma n_1^2 & 2\gamma s_1 n_1 \\ 0 & \gamma n_1 n_2 & \gamma(s_1 n_2+n_1 s_2) \\ 0 & \gamma n_1 n_3 & \gamma(s_1 n_3+n_1 s_3) \\ \alpha+\gamma n_1^2 & 2\gamma s_1 n_1 & 2\alpha s^2+\gamma s^2 n_1^2+\gamma s_1^2 \\ \gamma n_1 n_2 & \gamma(s_1 n_2+n_1 s_2) & \gamma(s^2 n_1 n_2+s_1 s_2) \\ \gamma n_1 n_3 & \gamma(s_1 n_3+n_1 s_3) & \gamma(s^2 n_1 n_3+s_1 s_3) \end{bmatrix},$$

$$A_2=\begin{bmatrix} 2\alpha s^2+\gamma s^2 n_1^2+\gamma s_1^2 & 2\gamma s^2 s_1 n_1 & \alpha s^4+\gamma s_1^2 s^2 \\ \gamma(s^2 n_1 n_2+s_1 s_2) & \gamma s^2(s_1 n_2+n_1 s_2) & \gamma s_1 s_2 s^2 \\ \gamma(s^2 n_1 n_3+s_1 s_3) & \gamma s^2(s_1 n_3+n_1 s_3) & \gamma s_1 s_3 s^2 \\ 2\gamma s^2 s_1 n_1 & \alpha s^4+\gamma s_1^2 s^2 & 0 \\ \gamma s^2(s_1 n_2+n_1 s_2) & \gamma s_1 s_2 s^2 & 0 \\ \gamma s^2(s_1 n_3+n_1 s_3) & \gamma s_1 s_3 s^2 & 0 \end{bmatrix},$$

$$A_3=\begin{bmatrix} 0 & \gamma n_1 n_2 & \gamma(s_1 n_2+n_1 s_2) \\ 0 & \alpha+\gamma n_2^2 & 2\gamma s_2 n_2 \\ 0 & n_2 n_3 & \gamma(s_2 n_3+n_2 s_3) \\ \gamma n_1 n_2 & \gamma(s_1 n_2+n_1 s_2) & \gamma(s^2 n_1 n_2+s_1 s_2) \\ \alpha+\gamma n_2^2 & 2\gamma s_2 n_2 & 2\alpha s^2+\gamma s^2 n_2^2+\gamma s_2^2 \\ \gamma n_2 n_3 & \gamma(s_2 n_3+n_2 s_3) & \gamma(s^2 n_2 n_3+s_2 s_3) \end{bmatrix},$$

$$A_4=\begin{bmatrix} \gamma(s^2 n_1 n_2+s_1 s_2) & \gamma s^2(s_1 n_2+n_1 s_2) & \gamma s_1 s_2 s^2 \\ 2\alpha s^2+\gamma s^2 n_2^2+\gamma s_2^2 & 2\gamma s^2 s_2 n_2 & \alpha s^4+\gamma s_2^2 s^2 \\ \gamma(s^2 n_2 n_3+s_2 s_3) & \gamma s^2(s_2 n_3+n_2 s_3) & \gamma s_2 s_3 s^2 \\ \gamma s^2(s_1 n_2+n_1 s_2) & \gamma s_1 s_2 s^2 & 0 \\ 2\gamma s^2 s_2 n_2 & \alpha s^4+\gamma s_2^2 s^2 & 0 \\ \gamma s^2(s_2 n_3+n_2 s_3) & \gamma s_2 s_3 s^2 & 0 \end{bmatrix},$$

$$A_5=\begin{bmatrix} 0 & \gamma n_1 n_3 & \gamma(s_1 n_3+n_1 s_3) \\ 0 & \gamma n_2 n_3 & \gamma(s_2 n_3+n_2 s_3) \\ 0 & \alpha+\gamma n_3^2 & 2\gamma s_3 n_3 \\ \gamma n_1 n_3 & \gamma(s_1 n_3+n_1 s_3) & \gamma(s^2 n_1 n_3+s_1 s_3) \\ \gamma n_2 n_3 & \gamma(s_2 n_3+n_2 s_3) & \gamma(s^2 n_2 n_3+s_2 s_3) \\ \alpha+\gamma n_3^2 & 2\gamma s_3 n_3 & 2\alpha s^2+\gamma s^2 n_3^2+\gamma s_3^2 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} \gamma(s^2 n_1 n_3 + s_1 s_3) & \gamma s^2(s_1 n_3 + n_1 s_3) & \gamma s_1 s_3 s^2 \\ \gamma(s^2 n_2 n_3 + s_2 s_3) & \gamma s^2(s_2 n_3 + n_2 s_3) & \gamma s_2 s_3 s^2 \\ 2\alpha s^2 + \gamma s^2 n_3^2 + \gamma s_3^2 & 2\gamma s^2 s_3 n_3 & \alpha s^4 + \gamma s_3^2 s^2 \\ \gamma s^2(s_1 n_3 + n_1 s_3) & \gamma s_1 s_3 s^2 & 0 \\ \gamma s^2(s_2 n_3 + n_2 s_3) & \gamma s_2 s_3 s^2 & 0 \\ 2\gamma s^2 s_3 n_3 & \alpha s^4 + \gamma s_3^2 s^2 & 0 \end{bmatrix}.$$

The rank of the matrix Q is equal to six. Indeed. Let us consider the matrix

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha + \gamma n_1^2 & \gamma n_1 n_2 & \gamma n_1 n_3 \\ \gamma n_1 n_2 & \alpha + \gamma n_2^2 & \gamma n_2 n_3 \\ \gamma n_1 n_3 & \gamma n_2 n_3 & \alpha + \gamma n_3^2 \end{bmatrix},$$

$$\det \begin{bmatrix} \alpha + \gamma n_1^2 & \gamma n_1 n_2 & \gamma n_1 n_3 \\ \gamma n_1 n_2 & \alpha + \gamma n_2^2 & \gamma n_2 n_3 \\ \gamma n_1 n_3 & \gamma n_2 n_3 & \alpha + \gamma n_3^2 \end{bmatrix} = \alpha^2(\alpha + \gamma),$$

is different from zero, therefore the column-vectors of the matrix M_1 are linearly independent. Similarly the column-vectors of the matrix

$$M_2 = \begin{bmatrix} \alpha s^4 + \gamma s_1^2 s^2 & \gamma s_1 s_2 s^2 & \gamma s_1 s_3 s^2 \\ \gamma s_1 s_2 s^2 & \alpha s^4 + \gamma s_2^2 s^2 & \gamma s_2 s_3 s^2 \\ \gamma s_1 s_3 s^2 & \gamma s_2 s_3 s^2 & \alpha s^4 + \gamma s_3^2 s^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

are linearly independent.

$$\det \begin{bmatrix} \alpha s^4 + \gamma s_1^2 s^2 & \gamma s_1 s_2 s^2 & \gamma s_1 s_3 s^2 \\ \gamma s_1 s_2 s^2 & \alpha s^4 + \gamma s_2^2 s^2 & \gamma s_2 s_3 s^2 \\ \gamma s_1 s_3 s^2 & \gamma s_2 s_3 s^2 & \alpha s^4 + \gamma s_3^2 s^2 \end{bmatrix} = s^{12}\alpha^2(\alpha + \gamma) \neq 0.$$

Corollary. *Let $H^s(G)$ be the Sobolev space with the norm $|| \cdot ||_{s,G}$, $s > 3\frac{1}{2}$, $t > 1\frac{1}{2}$. The operator*

$$(P, \tilde{B}) : H^s(\Omega) \times H^t(\Omega) \\ \to H^{s-4}(\Omega) \times H^{t-2}(\Omega) \times H^{s-\frac{1}{2}}(\partial\Omega) \times H^{s-\frac{3}{2}}(\partial\Omega) \times H^{t-\frac{1}{2}}(\partial\Omega),$$

$$(u, \theta) \to (f, g, u^0, u^1, \theta_0)$$

is a Fredholm operator.

If f, g, u^0, u^1, θ_0 are C^∞ and (u, θ) satisfy (1-2), then $(u, \theta) \in C^\infty$.
For every solution of (1-2)

$$||u||_{s,\Omega} + ||\theta||_{t,\Omega} \leq C(||f||_{s-4,\Omega} + ||g||_{s-2,\Omega} + ||u^0||_{s-\frac{1}{2},\partial\Omega} + ||u^1||_{s-\frac{3}{2},\partial\Omega} + ||\theta_0||_{t-\frac{1}{2},\partial\Omega} + ||u||_{s-1,\Omega} + ||\theta||_{t-1,\Omega}),$$

where $C = C(s, t)$=const.

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O pewnym zagadnieniu brzegowym dla układu równań różniczkowych cząstkowych opisujących termosprężystość materiałów złożonych

Streszczenie. W pracy rozważamy zagadnienie Dirichleta dla równań teorii termosprężystości materiałów złożonych. Pokazaliśmy, że zagadnienie to generuje operator fredholmowski działający pomiędzy odpowiednimi przestrzeniami Sobolewa.
Słowa kluczowe: zagadnienie Dirichleta, termosprężystość, przestrzenie Sobolewa, materiały złożone