Opuscula Math. 36, no. 4 (2016), 481–488
http://dx.doi.org/10.7494/OpMath.2016.36.4.481 Opuscula Mathematica

ON THE BAIRE CLASSIFICATION OF CONTINUOUS MAPPINGS DEFINED ON PRODUCTS OF SORGENFREY LINES

Olena Karlova and Olga Fodchuk

Communicated by P.A. Cojuhari

Abstract. We study the Baire measurability of functions defined on $\mathbb{R}^T$ which are continuous with respect to the product topology on a power $\mathbb{S}^T$ of Sorgenfrey lines.

Keywords: Baire-one function, Sorgenfrey line, equiconnected space.

Mathematics Subject Classification: 26A21, 54C05.

1. INTRODUCTION

The collection of all continuous mappings between topological spaces X and Y will be denoted by $\mathrm{C}(X, Y)$. A mapping $f : X \to Y$ belongs to the *first Baire class*, $f \in \mathrm{B}_1(X, Y)$, if there exists a sequence $(f_n)_{n=1}^{\infty}$ of mappings from $\mathrm{C}(X, Y)$ which is convergent to f pointwise on X.

The Sorgenfrey line $\mathbb{S}$ is the set of all reals equipped with the topology $\mathcal{S}$ generated by the basis of all half-open intervals $[a, b)$. Since the topology $\mathcal{S}$ is finer than the standard topology $\mathcal{E}$ on the real line, each continuous function $f : \mathbb{R} \to Y$ with values in arbitrary space Y is continuous in the topology $\mathbb{S}$ too. It is easy to see that the converse proposition is not true: the characteristic function $\chi_{[0,1)}$ of the half-open interval $[0, 1)$ is continuous in $\mathcal{S}$ but is discontinuous at the points $x = 0$ and $x = 1$ in the topology $\mathcal{E}$; at the same time is is easy to see that $\chi_{[0,1)}$ belongs to the first Baire class on $\mathbb{R}$. Bade [1] proved that each real-valued continuous function on $\mathbb{S}^2$ belongs to the first Baire class in the topology $\mathcal{E}$. Moreover, Bade noticed that Mrówka [7] obtained the inclusion $\mathrm{C}(\mathbb{S}^{\mathfrak{n}}, \mathbb{R}) \subseteq \mathrm{B}_1(\mathbb{R}^{\mathfrak{n}}, \mathbb{R})$ for every cardinal $\mathfrak{n}$. Since all functions in the above-mentioned results take values in the real line, it is natural to consider other range spaces which lead to the following questions.

Question 1.1. Let T be a set and $f : \mathbb{S}^T \to Y$ be a continuous mapping. Does the inclusion $f \in \mathrm{B}_1(\mathbb{R}^T, Y)$ hold if

a) Y is a topological vector space,
b) Y is a locally convex space,
c) Y is a metrizable topological vector space?

In this paper we show that the answer to Question 1.1c) is positive for any T; the answer to b) is positive for $|T| \le \aleph_0$; and the answer to Question 1.1a) is positive in the case $|T| < \aleph_0$.

2. CLASSIFICATION OF MAPPINGS ON QUARTER-STRATIFIABLE SPACES

Definition 2.1. A topological space X is said to be *equiconnected* if there exists a continuous function $\lambda : X \times X \times [0,1] \to X$ such that

(i) $\lambda(x, y, 0) = x$;
(ii) $\lambda(x, y, 1) = y$;
(iii) $\lambda(x, x, t) = x$

for all $x, y \in X$ and $t \in [0, 1]$.

The class of all equiconnected spaces contains the class of all topological vector spaces: the equality $\lambda(x, y, t) = (1 - t)x + ty$ for $x, y \in X$ and $t \in [0, 1]$ defines the required continuous function.

Now we recall the concept of the λ-sum in an equiconnected space (X, λ) (see [4]). For every $n \in \mathbb{N}$ we put

$$S_n = \{(\alpha_k)_{k=1}^n \in \mathbb{R}^n : \alpha_1 + \dots + \alpha_n = 1, \quad \alpha_1, \dots, \alpha_n \ge 0\}.$$

We define inductively a sequence of mappings $\lambda_n : X^n \times S_n \to X$. For $n = 1$ we set

$$\lambda_1(x, 1) = x$$

for all $x \in X$. If $n \in \mathbb{N}$, $x_1, \dots, x_{n+1} \in X$ and $(\alpha_1, \dots, \alpha_{n+1}) \in S_{n+1}$, then we set

$$\begin{aligned} &\lambda_{n+1}(x_1, \dots x_{n+1}, \alpha_1, \dots, \alpha_{n+1}) \\ &= \lambda_n\Big(\lambda\Big(x_1, x_2, \frac{\alpha_2}{\alpha_1 + \alpha_2}\Big), x_3, \dots x_{n+1}, \alpha_1 + \alpha_2, \alpha_3 \dots, \alpha_{n+1}\Big), \end{aligned}$$

in the case $\alpha_1 + \alpha_2 > 0$, and

$$\lambda_{n+1}(x_1, \dots x_{n+1}, \alpha_1, \dots, \alpha_{n+1}) = \lambda_n(x_2, x_3, \dots x_{n+1}, \alpha_2, \alpha_3 \dots, \alpha_{n+1}),$$

in the case $\alpha_1 + \alpha_2 = 0$.

Definition 2.2. For any $n \in \mathbb{N}$, $(\alpha_1, \dots, \alpha_n) \in S_n$ and for any $x_1, \dots x_n$ from an equiconnected space (X, λ) the element $\lambda_n(x_1, \dots x_n, \alpha_1, \dots, \alpha_n)$ is called *the convex combination of elements $x_1, \dots x_n$ with coefficients $\alpha_1, \dots, \alpha_n$*.

Definition 2.3. Let $(I, \leq)$ be a completely ordered set, $(x_i)_{i\in I}$ be a family of points of an equiconnected space (X, λ) and let $(\alpha_i)_{i\in I}$ be a collection of non-negative scalars with

(1) $\{i \in I : \alpha_i \neq 0\} = \{i_k : 1 \leq k \leq n\}$;
(2) $i_1 < i_2 < \ldots < i_n$;
(3) $\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n} = 1$.

Then $\lambda_n(x_{i_1}, \ldots, x_{i_n}, \alpha_{i_1}, \ldots, \alpha_{i_n})$ is called *the λ-sum of $(x_i)_{i\in I}$ with the coefficients* $(\alpha_i)_{i\in I}$ and is denoted by ${\sum_{i\in I}}^{\lambda}\alpha_i x_i$.

We observe that ${\sum_{i\in I}}^{\lambda}\alpha_i x_i = \sum_{i\in I}\alpha_i x_i$ for any topological vector space X.
If A is a subset of an equiconnected space (X, λ), then

$$\lambda^0(A) = A, \quad \lambda^n(A) = \lambda(\lambda^{n-1}(A), A, [0,1]) \text{ for } n \in \mathbb{N},$$
$$\lambda^\infty(A) = \bigcup_{n=1}^{\infty} \lambda^n(A).$$

Definition 2.4. An equiconnected space (X, λ) is *locally convex* [2] if for any $x \in X$ and a neighborhood U of x there is a neighborhood V of x such that $\lambda^\infty(V) \subseteq U$.

Definition 2.5. A topological space $(X, \mathcal{S})$ is called *metrically quarter-stratifiable* (see [2, Definition 2.1] and [2, Theorem 2.2]) if it admits a weaker metrizable topology τ (called its *stratifying topology*) with a sequence of τ-open coverings $\mathcal{U}_n = (U_{i,n} : i \in I_n)$ of X and a sequence $((x_{i,n} : i \in I_n))_{n=1}^\infty$ of families of points from X such that for every $x \in X$ we have

$$\forall (i_n)_{n=1}^\infty \left(x \in \bigcap_{n=1}^{\infty} U_{i_n,n} \implies x_{i_n,n} \to x \right). \tag{2.1}$$

Let us notice that (2.1) is equivalent to the following:

$$\begin{aligned} \forall\, U - \text{a neighborhood of } x \text{ in } (X, \mathcal{S}) \ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ \forall i \in I_n \\ (x \in U_{i,n} \implies x_{i,n} \in U). \end{aligned} \tag{2.2}$$

Indeed, it is easy to see that (2.2)$\Rightarrow$(2.1). Now let $x \in X$ and (2.1) holds. Assume that (2.2) is not valid and take a neighborhood U of x, an increasing sequence $(k_n)_{n=1}^\infty$ of numbers and a sequence $(j_n)_{n=1}^\infty$ of indexes $j_n \in I_{k_n}$ such that $x \in U_{j_n,k_n}$ but $x_{j_n,k_n} \notin U$ for every $n \in \mathbb{N}$. Since $(U_{i,n} : i \in I_n)$ is a covering of X for every n, we choose $s_n \in I_n$ such that $x \in U_{s_n,n}$. For all $n \in \mathbb{N}$ we set $i_n = j_m$ if $n = k_m$ for some $m \in \mathbb{N}$ and $i_n = s_n$, otherwise. Then the sequence $(i_n)_{n=1}^\infty$ does not satisfy the implication from (2.1), which implies a contradiction.

Definition 2.6. A topological space X is *strongly countably dimensional* if there exist a sequence $(X_n)_{n=1}^\infty$ of closed subspaces of X such that $X = \bigcup_{n=1}^\infty X_n$ and $\dim X_n < \infty$ for every $n \in \mathbb{N}$.

Theorem 2.7. *Let $(X,\mathcal{S})$ be a metrically quarter-stratifiable space with its stratifying metrizable topology τ, (Y,λ) be an equiconnected space. If one of the conditions holds*

(i) (X,τ) *is strongly countably dimensional, or*
(ii) Y *is locally convex,*

then

$$\mathrm{C}((X,\mathcal{S}),Y)\subseteq \mathrm{B}_1((X,\tau),Y).$$

Proof. Let $f:(X,\mathcal{S})\to Y$ be a continuous mapping.

(i) Let $(X_m)_{m=1}^\infty$ be a sequence of closed subspaces of (X,τ) such that $X=\bigcup_{m=1}^\infty X_m$ and $\dim X_m<\infty$ for every $m\in\mathbb{N}$. Since X is metrically quarter-stratifiable, we take a sequence $((U_{i,n}:i\in I_n))_{n=1}^\infty$ of τ-open coverings of X and a sequence $((x_{i,n}:i\in I_n))_{n=1}^\infty$ of families of points from X. By [3, Theorem 5.1.10], for every $n\in\mathbb{N}$ we may choose a locally finite refinement $\mathcal{V}_n=(V_{j,n}:j\in J_n)$ of the open covering $(U_{i,n}:i\in I_n)$ of the paracompact strongly countably dimensional space (X,τ) such that for all $m\in\mathbb{N}$ and $x\in X_m$ there exists a neighborhood U of x with

$$|\{j\in J_n: U\cap V_{j,n}\neq\emptyset\}|\leq m.$$

Fix $n\in\mathbb{N}$ and take a locally finite partition of unity $(\varphi_{j,n}:j\in J_n)$ on (X,τ) subordinated to $\mathcal{V}_n$. For every $j\in J_n$ we denote by $u_{j,n}$ an element $x_{i,n}$ with $V_{j,n}\subseteq U_{i,n}$. Assume that the set J_n is completely ordered and define $f_n:X\to Y$ by the formula

$$f_n(x)=\sum_{j\in J_n}{}^{\lambda}\varphi_{j,n}(x)f(u_{j,n}). \tag{2.3}$$

It follows from [4, Theorem 3.2] that f_n is continuous on (X,τ).

We prove now that $f_n(x)\to f(x)$ for every $x\in X$. Fix $x_0\in X$ and consider a neighborhood W of $y_0=f(x_0)$ in Y. Let m be a number such that

$$|\{j\in J_n: x_0\in V_{j,n}\}|\leq m$$

for every $n\in\mathbb{N}$. The continuity of λ_m, the equality

$$\lambda_m(y_0,\dots,y_0,\alpha_1,\dots,\alpha_m)=y_0$$

for all $(\alpha_1,\dots,\alpha_m)\in S_m$ and compactness of S_m imply the existence of a neighborhood W_1 of y_0 in Y with

$$\lambda_m(y_1,\dots,y_m,\alpha_1,\dots,\alpha_m)\in W$$

for all $y_1,\dots,y_m\in W_1$ and $(\alpha_1,\dots,\alpha_m)\in S_m$. Since the mapping f is continuous at the point x_0, there exists an $\mathcal{S}$-open neighborhood U of x_0 such that $f(x)\in W_1$ for every $x\in U$. Using (2.2) we choose a number n_0 such that $u_{j,n}\in U$ for all $n\geq n_0$ and $j\in J_n$ with $x_0\in V_{j,n}$.

We show that $f_n(x_0)\in W$ for every $n\geq n_0$. Let $n\geq n_0$ be fixed and

$$J=\{j\in J_n: x_0\in V_{j,n}\}=\{j_1,j_2,\dots j_k\},$$

where $j_1 < j_2 < \ldots < j_k$ and $k \le m$. Denote

$$\alpha_1 = \varphi_{j_1,n}(x_0), \ldots, \alpha_k = \varphi_{j_k,n}(x_0), y_1 = f(x_{j_1,n}), \ldots, y_k = f(x_{j_k,n}).$$

Then $y_1, \ldots y_k \in W_1$. Hence,

$$f_n(x_0) = \lambda_k(y_1, \ldots, y_k, \alpha_1, \ldots \alpha_k) = \lambda_m(y_1, \ldots, y_k, y_0, \ldots, y_0, \alpha_1, \ldots \alpha_k, 0, \ldots, 0) \in W$$

by [4, Proposition 2.3].

(ii) We consider sequences $((U_{i,n} : i \in I_n))_{=1}^{\infty}$ of τ-open coverings of X and $((x_{i,n} : i \in I_n))_{n=1}^{\infty}$ of families of points from X such that (2.2) holds. Since (X, τ) is a paracompact space, for every $n \in \mathbb{N}$ we take a locally finite refinement $(V_{j,n} : j \in J_n)$ of $(U_{i,n} : i \in I_n)$. For every $j \in J_n$ by $u_{j,n}$ we denote a point $x_{i,n}$ such that $V_{j,n} \subseteq U_{i,n}$. Let $(\varphi_{j,n} : j \in J_n)$ be a locally finite partition of unity on (X, τ) subordinated to the covering $(V_{j,n} : j \in J_n)$. For every $n \in \mathbb{N}$ and $x \in X$ we define a mapping $f_n : X \to Y$ by the equality (2.3) and observe that f_n is continuous on (X, τ) by [4, Theorem 3.2].

It remains to show that $f_n(x) \to f(x)$ on X. Fix $x_0 \in X$ and a neighborhood W of $y_0 = f(x_0)$. Since Y is locally convex, there exists a neighborhood W_1 of y_0 in Y such that

$$\lambda^{\infty}(W_1) \subseteq W.$$

Since f is continuous on the space $(X, \mathcal{S})$ at x_0, we may choose a neighborhood U of x_0 in $(X, \mathcal{S})$ such that $f(x) \in W_1$ for every $x \in U$. Using (2.2) we take a number n_0 with $u_{j,n} \in U$ for all $n \ge n_0$ and $j \in J_n$ with $x_0 \in V_{j,n}$. In order to show that $f_n(x_0) \in W$ for all $n \ge n_0$ we fix $n \ge n_0$ and set

$$J = \{j \in J_n : x_0 \in V_{j,n}\} = \{j_1, j_2, \ldots, j_k\},$$

where $j_1 < j_2 < \ldots < j_k$. Denote

$$\alpha_1 = \varphi_{j_1,n}(x_0), \ldots, \alpha_k = \varphi_{j_k,n}(x_0), y_1 = f(x_{j_1,n}), \ldots, y_k = f(x_{j_k,n}).$$

Then $y_1, \ldots y_k \in W_1$ and

$$f_n(x_0) = \lambda_k(y_1, ..., y_k, \alpha_1, ..., \alpha_k) \in \lambda^k(W_1) \subseteq W.$$

Therefore, $f \in \mathrm{B}_1((X, \tau), Y)$. □

3. MAPPINGS ON A PRODUCT OF THE SORGENFREY LINES

Lemma 3.1. *For any at most countable set T the product $\mathbb{S}^T$ is metrically quarter-stratifiable and a topology τ from Definition 2.5 is the Tykhonoff topology on $\mathbb{R}^T$.*

Proof. We consider the case $|T| = \aleph_0$. For all $n, k \in \mathbb{N}$ we set $I_{n,k} = \mathbb{Z}$, $I_n = \prod_{k=1}^{\infty} I_{n,k} = \mathbb{Z}^{\aleph_0}$ and $X_n = \mathbb{R}$. Now for all $n \in \mathbb{N}$ and $i = (i_k)_{k=1}^{\infty} \in I_n$ we put

$$U_{i,n} = \prod_{k=1}^{n} \Big(\frac{i_k - 1}{n}, \frac{i_k + 1}{n}\Big) \times \prod_{k=n+1}^{\infty} X_k,$$

$$x_{i,n} = \Big(\frac{i_1 + 1}{n}, \ldots, \frac{i_n + 1}{n}, 0, 0, \ldots\Big).$$

We verify that the defined sequences $((U_{i,n} : i \in I_n))_{n=1}^{\infty}$ of open coverings of $\mathbb{R}^{\aleph_0}$ and points $((x_{i,n} : i \in I_n))_{n=1}^{\infty}$ satisfy the properties indicated in Definition 2.5. Let $x = (\xi_1, \xi_2, \dots) \in \mathbb{R}^{\aleph_0}$ and let $U = U_1 \times \dots \times U_m \times X_{m+1} \times \dots$ be a basic neighborhood of x in $\mathbb{S}^{\aleph_0}$. For every $k = 1, \dots, m$ we take $n_k \in \mathbb{N}$ such that for all $n \geq n_k$ and $i_k \in I_{n,k}$ the inclusion $\xi_k \in (\frac{i_k-1}{n_k}, \frac{i_k+1}{n_k})$ implies that $\frac{i_k+1}{n_k} \in U_k$. We set $n_0 = \max\{m, n_1, \dots, n_m\}$. Assume that $n \geq n_0$ and $i \in I_n$ be such that $x \in U_{i,n}$. Then, obviously, $x_{i,n} \in U$. □

Theorem 2.7 and Lemma 3.1 immediately imply the following result.

Theorem 3.2. *Let Y be an equiconnected space. Then*

$$\mathrm{C}(\mathbb{S}^n, Y) \subseteq \mathrm{B}_1(\mathbb{R}^n, Y)$$

for any $n \in \mathbb{N}$. If Y is a locally convex equiconnected space, then

$$\mathrm{C}(\mathbb{S}^{\aleph_0}, Y) \subseteq \mathrm{B}_1(\mathbb{R}^{\aleph_0}, Y).$$

Theorem 3.3. *Let Y be a metrizable connected and locally arcwise connected space. Then*

$$\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$$

for any at most countable set T.

Proof. Consider a continuous mapping $f : \mathbb{S}^T \to Y$, where $|T| \leq \aleph_0$. Take an arbitrary open set $V \subseteq Y$ and a continuous function $g : Y \to \mathbb{R}$ with $V = g^{-1}((0, +\infty))$. Let $h = g \circ f$. Then $h \in \mathrm{B}_1(\mathbb{R}^T, \mathbb{R})$ by Theorem 3.2. Consequently, the preimage $f^{-1}(V) = h^{-1}((0, +\infty))$ is an F_σ-set in $\mathbb{R}^T$. Since the space $Z = f(\mathbb{S}^T)$ is separable as a continuous image of the separable space $\mathbb{S}^T$, Theorem 1 from [5] implies that $f \in \mathrm{B}_1(\mathbb{R}^T, Y)$. □

Theorem 3.4. *Let Y be a metrizable connected and locally arcwise connected space. Then*

$$\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$$

for any set T.

Proof. The case $|T| \leq \aleph_0$ is considered in Theorem 3.3.

Assume that $|T| > \aleph_0$ and let $f : \mathbb{S}^T \to Y$ be a continuous mapping. By [6] there exists a countable set $T_0 \subseteq T$ such that for all $x, u \in \mathbb{S}^T$ the equality $x|_{T_0} = u|_{T_0}$ implies the equality $f(x) = f(u)$. Define the continuous mapping $\varphi : \mathbb{S}^T \to \mathbb{S}^{T_0}$ by the rule $\varphi(x) = x|_{T_0}$ for all $x \in \mathbb{S}^T$ and put $g(u) = f(x)$ if $u = \varphi(x)$ for some $x \in \mathbb{S}^T$. Observe that the mapping $g : \mathbb{S}^{T_0} \to Y$ is well-defined and continuous. According to Theorem 3.3 we have $g \in \mathrm{B}_1(\mathbb{R}^{T_0}, Y)$. Take a sequence of continuous mappings $g_n : \mathbb{R}^{T_0} \to Y$ which is convergent to g on $\mathbb{R}^{T_0}$ pointwise. For all $n \in \mathbb{N}$ and $x \in \mathbb{R}^T$ we put $h_n(x) = g_n(\varphi(x))$. Then

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} g_n(\varphi(x)) = g(\varphi(x)) = f(x).$$

Hence, $f \in \mathrm{B}_1(\mathbb{R}^T, Y)$. □

Theorems 3.2–3.4 imply the ensuing corollary.

Corollary 3.5. *Let Y be a topological vector space. Then the inclusion $\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$ is valid if one of the following conditions hold:*

a) $|T| < \aleph_0$,
b) *Y is a locally convex space and $|T| \leq \aleph_0$,*
c) *Y is metrizable.*

The following question is open.

Question 3.6. Does the inclusion $\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$ hold for $|T| = \aleph_0$ and any topological vector space Y?

REFERENCES

[1] W. Bade, *Two properties of the Sorgenfrey plane*, Pacif. J. Math. (1971), 349–354.

[2] T. Banakh, *(Metrically) quarter-stratifiable spaces and their applications*, Math. Stud. **18** (2002) 2, 10–28.

[3] R. Engelking, *Theory of Dimensions, Finite and Infinite. Revised and Completed Edition*, Heldermann Verlag, Lemgo, 1995.

[4] O. Karlova, V. Maslyuchenko, V. Mykhaylyuk, *Equiconnected spaces and Baire classification of separately continuous functions and their analogs*, Cent. Eur. J. Math. **10** (2012) 3, 1042–1053.

[5] O. Karlova, V. Mykhaylyuk, *Functions of the first Baire class with values in metrizable spaces*, Ukr. Math. J. **58** (2006) 4, 567–571 [in Ukrainian].

[6] S. Mazur, *On continuous mappings on Cartesian products*, Fund. Math. 39 (1952), 229–238.

[7] S. Mrówka, *Some problems related to N-compact spaces*, preprint.

Olena Karlova (corresponding author)
maslenizza.ua@gmail.com

Chernivtsi National University
Faculty of Mathematics and Informatics
Department of Mathematical Analysis
Kotsyubyns'koho str., 2, Chernivtsi, Ukraine

Olga Fodchuk

Chernivtsi National University
Faculty of Mathematics and Informatics
Department of Mathematical Analysis
Kotsyubyns'koho str., 2, Chernivtsi, Ukraine

Received: August 26, 2015.
Revised: January 26, 2016.
Accepted: January 28, 2016.