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Reliability concept under general uncertainty

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Abstract

The Toyota crisis is tearing off the brand image of quality and reliability and therefore it is logical to question whether the dominating position of probability theory, on which Japanese quality and reliability engineering practices are established, should be examined. In general, reliability analysis is an exercise under uncertain environment. Foundationally speaking, uncertain modeling is a matter of choosing what kind of uncertain measure as its standing point. In this paper, we introduce the uncertainty reliability concept on the platform of the axiomatic uncertain measure theory and compare it to probabilistic reliability concept based on Kolmogorov's probability measure theory, on which the traditional quality and reliability engineering is established. It is expecting that a foundational work can be established for a more rigorous reliability engineering and risk analysis under general uncertainty environments.

1. Introduction

Recent Toyota crisis does not only breaking down the image of Toyota brand name but also shaking the "Made in Japan" image of quality and reliability. It is unfair to blame Japanese companies ignored quality and reliability problems because the essential DNA chain of Japanese manufacturing is quality and reliability engineering. However, "Made in Japan" crisis went widely spreading, including Toyota Prius brake fault problem, Honda Jazz electric window-sparking fire problem, Sony Camera problem, Passenger seat problem for Boeing 747 etc, have shocked business, industries, etc. According to economist opinion, just the "Toyota recall" event may decrease Japan GDP up to 0.12%. We, as active members of quality and reliability research communities, should not laugh at Japanese colleagues rather keep calm mind to re-examine whether the theoretical foundation of engineering is imperfect or not. It is well-known that quality and reliability engineering is established on [9] probability measure theory because the randomness is the only form of uncertainty well-admitted.

Unfortunately, the real world is not as simple as peoples thought about. Uncertainty is intrinsic and diversified in form. For example, the vagueness is another form of uncertainty, which is more and more aware of in today's industrial environments, just as [2] commented, "In a global market, companies must deal with a high rate of changes in business environment. The parameters, variables and restrictions of the production system are inherently vagueness." Therefore quality and reliability engineering is no longer a blind exercise of applying the traditional techniques from existing probabilistic reliability engineering literature. *Without a thoroughly understanding of uncertainty, and its characteristics, the abstracting real world uncertainty into correct concepts and uncertain laws, it is inevitable to keep the engineering exercises away from the reality of safety, quality and reliability.*

As a matter of facts, the terms of randomness, fuzziness, greyness, or roughness do not reveal the supposed scientific connotations respectively. But more and more term creations cause deeper and deeper confusions. They may be replaced with A, B, C and etc without creating any confusion.
The fundamental problem here is what "measure" will be utilized to characterize the relevant concepts. For example, the phenomenon "fuzzy" exists in practice, but it was never fully and accurately quantified by either possibility measure or credibility measure. So the term "fuzzy" makes no scientific sense when you say it, because the existing measures cannot characterize it, i.e., the concept extraction process is incorrect. Even we admit "fuzzy" phenomenon, but when people call it "grey" phenomenon, you have no reason to stop them, see [11], [12].
Mathematically speaking, we would like to call them "uncertainty" because the "uncertainty" phenomenon could be quantified by uncertain measure. Measure defines an event measuring grade system for abstracting a conceptual uncertainty environment. A measure system's establishment is not a copy of a real world phenomenon, however, it is some abstraction of a real world phenomenon in certain degree and it has certain reflection of reality. Therefore a reasonable measure theory may have deep and wide applications in scientific fields. The theory of probability was a purely mathematical development with no direct links to science or reality. But the axiomatic mathematical foundation of probability eventually were picked up and applied in science.
The creation of measure theory is a human brain activity, and therefore heavily reflects the human thinking characteristic. "The law of contradiction" and "law of excluded middle" dominate human thinking and thus have the mapping in measure system specifications.
Logically, it is obvious that probabilistic modeling is only a good approximation to real world problem when randomness governs the phenomenon. If other forms of uncertainty appear, probabilistic modeling is definitely questionable. Therefore, developing the appropriate models for modeling general uncertainty including vagueness and randomness is necessary, e.g., [4]-[8] and [14], [15].

2. Probabilistic reliability concept

A fundamental question is what kind of function can define a probability measure and thus define distribution function and reliability of lifetime accordingly? To address this question, we need to start with [9] three axioms of probability.
Let Ω be a nonempty set (space), and $\mathcal{F}(\Omega)$ the σ -algebra on Ω . Each element, let us say, $A\subset\Omega$, $A\in\mathcal{F}(\Omega)$ is called an uncertain event. A number denoted as $P\{A\}$, $0\le P\{A\}\le 1$, is assigned to event $A\in\mathcal{F}(\Omega)$, which indicates the uncertain measuring grade with which event $A\in\mathcal{F}(\Omega)$ occurs. The normal set function $P\{A\}$ satisfies following axioms given by [9]:

Axiom K.1. (Nonnegativity) The probability of an event is a nonnegative real number, i.e., $P\{A\}\ge 0$, $\forall A\in\mathcal{F}$.

Axiom K.2. (Unit measure) The probability of the entire sample space is 1, i.e., $P\{\Omega\}=1$

Axiom K.3. (σ -Additivity) Any countable sequence of pairwise disjoint events $A_1,A_2,\cdots\in\mathcal{F}$ $A_i\cap A_j=\varnothing$, $(i\ne j=1,2,\cdots)$ satisfies,.

$$P\left\{\bigcup_{n=1}^{\infty}A_n\right\}=\sum_{n=1}^{\infty}P\{A_n\} \quad (1)$$

The direct consequences of the three axioms are stated as following:

Theorem 2.1. (Additive law of probability) Let $A_1,A_2\in\mathcal{F}$, then

$$P\{A_1\bigcup A_2\}=P\{A_1\}+P\{A_2\}-P\{A_1\bigcap A_2\} \quad (2)$$

Theorem 2.2. (Self-Duality) For $\forall A\in\mathcal{F}$,

$$P\{A^c\}=1-P\{A\}. \quad (3)$$

Definition 2.3. [17] Any set function $P:\mathcal{F}\to[0,1]$ satisfies *Axioms K.1-K.3* is called a probability measure. The triple $(\Omega,\mathcal{F},P)$ is called the uncertain measure space.

Definition 2.4. An random variable X is a measurable mapping, i.e., $X:(\Omega,\mathcal{F}(\Omega))\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ -algebra on $\mathbb{R}=(-\infty,+\infty)$.

Definition 2.5. A lifetime of an individual is a nonnegative random variable T which is a measurable mapping, i.e., $T:(\Omega,\mathcal{F}(\Omega))\rightarrow(\mathrm{R}^{+},\mathcal{B}(\mathrm{R}^{+}))$, where $\mathcal{B}(\mathrm{R}^{+})$ denotes the Borel σ-algebra on $\mathrm{R}^{+}=(0,+\infty]$.

To understand the measurability of a random variable, particularly, the role played by the σ-algebra $\mathcal{F}(\Omega)$, let us recall however the measurability structured for a random variable. Let $(\Omega,\mathcal{F},P)$ be a probability space and $(\mathrm{R},\mathcal{B}(\mathrm{R}))$ be a measurable space on real-line, then a real-valued function X is random variable if and only if for all $r\in\mathrm{R}$, the pre-image $\{\omega\in\Omega: X(\omega)\leq r\}\in\mathcal{F}$. For each value $r\in\mathrm{R}$ taken by a real-valued random variable X, the event $B=(-\infty,r]$ is an element of the Borel σ-algebra of real-line R, the pre-image of event B under random variable X is

$$\{\omega\in\Omega: X(\omega)\in B\}=\{\omega\in\Omega: X(\omega)\leq r\} \quad (4)$$

event $\{\omega\in\Omega: X(\omega)\leq r\}$ is an element of σ-algebra $\mathcal{F}$ of Ω, where the probability measure P defined on this set class, i.e., σ-algebra $\mathcal{F}$, i.e., $P:\mathcal{F}\rightarrow[0,1]$.

Therefore every element (event) of $\mathcal{F}$ is assigned with a probability grade, i.e., event $\{\omega\in\Omega: X(\omega)\leq r\}$ is assigned a probability grade, which is $P\{\omega\in\Omega: X(\omega)\leq r\}$.

Overall, σ-algebra $\mathcal{F}$ facilitates the formal definition of a random variable in terms of membership of the pre-image $\{\omega\in\Omega: X(\omega)\leq r\}$ to the σ-algebra $\mathcal{F}$, in which the probability measuring grade defined and every event of σ-algebra $\mathcal{F}$ is assigned. As [3] pointed, each random variable on the probability space $(\Omega,\mathcal{F},P)$ induces a probability space $(\mathrm{R},\mathcal{B}(\mathrm{R}),\mu)$ by means of the following correspondence.

$$\forall B\in\mathcal{B}(\mathrm{R}):\ \mu(B)=P\{X^{-1}(B)\}=P\{X\in B\} \quad (5)$$

Chung [3] further denotes $\mu=P\circ X^{-1}$ and specifically, the probability distribution is defined by the induced measure μ,

$$F(r)=\mu\{(-\infty,r]\}=P\{X\leq r\} \quad (6)$$

In conclusion, the random variable X defined on a given probability space $(\Omega,\mathcal{F}(\Omega),P)$ is a measurable mapping to $(\mathrm{R},\mathcal{B}(\mathrm{R}))$ and thus induces the distribution function, F, which is used to characterize the random variable.

Conversely, a nonnegative real-valued function F, given the function satisfying some conditions, a measure can be defined by F, and accordingly a random variable X on an appropriate probability space.

Definition 2.6. The function, denoted by F is a probability distribution function if and only F satisfies following three conditions:
(1) $\lim_{x\rightarrow-\infty}F(x)=0,\ \lim_{x\rightarrow+\infty}F(x)=1$;
(2) $F(x)$ is non-decreasing in x;
(3) F is right-continuous, i.e., $\forall x_0\in\mathrm{R}$, $\lim_{x\downarrow x_0}F(x)=F(x_0)$.

Definition 2.7. A function Ψ is normed if and only if it is mapped from real-line to unit interval [0,1], i.e.,.

$$\Upsilon:\mathrm{R}\rightarrow[0,1] \quad (7)$$

Remark 2.8. It is obvious that distribution is a right-continuous non-decreasing normed function defined on R.

Definition 2.9. (Ash [1]) A Lebesgue-Stieltjes measure is a set function μ on Borel σ-algebra $\mathcal{B}(\mathrm{R})$ such that $\mu\{I\}<+\infty$ for each bounded interval $I\subset\mathrm{R}$.

Definition 2.10. (Ash [1]) Let $\mathcal{F}$ be a set class of a space (set) S. Then $\mathcal{F}$ is termed as an algebra (field) if and only if $\mathrm{S}\in\mathcal{F}$ and $\mathcal{F}$ is closed under complementation and finite union.

Theorem 2.11. (Ash [1]) Let F be a right-continuous non-decreasing normed function on the compact space $\overline{\mathrm{R}}$, define

$$\mu\{(a,b]\}=F(b)-F(a),\ \forall a,b\in\overline{\mathrm{R}} \quad (8)$$

Further, define

$$\mu\left\{\bigcup_{k=1}^{n}I_k\right\}=\sum_{k=1}^{n}\mu\{I_k\},\ n>1,\ \text{integer} \quad (9)$$

for any disjoint right-semclosed interval sequence $\{I_1,I_2,\cdots,I_n\}$. Then μ is a finitely additive set function on the algebra $\mathcal{F}_0(\overline{\mathrm{R}})$ of all right-semi-closed intervals of $\overline{\mathrm{R}}$.

Lemma 2.12. [1] The set function μ is countably additive on algebra $\mathcal{F}_0(\overline{\mathrm{R}})$.

Theorem 2.13. [1] Let F be a right-continuous non-decreasing normed function defined on R, and let $\mu\{(a,b]\}=F(b)-F(a),\ \forall a,b\in\mathrm{R}$. There is a unique extension of μ to a Lebegue-Stieltjes measure on $\mathcal{B}(\mathrm{R})$.

Proof. Ash [1] showed this the in following way: let $\mathcal{F}_0(\bar{\mathrm{R}})$ be the algebra of all right-semi-closed intervals of R and extend μ to the algebra (set class of all right-semi-closed intervals of R), $\mathcal{F}_0(\mathrm{R})$. By defining the map

$$\begin{aligned}&(a,b]\to(a,b],\ \text{if } a,b\in\mathrm{R} \text{ or if } b\in\mathrm{R}, a=-\infty,\\&(a,\infty]\to(a,\infty],\ \text{if } a\in\mathrm{R} \text{ or if } a=-\infty.\end{aligned}\tag{10}$$

we establishes a one-to-one, μ -prserving correspondence between a subset of $\mathcal{F}_0(\bar{\mathrm{R}})$ and $\mathcal{F}_0(\mathrm{R})$. In terms of *Lemma 2.12*, [1] μ is countably additive on algebra $\mathcal{F}_0(\mathrm{R})$. By the Carathéodory extension theorem, μ has a unique extension to $\mathcal{B}(\mathrm{R})$. Note that

$$\mu\{(a,b]\}=F(b)-F(a),\ \forall a,b\in\mathrm{R}\tag{11}$$

According to Definition 2.9, μ is a Lebesgue-Stieltjes measure on $\mathcal{B}(\mathrm{R})$.

Remark 2.14. Let F be a distribution, then F is continuous at x if and only if $\mu\{\{x\}\}=0$, the magnitude of F at x coincides with the measure of $\{x\}$.

Remark 2.15. The functional form of μ may be

$$\mu\{I\}=\int_I f(s)ds\tag{12}$$

where f is an integrable nonnegative function and I is an arbitrary semi-closed interval of R. Eq. (8) helps to trace back the link between μ and Lebesgue-Stieltjes measure, even Lebesgue (if $f(s)=1$). (Also see [1]).

Now, we are ready to discuss probabilistic reliability concept. Let T be the lifetime (a non-negative random variable) of an individual having a distribution function F.

Definition 2.16. [10] The survival function at time t is the probability of an individual surviving till time t is

$$S(t)=\Pr\{T\geq t\}=1-F(t)\tag{13}$$

In quality engineering, $S(t)$ is referred to as the reliability function.

Definition 2.17. [10] The hazard function is defined by

$$h(t)=\lim_{\Delta t\to 0}\frac{\Pr\{t\leq T<t+\Delta t\mid T\geq t\}}{\Delta t}\tag{14}$$

As Lawless [10] pointed out, “The harzard function specifies the instantaneous rate of death or failure at time *t*, given that the individual survives up till *t*.” “All in all, the main point to be remembered is that the hazard function represents an aspect of a distribution that has direct physical meaning and that information about the nature of hazard function is helping in selecting a model.” Now, the reliability function can be fully explored on establishment of the distribution function *F*.

Theorem 2.18. The function, denoted by *S* is a reliability distribution function if and only if *S* satisfies following three conditions:
(1) $\lim_{t\to 0}S(t)=1,\ \lim_{t\to+\infty}S(t)=0$;
(2) *S*(*t*) is non-increasing in *t*;
(3) S is left-continuous, i.e., $\forall t_0\in\mathrm{R}^+$, $\lim_{t\uparrow t_0}S(t)=S(t_0)$.

Now, we are ready to explore reliability concept under general uncertainty environments by starting the uncertain measure specification in next section.

3. Uncertainty reliability concept

Uncertain measure [12], [13] is an axiomatically defined set function mapping from a σ-algebra of a given space (set) to the unit interval [0,1], which provides a measuring grade system of an uncertain phenomenon and facilitates the formal definition of an uncertain variable.

Let Ξ be a nonempty set (space), and $\mathcal{A}(\Xi)$ the σ -algebra on Ξ. Each element, let us say, $A\subset\Xi$, $A\in\mathcal{A}(\Xi)$ is called an uncertain event. A number denoted as $\lambda\{A\}$, $0\leq\lambda\{A\}\leq 1$, is assigned to event $A\in\mathcal{A}(\Xi)$, which indicates the uncertain measuring grade with which event $A\in\mathcal{A}(\Xi)$ occurs. The normal set function $\lambda\{A\}$ satisfies following axioms given by [12], [13]:

Axiom L.1. (Normality) $\lambda\{\Xi\}=1$.

Axiom L.2. (Monotonicity) $\lambda\{\cdot\}$ is non-decreasing, i.e., whenever $A \subset B$, $\lambda\{A\} \le \lambda\{B\}$.

Axiom L.3. (Self-Duality) $\lambda\{\cdot\}$ is self-dual, i.e., for any $A \in \mathcal{A}(\Xi)$, $\lambda\{A\} + \lambda\{A^c\} = 1$.

Axiom L.4. (σ- Subadditivity) $\lambda\left\{\bigcup_{i=1}^{\infty} A_i\right\} \le \sum_{i=1}^{\infty} \lambda\{A_i\}$ for any countable event sequence $\{A_i\}$.

Definition 3.1. [12], [13] Any set function $\text{D}: \mathcal{A}(\text{X}) ® [0,1]$ satisfies *Axioms L.1-L.4* is called an uncertain measure. The triple $(\text{X}, \mathcal{A}(\text{X}), \text{D})$ is called the uncertain measure space.

Definition 3.2. [12], [13] An uncertain variable ξ is a measurable mapping, i.e., $\xi : (\Xi, \mathcal{A}(\Xi)) \to (\text{R}, \mathcal{B}(\text{R}))$, where $\mathcal{B}(\text{R})$ denotes the Borel σ-algebra on $\text{R} = (-\infty, +\infty)$.

Remark 3.3. The fundamental difference between a random variable and an uncertain variable is the measure space on which they are defined. In the triples, the first two factors are similar in formation: the set and the σ- algebra on the set. However, the third factor in the triples: the measures defined on the σ- algebras are not similar. The former (i.e. the probability measure) obeys σ- additivity and the later (i.e. the uncertain measure) obeys σ- subadditivity. The way for specifying measure inevitably has impacts on the behaviour of the measurable function on the triple.

Definition 3.4. [12], [13] The uncertain distribution $\Psi : \text{R} \to [0,1]$ of an uncertain variable ξ on $(\text{X}, \mathcal{A}(\text{X}), \text{D})$ is

$$\Psi(x) = \lambda\{\tau \in \Xi \,|\, \xi(\tau) \le x\} \tag{15}$$

For the uncertain measure, as an axiomatic measure development, the set class, σ - algebra $\mathcal{A}(\Xi)$ plays the critical roles in defining set function - uncertain measure Das well as those in defining the measurability of uncertain variable. The roles are identically the same as to the roles played by a σ -algebra in probability measure development.

Definition 3.5. [12], [13] An n-dimensional uncertain vector from an uncertain measure space $(\Xi, \mathcal{A}(\Xi), \lambda)$ to the set of n-dimensional real-valued vector, i.e., for Borel set B of R^n, the set

$$\{\underline{\xi} \in B\} = \{\tau \in \Xi \,|\, \underline{\xi}(\tau) \in B\} \tag{16}$$

is an event.

Theorem 3.6 [12], [13] Let $\underline{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)^T$ be an uncertain vector, and $f : \text{R}^n \to \text{R}$ a measurable function. Then $f(\underline{\xi})$ is an uncertain variable such that

$$\lambda\{f(\underline{\xi}) \in B\} = \lambda\{\underline{\xi} \in f^{-1}(B)\} \tag{17}$$

for any Borel set B of R^n.
Parallel to probabilistic reliability developments, we define uncertainty reliability concepts as follows.

Definition 3.7. The lifetime of an individual under uncertainty environment is a nonnegative uncertain variable. The uncertain distribution is

$$\Psi(t) = \lambda\{\tau \in \Xi \,|\, \xi(\tau) \le t\},\ t \ge 0 \tag{18}$$

Definition 3.8. The uncertain survival function at time t is the uncertain measure of an individual surviving till time t

$$R(t) = \lambda\{\tau \in \Xi \,|\, \xi(\tau) \ge t\},\ t \ge 0\,. \tag{19}$$

Definition 3.9. The uncertain hazard function at time t is the instantaneous uncertain measure of an individual surviving till time t

$$h(t) = \lim_{\Delta t \to 0} \frac{\lambda\{\tau \in \Xi : t \le \xi(\tau) < t + \Delta t \,|\, \xi(\tau) \ge t\}}{\Delta t}\,. \tag{20}$$

It is obvious that uncertainty lifetime, uncertainty lifetime distribution, reliability function and hazard function are defined similarly to those of probabilistic concepts in formality. It is nature to ask what is the unique characteristic of the uncertain hazard function, following the similar relationship to the uncertain reliability as in probabilistic hazard function and the probabilistic reliability function? The answer is no. Next section will address the reason.

4. Intrinsic feature of uncertainty reliability

As we stressed in the introduction section, the measure specification is essentially determining everything. Similar to probability theory, in uncertainty theory an uncertain variable is a measurable mapping which is characterized by the membership of the pre-image of event (a Borel set) $B=(-\infty,r]$ under the uncertain variable ξ to the σ-algebra $\mathcal{A}(\Xi)$. In other words,

$$\forall B\in\mathcal{B}(\mathbb{R}),\{\tau\in\Xi:\xi\in B\}\in\mathcal{A}(\Xi). \tag{21}$$

The measurability of uncertain variable ξ definitely induces a measure on the measurable space $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. Let us denote the induced measure ν. Similar to the probabilistic case in Eq. (5), for $\forall B\in\mathcal{B}(\mathbb{R})$, the induced measure is $\nu=\text{ƛ}\circ\xi^{-1}$

$$\nu\{B\}=\text{ƛ}\{\tau\in\Xi:\xi\in B\}=\text{ƛ}\{\tau\in\Xi:\xi(\tau)\le r\} \tag{22}$$

Similarly, the uncertain distribution is defined by the induced measure

$$\Psi(r)=\nu\{(-\infty,r]\}=\Delta\{\tau\in\Xi:\xi(\tau)\le r\} \tag{23}$$

The uncertain distribution defined by the uncertain variable on the uncertain space $(\Xi,\mathcal{A}(\Xi),\Delta)$ in terms of the induced uncertain measure ν on $\mathcal{B}(\mathbb{R})$.

Now, let us define the uncertain lifetime distribution and further investigate its (induced) uncertain measure so that the intrinsic features can be exposed.

Definition 4.1. The function on $\mathbb{R}^+$, denoted by Ψ, is an uncertain lifetime distribution function if and only Ψ satisfies following two conditions:

(1) $\lim_{x\downarrow 0}\Psi(x)=0,\ \lim_{x\to+\infty}\Psi(x)=1$,

(2) $\Psi(x)$ is non-decreasing in *x*.

Theorem 4.2. [16] Let $\Psi:\mathbb{R}\to[0,1]$ be a non-decreasing function with

$$\Psi(-\infty)=0,\Psi(+\infty)=1. \tag{24}$$

Then set function $\nu:\mathcal{B}(\mathbb{R})\to[0,1]$, for any Borel set B:

$$\nu\{B\}=\begin{cases}\nu^*\{B\} & \text{if } \nu^*\{B\}<0.5\\ 1-\nu^*\{B^c\} & \text{if } \nu^*\{B^c\}<0.5\\ 0.5 & \text{otherwise}\end{cases} \tag{25}$$

where

$$\nu^*\{B\}=\nu_1\{B\}\wedge\nu_2\{B\}\wedge\nu_3\{B\},\forall B\in\mathcal{B}(\mathbb{R}) \tag{26}$$

with $\nu_i:\mathcal{B}(\mathbb{R})\to[0,1],\ i=1,2,3$,

$$\nu_1\{B\}=\begin{cases}1-\lim_{x\uparrow\inf\{B\}}\Psi(x) & \text{if }\inf\{B\}\in B\\ 1-\Psi(\inf\{B\}) & \text{otherwise}\end{cases}, \tag{27}$$

$$\nu_2\{B\}=\Psi(\sup\{B\}), \tag{28}$$

and

$$\nu_3\{B\}=\inf_{(a,b]\subset B^c}\{\Psi(a)+1-\Psi(b)\}, \tag{29}$$

is an uncertain measure on the Borel σ-algebra, $\mathcal{B}(\mathbb{R})$.

Proof. See [16].

Remark 4.3. It can be further verified that the uncertain distribution Ψ is neither required to be left continuous nor right-continuous. Ψ can have many finite jumps and “removable” points. Let us state the definition of an essential form of an uncertain lifetime distribution.

Definition 4.4. (The essential form of an uncertain lifetime distribution) Let ξ be an uncertain variable with essential form, which takes values from ascending ordered domain set $D=\{c_0=0,c_1,\cdots,c_n=+\infty\}$ with the uncertain distribution Ψ defined by

$$\begin{aligned}&\Psi(c_0)=\text{ƛ}\{\xi\le c_0=0\}=0,\\&\Psi(c_1-)=\psi_{1-},\ \Psi(c_1)=\psi_1,\ \Psi(c_1+)=\psi_{1+},\\&\cdots\cdots,\\&\Psi(c_i-)=\psi_{i-},\ \Psi(c_i)=\psi_i,\ \Psi(c_i+)=\psi_{i+},\\&\cdots\cdots,\\&\Psi(c_n-)=\psi_{n-},\Psi(c_n=+\infty)=1.00\end{aligned} \tag{30}$$

such that $\psi_{i-}<\psi_i<\psi_{i+},\ i=1,2,\cdots,n$. Furthermore, it requires

$$\pi_i=\text{ƛ}\{\xi=c_i\}\in(0,\psi_{i+}-\psi_{i-}),\ i=1,2,\cdots,n.$$

Definition 4.5. If the an uncertainty distribution takes the form

$$\Psi_d(z)=\begin{cases}0 & z<c_0=0\\ \pi_1 & z=c_1\\ \pi_{1+} & c_1<z<c_2\\ \pi_2 & z=c_2\\ \vdots & \vdots\\ \pi_i & z=c_i\\ \pi_{i+} & c_i<z<c_{i+1}\\ \pi_{i+1} & z=c_{i+1}\\ \vdots & \vdots\\ 1.0 & z\uparrow c_m=+\infty\end{cases} \tag{31}$$

where $0<\pi_i<\pi_{i+}<\pi_{i+1}<1$, then it is called as a discrete uncertain distribution.

Definition 4.6. A function $\Psi:\mathbb{R}^+\to[0,1]$, is a continuous uncertain lifetime distribution if and only if Ψ satisfies following three conditions:
(1) $\lim_{x\downarrow 0}\Psi(x)=0,\ \lim_{x\to+\infty}\Psi(x)=1$,
(2) $\Psi(x)$ is non-decreasing in *x*,
(3) For $\forall x\in\mathbb{R}$, $\lim_{y\uparrow x}\Psi(y)=\lim_{y\downarrow x}\Psi(y)=\Psi(x)$.

Table 1. Basic comparisons between uncertain lifetime and random lifetime

	Uncertain lifetime	Random lifetime
Symbol	ξ	X
Mapping	$\xi:(\Xi,\mathcal{A},\lambda)\to(\mathrm{R}^+,\mathcal{B},\nu)$	$\xi:(\Omega,\mathcal{F},P)\to(\mathrm{R}^+,\mathcal{B},\mu)$
	$\nu=\lambda\circ\xi^{-1}$	$\mu=P\circ X^{-1}$
	$\Psi(x)=\lambda\{\xi\le x\}$	$F(x)=P\{\xi\le x\}$
Distribution	$\Upsilon:\mathrm{R}^+\to[0,1]$	$F:\mathrm{R}^+\to[0,1]$
	$\Psi(0)=0,\Psi(+\infty)=1$	$F(0)=0,F(+\infty)=1$
	No limitations on Ψ i.e., finite jumps and removable points are allowed.	$\lim_{x\to x_0+0}F(x)=F(x_0)$ i.e., right-continuity is required.
Absolutely continuous	$\Psi(x)=\int_{(-\infty,x]}\lambda(s)ds$ λ continuous	$F(x)=\int_{(-\infty,x]}f(s)ds$ f continuous
Singleton	$\lambda\{\{x\}\}$ is not necessarily zero even Ψ is absolutely continuous	$\Pr\{\{x\}\}=0$ if F is absolutely continuous

Theorem 4.7. (Characterization theorem of uncertain reliability) $\bar{\Psi}$ is called an uncertain reliability function, which takes values from ascending ordered domain set $\mathrm{D}=\{c_0=0,c_1,\cdots,c_n=+\infty\}$ with the uncertain distribution Ψ defined by

$$\begin{aligned}&\bar{\Psi}(c_0)=1-\lambda\{\xi\le c_0=0\}=1,\\ &\bar{\Psi}(c_1-)=1-\psi_{1-},\ \bar{\Psi}(c_1)=1-\psi_1,\ \bar{\Psi}(c_1+)=1-\psi_{1+},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ &\bar{\Psi}(c_i-)=1-\psi_{i-},\ \bar{\Psi}(c_i)=1-\psi_i,\ \bar{\Psi}(c_i+)=1-\psi_{i+},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ &\bar{\Psi}(c_n-)=1-\psi_{n-},\bar{\Psi}(c_n=+\infty)=0.00\end{aligned} \tag{32}$$

such that $\psi_{i-}<\psi_i<\psi_{i+}$, $i=1,2,\cdots,n$. Furthermore, it requires

$$\pi_i=\lambda\{\xi=c_i\}\in(0,\psi_{i+}-\psi_{i-}),\ i=1,2,\cdots,n.$$

Remark 4.8. Accordingly, the uncertain reliability function is neither required to be left-continuous nor right-continuous and it can have as many as possible finite jumps and "removable" points, while the probabilistic reliability function only requires left-continuity. We should be fully aware that the less-constrained uncertain reliability function will bring more technical challenges in deriving the expression of uncertain hazard function in Eq. (20) and the parameter estimation for the uncertain reliability as well.

Example 4.9. A new trial weapon system has a discrete uncertain lifetime distribution as defined by

$$\Psi_\xi(z)=\begin{cases}0 & z<0\\ 0.25 & z=0\\ 0.45 & 0<z<1\\ 0.575 & z=1\\ 0.7 & 1<z<2\\ 0.77 & z=2\\ 0.84 & 2<z<3\\ 0.85 & z=3\\ 0.95 & 3<z<4\\ 1.0 & z\ge 4\end{cases} \tag{33}$$

Find its expected uncertainty life.

Note that the uncertainty expectation for an uncertain discrete distribution (with finite jumps and removable points) is

$$\mathrm{E}_\Psi[\xi]=\sum_{i=0}^{n}w_i c_i \tag{34}$$

where

$$
\begin{aligned}
w_i &= \max_{0\le j\le n}\left\{\pi_j,\pi_{j+} \mid c_j \le c_i\right\}\wedge 0.5 \\
&- \max_{0\le j\le n}\left\{\pi_j,\pi_{j+} \mid c_j < c_i\right\}\wedge 0.5 \\
&+ \max_{0\le j\le n}\left\{\pi_j,\pi_{j+} \mid c_j \ge c_i\right\}\wedge 0.5 \\
&- \max_{0\le j\le n}\left\{\pi_j,\pi_{j+} \mid c_j > c_i\right\}\wedge 0.5
\end{aligned} \tag{35}
$$

$i = 0,1,2,3,4$. Then

$$
\begin{aligned}
\mathrm{E}[\xi] &= 0.25\times 0 + 0.05\times 1 + 0.00\times 2 \\
&+ 0.00\times 3 + 0.05\times 4 = 0.25
\end{aligned}
$$

It means the expected uncertainty life is 0.25 year.

Remark 4.10. The example clearly delivers a message to reliability researchers as well as engineers: uncertain lifetime distribution could be intrinsically different from that of probabilistic lifetime distribution, particularly, in its essential form. In probabilistic lifetime analysis, there is no way to analyze the life phenomenon with finite jumps and removable features. Therefore uncertain reliability is not a mathematical game but an new approach to resolve the problems from real world.

Finally, we introduce [13] hazard function concept which can be regard an alternative quality index under general uncertainty.

Definition 4.11. Let ξ be an uncertainty lifetime of an individual (e.g., system/element). If ξ has an uncertainty distribution $\Psi(t)$, then the hazard distribution (or failure distribution) at time *t* is

$$
S(x\mid t)=\begin{cases} 0 & \text{if} \quad \Psi(x)\le\Psi(t) \\ \dfrac{\Psi(x)}{1-\Psi(t)}\vee 0.5 & \text{if } \Psi(t)\le\Psi(x)\le(1+\Psi(t))/2 \\ \dfrac{\Psi(x)-\Psi(t)}{1-\Psi(t)} & \text{if } (1+\Psi(t))/2\le\Psi(x) \end{cases} \tag{36}
$$

It is obvious that $S(x|t)$ is the conditional uncertainty distribution of uncertainty lifetime ξ given $\{\xi > t\}$.

5. Conclusion

In this paper, we give a systematic examination of reliability concept under uncertain measure foundation. The theoretical comparisons reveal that the characteristic of the probabilistic reliability function is different from that of uncertainty reliability function. Therefore it is evident that identification of the formality of the uncertainty of the individual manufacturing environment is critical to apply quality and reliability techniques appropriately. Without identifying the form of uncertainty and the uncertain distribution for specifying it, quality and reliability engineering efforts will be aimless.

However, we have to admit that this paper is a beginning efforts, more research needs to be done. For example, what is the uncertain hazard function when the uncertain lifetime takes the essential form (i.e., with finite jumps and removable points). Does Liu's hazard distribution (in Eq. (36)) lead to a hazard function consistent to the one derived from Eq. (20), when the uncertainty lifetime takes the essential form? Our next paper will address these questions.

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