Opuscula Math. 36, no. 6 (2016), 787–797
http://dx.doi.org/10.7494/OpMath.2016.36.6.787 Opuscula Mathematica

ELEMENTARY OPERATORS – STILL NOT ELEMENTARY?

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Communicated by P.A. Cojuhari

Abstract. Properties of elementary operators, that is, finite sums of two-sided multiplications on a Banach algebra, have been studied under a vast variety of aspects by numerous authors. In this paper we review recent advances in a new direction that seems not to have been explored before: the question when an elementary operator is spectrally bounded or spectrally isometric. As with other investigations, a number of subtleties occur which show that elementary operators are still not elementary to handle.

Keywords: spectral isometries, elementary operators, Jordan isomorphisms.

Mathematics Subject Classification: 47B47, 46H99, 47A10, 47A65, 47B48.

1. INTRODUCTION

Throughout we shall denote by A a unital Banach algebra over the complex field $\mathbb{C}$. A linear mapping $S\colon A \to A$ is said to be an *elementary operator* if there exist $a_1, \dots, a_n, b_1, \dots, b_n \in A$ such that $Sx = \sum_{j=1}^{n} a_j x b_j$ for all $x \in A$. Numerous properties of elementary operators have been studied by a variety of authors over many decades; among them are a detailed analysis of their spectra; their Fredholm properties; compactness, weak compactness and related properties such as strict singularity; norm properties; positivity (where appropriate); etc., etc. The two proceedings volumes [17] and [12] contain a wealth of references and several survey articles.

Despite their seemingly simple definition elementary operators often exhibit a somewhat intricate behaviour and many rather straightforward questions are hard to answer or even lead to open problems. On the other hand, various classes of more general operators can be approximated by elementary operators, see, e.g., [2, Chapter 5]. Therefore it appears well worth to continue to study this class of operators.

In this short note we focus on recent results that provide criteria when an elementary operator is spectrally bounded or spectrally isometric (for the definitions, see below). The aim is to illustrate the question in the title: why elementary operators still resist

a smooth, general theory and which special methods need to be applied depending on the problems one intends to tackle.

2. SPECTRALLY BOUNDED OPERATORS

Let A and B be unital, complex Banach algebras and $E \subseteq A$ be a closed subspace. A linear mapping $T\colon E \to B$ is called *spectrally bounded* if there is $M \geq 0$ such that $r(Tx) \leq M\, r(x)$ for all $x \in E$, where $r(\cdot)$ denotes the spectral radius. These operators are abundant:

Example 2.1.

(i) Every Jordan epimorphism is spectrally bounded.
(ii) The trace on $M_n(\mathbb{C})$ is spectrally bounded.
(iii) If $A = C(X)$, *every* bounded operator is spectrally bounded.
(iv) $M_{a,b}\colon A \to A$, $x \mapsto axb$ is spectrally bounded if and only if ba is central modulo the radical [28].

The last-quoted paper [28] by Pták is one of many articles by a variety of authors in which the origins of spectrally bounded operators can be found. The concept itself was introduced in [18] with the goal to unify various ideas and to progress towards a more systematic treatment which was then started in [22]. Pták's aim was "... to present... an extension to noncommutative algebras of the classical results of I. M. Singer and J. Wermer about derivations." (quoted from [28]). In the course of this, he showed the following. Let $L_a\colon A \to A$ and $R_a\colon A \to A$ denote $x \mapsto ax$ and $x \mapsto xa$, respectively; that is, the left and the right multiplication, respectively, by $a \in A$. Then $xa - ax \in \operatorname{rad}(A)$, the Jacobson radical of A, for all $x \in A$ if and only if $r(ax) \leq r(a)\, r(x)$ for all $x \in A$; the latter condition is easily seen to be equivalent to spectral boundedness of L_a and, since $r(ax) = r(xa)$ for all x, equally equivalent to spectral boundedness of R_a. It follows, see also [11], that the inner derivation $R_a - L_a$ maps A into its radical $\operatorname{rad}(A)$ if and only if it is spectrally bounded. In fact, it was shown in [9] that an arbitrary (not necessarily continuous) derivation d on A maps into $\operatorname{rad}(A)$ if and only if d is spectrally bounded. This allows for an alternative formulation of the noncommutative Singer–Wermer conjecture which is still an open problem; for more details, see [21].

In general the relation of a linear mapping, such as a derivation, to the ideal $\operatorname{rad}(A)$ can be quite delicate. But since it does not carry any spectral information (for all $x \in A$, $r(x) = r(x + \operatorname{rad}(A))$) and $\operatorname{rad}(A)$ is invariant under every elementary operator, we can, and will, henceforth assume that A is semisimple, that is, $\operatorname{rad}(A) = 0$. Therefore, with the notation of Example 2.1 (iv) above, $M_{a,b} = L_a R_b$, we have the first and most basic result on spectral boundedness of an elementary operator.

Proposition 2.2. *Let A be a unital semisimple Banach algebra. Let $a, b \in A$. Then $M_{a,b}$ is spectrally bounded if and only if $ba \in Z(A)$, the centre of A.*

This follows now immediately from Pták's result and the fact that $r(axb) = r(bax)$ for all $x \in A$. Despite the fact that this is a very basic observation, its proof is

not entirely trivial and needs both the subharmonicity of the spectral radius as well as Jacobson's density theorem applied to the induced two-sided multiplication in irreducible representations of A.

Unless the domain of T is contained in $C(X)$, for a compact Hausdorff space X, compare Example 2.1 (iii) above (so that spectral radius and norm coincide), there is no relation between boundedness and spectral boundedness of the linear mapping T in general. An important result due to Aupetit [3, Theorem 5.5.2], which was used in [5] to determine the separating space of Lie homomorphisms, states that every spectrally bounded operator onto a semisimple Banach algebra is automatically bounded.

A *Jordan homomorphism* is a linear mapping $T\colon A \to B$ such that $T(x^2) = (Tx)^2$ for all $x \in A$. It follows that T preserves the Jordan product $x \circ y = \frac{1}{2}(xy+yx)$, $x, y \in A$. A purely algebraic property of a Jordan epimorphism T (that is, a surjective Jordan homomorphism) is that $T1 = 1$ and Tx is invertible in B for every invertible $x \in A$. Therefore, every Jordan epimorphism T is a spectral contraction (i.e., $r(Tx) \leq r(x)$ for all $x \in A$), compare with Example 2.1 (i) above. Various conditions on the domain algebra imply that Jordan epimorphisms are the only surjective spectrally bounded operators. Here is a sample.

Theorem 2.3 ([23])**.** *Let A be a properly infinite von Neumann algebra. Let $T\colon A \to B$ be a unital spectrally bounded operator onto a semisimple unital Banach algebra B. Then T is a Jordan epimorphism.*

The absence of finite traces on A is decisive in the above result, which also holds for unital purely infinite C^*-algebras of real rank zero [16]. In fact, in [19], we formulated the following, still open problem.

Problem 2.4. Let A be a unital C^*-algebra without tracial states. Is every unital spectrally bounded operator from A onto a semisimple unital Banach algebra B a Jordan epimorphism?

Traces are natural candidates for spectrally bounded operators; cf. Example 2.1 (ii) above. For finite-dimensional algebras, the situation is rather clear.

Proposition 2.5. *Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a unital surjective linear mapping. Then T is spectrally bounded if and only if there are a constant $\gamma > 0$ and a Jordan automorphism S on $M_n(\mathbb{C})$ such that $Tx = \gamma\, Sx + (1-\gamma)\,\mathrm{tr}(x)$ for every $x \in M_n(\mathbb{C})$, where* tr *stands for the normalised trace on $M_n(\mathbb{C})$.*

A proof of this result can, e.g., be found in [21]. For infinite-dimensional algebras with "similar properties" as $M_n(\mathbb{C})$, the situation appears to be far more complicated. The following open question was raised in [21].

Problem 2.6. Does the statement of Proposition 2.5 extend to the case where $M_n(\mathbb{C})$ is replaced by a type II_1 factor A and the trace is the unique normalised trace on A?

3. SPECTRAL ISOMETRIES

Isometries are the best-behaved bounded linear operators. Hence, there is a vast literature and a lot of structural information available on them; see, e.g., [13] and [14]. Defining a *spectral isometry* as a linear operator $T\colon A \to B$ with the property $r(Tx) = r(x)$ for all $x \in A$, would one not expect a similar amount of detailed information for spectral isometries? Clearly, by the above discussion, every *Jordan isomorphism* (i.e., bijective Jordan homomorphism) preserves invertibility in both directions and thus is a unital surjective spectral isometry. It follows easily from Proposition 2.6 that every unital surjective spectral isometry $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a Jordan isomorphism; see [21] and, for an independent argument, [4]. One can deduce from this that a unital spectral isometry from a finite-dimensional semisimple Banach algebra onto a semisimple unital Banach algebra has the same property; but this requires some more work, see [24]. In fact, there is a general conjecture which we first posed at the Banach algebra conference in Odense in 2001.

Problem 3.1. Let A and B be unital C^*-algebras. Let $T\colon A \to B$ be a unital surjective spectral isometry. Does it follow that T is necessarily a Jordan isomorphism?

This conjecture is modelled after Kadison's well-known extension of the Banach–Stone theorem, [15], stating that, when $T\colon A \to B$ is a surjective linear isometry between two unital C^*-algebras A and B, then $T1$ is a unitary in B and the mapping $x \mapsto (T1)^{-1}Tx$, $x \in A$ is a Jordan *-isomorphism (that is, it preserves additionally selfadjoint elements). Very recently the first part of Kadison's theorem has been established in full for arbitrary Banach algebras.

Proposition 3.2 ([26])**.** *Let A and B be unital semisimple Banach algebras. Let $T\colon A \to B$ be a surjective spectral isometry. Then $T1$ belongs to the centre of B and its spectrum lies in the unit circle in $\mathbb{C}$ (that is, is a unitary in the Banach algebra sense).*

Resulting from this, for every surjective spectral isometry T, $x \mapsto (T1)^{-1}Tx$, $x \in A$ is a *unital* surjective spectral isometry, so one can always make this assumption without restricting the generality. We also note that every spectral isometry with semisimple domain is injective; this has been recorded in many places, e.g., [22] or [25].

Evidently all structural results for unital spectrally bounded operators yield (partial) answers to the above Problem 3.1. Here we record a result that was obtained in a different way and hence is special to spectral isometries.

Theorem 3.3 ([25])**.** *Let $T\colon A \to B$ be a unital surjective spectral isometry between the unital C^*-algebras A and B. Suppose that A is separable and has Hausdorff spectrum. Then T is a Jordan isomorphism.*

There is also some evidence that Problem 3.1 could have an affirmative answer for all semisimple Banach algebras; see, e.g., [1] and [10]. In general, however, the question is wide open and one might therefore try to find a counterexample. This was the motivation for the studies in [27] since elementary operators are given in a concrete way and therefore might be easier to handle.

4. ELEMENTARY OPERATORS

Let $\mathscr{E}\ell(A)$ denote the algebra of all elementary operators on the unital Banach algebra A. Every $S \in \mathscr{E}\ell(A)$ is of the form $S = \sum_{j=1}^{n} M_{a_j,b_j}$ for n-tuples $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{b} = (b_1, \ldots, b_n) \in A^n$. We will abbreviate this fact by $S = S_{\boldsymbol{a},\boldsymbol{b}}$ whenever convenient. However, such a representation as a sum of two-sided multiplications is by no means unique. Therefore, in order to have some sort of invariant, we introduce the *length of* S, $\ell(S)$, as follows. If $S = 0$ then $\ell(S) = 0$. If $S \neq 0$ then $\ell(S)$ is the smallest $n \in \mathbb{N}$ such that S can be written as a sum of n two-sided multiplications. We shall denote the set of all elementary operators of length at most n by $\mathscr{E}\ell_n(A)$. A necessary condition for $\ell(S) = n$ is that, in the representation $S = \sum_{j=1}^{n} M_{a_j,b_j}$, the sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ individually are linearly independent. For certain algebras A, for instance if A is an irreducible algebra of bounded linear operators on a Banach space, this condition is also sufficient. The main question we shall pursue in this section is which sets of coefficients $\boldsymbol{a}$, $\boldsymbol{b}$ will give us spectrally bounded and spectrally isometric elementary operators.

The following result from [8] extends the necessary condition in Proposition 2.2.

Proposition 4.1. *Let A be a semisimple unital Banach algebra. Let $S = S_{\boldsymbol{a},\boldsymbol{b}} \in \mathscr{E}\ell_n(A)$ be spectrally bounded. Then $\sum_{j=1}^{n} b_j a_j \in Z(A)$, the centre of A.*

Clearly this condition is not sufficient. For example, $L_a - R_a$ is spectrally bounded on a semisimple Banach algebra A if and only if $L_a - R_a = 0$, equivalently, $a \in Z(A)$; see the discussion in Section 2 and [11]. However $a - a \in Z(A)$ for every $a \in A$.

The following sufficient condition for spectral boundedness was obtained in [8, Corollary 2.6] on the basis of results in [7].

Proposition 4.2. *Let A be a semisimple unital Banach algebra. Let $S \in \mathscr{E}\ell_n(A)$ and suppose that $S = S_{\boldsymbol{a},\boldsymbol{b}}$ with $b_i a_i \in Z(A)$ for all $1 \leq i \leq n$ and $b_i a_j = 0$ for all $i < j$. Then S is a spectrally bounded.*

The conditions in the above result in particular imply that each two-sided multiplication M_{a_j,b_j} in the representation $S = \sum_{j=1}^{n} M_{a_j,b_j}$ is spectrally bounded and there is some "orthogonality" between their ranges. However, the conditions are not necessary: by [11], $L_a - R_b = M_{a,1} - M_{1,b}$ is spectrally bounded if and only if both a and b are central while the condition stated in Proposition 4.2 clearly fails.

So far, necessary and sufficient conditions for $S \in \mathscr{E}\ell(A)$ to be spectrally bounded in general are only known if $\ell(S) \leq 2$; see [6]. In the remainder of this article, we will discuss the results obtained in [27] for the case $\ell(S) = 3$. As it turns out, the additional assumption that S is unital changes the picture drastically.

Going back to $\ell(S) = 1$, i.e., $S = M_{a,b}$, the hypothesis $S1 = 1$ together with S spectrally bounded immediately entails that a is invertible with $b = a^{-1}$. Indeed, $ab = 1$ yields $aba = a$ which, as $ba \in Z(A)$ by Propostion 2.2, implies $ba^2 = a$. Upon multiplying by b on the right we have $ba^2b = ab = 1$ so $ba = 1$ and $b = a^{-1}$. As a consequence, we note the following.

Proposition 4.3 ([27, Proposition 3.1])**.** *Let A be a unital semisimple Banach algebra and let $a, b \in A$. The following conditions are equivalent:*

(a) *$M_{a,b}$ is unital and spectrally bounded;*
(b) *$M_{a,b}$ is a unital spectral isometry;*
(c) *a is invertible with $b = a^{-1}$.*

In each case, $M_{a,b}$ is automatically surjective.

This result extends to length 2 as follows; note that surjectivity is no longer automatic. Combining Theorem 4.2 with Proposition 4.3 in [27] we have the following description.

Theorem 4.4. *Let A be a semisimple unital Banach algebra. Suppose $S \in \mathscr{E}\ell_2(A)$ is unital. The following conditions are equivalent.*

(a) *S is spectrally bounded;*
(b) *S is spectrally isometric;*
(c) *S is multiplicative.*

Moreover, S is surjective if and only if $b_1a_1 + b_2a_2 = 1$.

The method of proof involves studying the behaviour of S in primitive quotients. Let $P \subseteq A$ be a primitive ideal in A. Let $S \in \mathscr{E}\ell_n(A)$. As $SP \subseteq P$ we obtain an induced elementary operator $S_P \in \mathscr{E}\ell_n(A/P)$ via $S_P x^P = (Sx)^P$, where $x^P = x + P$ denotes the coset of $x \in A$. Clearly, if $S = S_{\boldsymbol{a},\boldsymbol{b}}$ then $S_P = S_{\boldsymbol{a}^P,\boldsymbol{b}^P}$, and S is unital if and only if S_P is unital for every primitive ideal P. As A/P is an irreducible algebra of operators on some Banach space E, we have more tools available such as Jacobson's density theorem [3, Theorem 4.2.5] and Sinclair's addendum for invertible elements [3, Corollary 4.2.6] which allow us to control the behaviour of the coefficients of S_P. Since A is semisimple, we can piece the information from the quotients A/P together to obtain a global description. For the details, see [6–8].

It turns out that, even for arbitrary length, the sufficient conditions in Proposition 4.2 together with the assumption of preserving the identity have strong consequences.

Theorem 4.5 ([27, Theorem 4.4])**.** *Let A be a semisimple unital Banach algebra. Let $S \in \mathscr{E}\ell_n(A)$ be unital. Suppose that $S = S_{\boldsymbol{a},\boldsymbol{b}}$ with $e_i = b_ia_i \in Z(A)$ for all $1 \leq i \leq n$ and $b_ia_j = 0$ for all $i < j$. Then S is a spectral isometry. Moreover, the following are equivalent:*

(a) *S is surjective.*
(b) *$\sum_{i=1}^n e_i = 1$.*
(c) *$S = M_{w,w^{-1}}$ for an invertible element $w \in A$.*

We will sketch the proof of the first part of the statement in the case $\ell(S) = 3$ in order to illustrate the techniques and afterwards show the limitations making it difficult to move on to if-and-only-if conditions for longer length, at least at present.

Starting from the main additional assumption, $S1 = 1$, we have in succession the following identities:

$$\begin{aligned} a_1b_1 + a_2b_2 + a_3b_3 &= 1, \\ a_1b_1a_2 + a_2b_2a_2 + a_3b_3a_2 &= a_2, \\ b_2a_2b_2a_2 &= b_2a_2, \end{aligned}$$

that is, $e_2 = b_2a_2$ is a central idempotent. Similarly, upon multiplying the first identity above on the left by b_1 we obtain $b_1a_1b_1 = b_1$ so that $e_1 = b_1a_1$ is a central idempotent and $b_1 = e_1b_1$. In a similar vein, $e_3 = b_3a_3$ is a central idempotent with $a_3 = e_3a_3$. Let P be a primitive ideal of A. For all $1 \leq i \leq 3$, $e_i^P \in \{0, 1\}$ since $Z(A/P) = \mathbb{C}1$. An inspection of the proof of Proposition 4.2 shows that $r(S_Px^P) \leq r(x^P)$ for all $x \in A$ (as the "constant of spectral boundedness" depends on $r(e_i^P)$ which is at most 1). Since $r(y) = \sup_P r(y^P)$ for each $y \in A$, [3, Theorem 4.2.1], we find that $r(Sx) \leq r(x)$ for all x. Thus, in order to conclude that S is a spectral isometry, it suffices to prove the reverse inequality.

Before proceeding to this step we note a very useful matrix notation for the elementary operator S. Considering A as a closed subalgebra of $M_3(A)$ via the embedding

$$a \longmapsto \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we have, for every $x \in A$,

$$\begin{aligned} r(Sx) &= r\Big(\begin{pmatrix} Sx & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\Big) = r\Big(\begin{pmatrix} \sum_{j=1}^3 a_jxb_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\Big) \\ &= r\Big(\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 0 \end{pmatrix}\Big) \\ &= r\Big(\begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 0 & 0 \\ b_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}\Big) \\ &= r\Big(\begin{pmatrix} b_1a_1 & 0 & 0 \\ b_2a_1 & b_2a_2 & 0 \\ b_3a_1 & b_3a_2 & b_3a_3 \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}\Big) \\ &= r\Big(\begin{pmatrix} e_1x & 0 & 0 \\ b_2a_1x & e_2x & 0 \\ b_3a_1x & b_3a_2x & e_3x \end{pmatrix}\Big). \end{aligned}$$

To continue the argument, once again we take a primitive ideal P of A and aim to show that $r(S_Px^P) \geq r(x^P)$ for all $x \in A$. We distinguish the various combinations which can occur from the fact that $e_i^P \in \{0, 1\}$. If at least two of the e_i's are zero the

elementary operator, S_P reduces to a two-sided multiplication which we already dealt with. If $e_i = 1$ for all i then $b_i a_j = 0$ for *all* $i \neq j$. For example, we have in succession

$$\begin{aligned} a_1b_1 + a_2b_2 + a_3b_3 &= 1, \\ b_2a_1b_1a_1 + b_2a_2b_2a_1 + b_2a_3b_3a_1 &= b_2a_1, \\ b_2a_1 + b_2a_1 &= b_2a_1, \end{aligned}$$

wherefore $b_2a_1 = 0$. As a result, the above matrix calculation applied to S_P yields $r(S_P x^P) = r(x^P)$ for all $x \in A$. If $e_1^P = 0$ or $e_3^P = 0$ then S_P is of length at most 2 and Theorem 4.4 applies. As $e_i = 0$ for all i is impossible (since $S_P 1 = 1$) the only case left is $e_1^P = e_3^P = 1$ and $e_2^P = 0$. This is the most interesting case, as we shall see below in Example 4.6.

Under this assumption, for every $\lambda \in \mathbb{C}$, we have

$$\lambda - \begin{pmatrix} x^P & 0 & 0 \\ b_2^P a_1^P x^P & 0 & 0 \\ b_3^P a_1^P x^P & b_3^P a_2^P x^P & x^P \end{pmatrix} = \begin{pmatrix} \lambda - x^P & 0 & 0 \\ -b_2^P a_1^P x^P & \lambda & 0 \\ -b_3^P a_1^P x^P & -b_3^P a_2^P x^P & \lambda - x^P \end{pmatrix}.$$

Let $\lambda \in \sigma(x^P)$ be such that $|\lambda| = r(x^P)$. Then λ belongs to the left approximate point spectrum of x^P and thus there is a sequence $(y_n^P)_{n\in\mathbb{N}}$ of unit elements in A/P with the property $(\lambda - x^P)y_n^P \to 0$ $(n \to \infty)$. As this entails that

$$\begin{pmatrix} \lambda - x^P & 0 & 0 \\ -b_2^P a_1^P x^P & \lambda & 0 \\ -b_3^P a_1^P x^P & -b_3^P a_2^P x^P & \lambda - x^P \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y_n^P & 0 & 0 \end{pmatrix} \longrightarrow 0,$$

we conclude that $r(S_P x^P) \geq |\lambda| = r(x^P)$. Consequently, in each of the above cases, for every $x \in A$, $r(Sx) \geq r(x^P)$ for all P and thus $r(Sx) \geq r(x)$ which is the desired estimate.

The second part of the theorem requires a subtle recursion argument for the coefficients of S; see [27, Theorem 4.4]. It is interesting that the surjectivity forces the elementary operator to be of length one. Without the surjectivity assumption, a length-three elementary operator which is both unital and spectrally isometric does not have to be multiplicative or even a Jordan homomorphism, in contrast to shorter length (Proposition 4.3 and Theorem 4.4). This is in accordance with the general conjecture (Problem 3.1); we will now supply a concrete example, taken from [27], illustrating that, even for elementary operators, the situation is rather delicate.

Example 4.6. Let $A = \mathscr{L}(\ell^2)$ and take two isometries s_1, s_2 in A such that $s_1 s_1^* + s_2 s_2^* = 1$ (which in particular implies that s_1 and s_2 have orthogonal ranges). This can always be done by writing $\ell^2 = \ell^2 \oplus \ell^2$ and letting s_1 be the isometric isomorphism with the first component and s_2 the isometric isomorphism with the second component (so that $s_i s_i^*$, $i = 1, 2$ are the corresponding orthogonal projections). Let $T = M_{s_1,s_1^*} + M_{s_2,s_2^*}$ which is a length-two unital elementary operator on A ($\ell(T) = 2$ since $\{s_1, s_2\}$ is linearly independent). It is easily seen that T is an isometry and multiplicative, cf. [27, Example 2.4], and hence a spectral isometry.

We now alter T to obtain a length-three elementary operator as follows. Take $z \in A$, $z \neq 0$ with $z^2 = 0$ and put $a_1 = s_1$, $a_2 = s_2 z$, $a_3 = s_2$, $b_1 = s_1^*$, $b_2 = z s_1^*$ and $b_3 = s_2^*$. It is readily verified that $\{a_1, a_2, a_3\}$ is linearly independent and so is $\{b_1, b_2, b_3\}$. Consequently $\ell(S) = 3$. Moreover, $b_i a_i \in \mathbb{C}1$ for all $1 \leq i \leq 3$, $b_i a_j = 0$ for all $1 \leq i < j \leq 3$, and $S1 = 1$. By Theorem 4.5, it follows that S is a unital spectral isometry. In the notation of the above theorem, $e_1 = e_3 = 1$ while $e_2 = 0$. Therefore, S is not surjective.

In order to show that S is not a Jordan homomorphism we need to produce $x \in A$ such that $(Sx)^2 - S(x^2) \neq 0$. Since $z^2 = 0$ and $z \neq 0$ there is $\eta \in \ell^2$ such that $\{\eta, z\eta\}$ is linearly independent. Put $\xi = s_1 \eta$ and note that $s_1^* \xi = \eta$. Take $x \in A$ such that $x\eta = 0$ and $xz\eta = \eta$. We compute

$$\big((Sx)^2 - S(x^2)\big)\xi = s_2 z x z x s_1^* \xi + s_2 x z x z s_1^* \xi = s_2 \eta \neq 0,$$

as all the other terms cancel each other out or vanish and s_2 is an isometry. □

On the other hand, this example entirely relies on the non-surjectivity of the elementary operator used as the next result shows.

Theorem 4.7 ([27])**.** *Let A be a unital C*-algebra. Then every unital surjective $S \in \mathscr{E}\ell_3(A)$ which is spectrally isometric is an algebra automorphism of A.*

Key to this theorem is a necessary condition for spectrally isometric length-three elementary operators valid in irreducible representations of high enough dimension which is not available for longer length, see [27] and [8], whereas low finite dimensions are taken care of by Proposition 2.5. At this moment is it unclear what a sufficient and necessary condition for an elementary operator of arbitrary length to be spectrally isometric could look like.

Acknowledgments

This paper reviews some of the topics of my invited talk "Status report on spectrally bounded and spectrally isometric operators" at the conference on Spectral Theory and Applications in Krakow in May 2015 from the point of view of their connections with the theory of elementary operators on Banach algebras. I would like to thank my Polish colleagues for their wonderful hospitality.

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Received: April 17, 2016.
Accepted: July 28, 2016.