Opuscula Math. 41, no. 1 (2021), 95–112
https://doi.org/10.7494/OpMath.2021.41.1.95 Opuscula Mathematica

ON THE CROSSING NUMBERS OF JOIN PRODUCTS OF $W_4 + P_n$ AND $W_4 + C_n$

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Communicated by Andrzej Żak

Abstract. The crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in the plane. The main aim of the paper is to give the crossing number of the join product $W_4 + P_n$ and $W_4 + C_n$ for the wheel W_4 on five vertices, where P_n and C_n are the path and the cycle on n vertices, respectively. Yue *et al.* conjectured that the crossing number of $W_m + C_n$ is equal to $Z(m+1)Z(n) + (Z(m)-1)\left\lfloor \frac{n}{2} \right\rfloor + n + \left\lceil \frac{m}{2} \right\rceil + 2$, for all $m, n \geq 3$, and where the Zarankiewicz's number $Z(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ is defined for $n \geq 1$. Recently, this conjecture was proved for $W_3 + C_n$ by Klešč. We establish the validity of this conjecture for $W_4 + C_n$ and we also offer a new conjecture for the crossing number of the join product $W_m + P_n$ for $m \geq 3$ and $n \geq 2$.

Keywords: graph, crossing number, join product, cyclic permutation, path, cycle.

Mathematics Subject Classification: 05C10, 05C38.

1. INTRODUCTION

For the first time, P. Turán [26] described the brick factory problem. He was forced to work in a brick factory and his job was to push a wagon-load of bricks along a track from a kiln to storage site. The factory contained several kilns and storage sites, with tracks criss-crossing the floor among them. Turán found it difficult to push the wagon across a track crossing, and in his mind he began to consider how the factory might be redesigned to minimize these crossings. Since then, the topic has steadily grown and the crossing number research has become entrenched as one of the core areas in topological graph theory. The problem of reducing the number of crossings is of interest in many areas. One of the most popular areas is VLSI-layout implementation of which revolutionized circuit design and had a strong impact on parallel computing. The crossing numbers have been also studied to improve the readability of hierarchical structures and automated graph drawings. The visualized graph should be easy to read and understand. For the sake of clarity of the graphical drawings, the reducing of crossings is probably the most important.

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane. (For the definition of a *drawing* see [12].) It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. Let D ($D(G)$) be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. It is easy to see that for any three mutually edge-disjoint subgraphs G_i, G_j, and G_k of G, the following equations hold:

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j),$$

$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k).$$

It is well known that computing the crossing number of a graph is an NP-complete problem by Garey and Johnson [7]. The exact values of the crossing numbers are known only for some graphs or some families of graphs. In [9, 10], Ho gave the characterization for a few multipartite graphs. The purpose of this article is to extend the known results concerning this topic for the wheel W_4 on five vertices based on its isomorphism with the complete tripartite graph $K_{1,2,2}$. The join product of two graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of disjoint edge sets of the graphs G_i, G_j, and the complete bipartite graph $K_{m,n}$. Let P_n and C_n be the *path* and the *cycle* on n vertices, respectively, and let D_n denote the *discrete graph* (sometimes called *empty graph*) on n vertices. We will often use the Kleitman's result [11] on crossing numbers of the complete bipartite graphs. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{with } \min\{m, n\} \leq 6.$$

Using Kleitman's result [11], the crossing numbers for the join product of two paths, the join product of two cycles, and also for the join product of a path and a cycle were studied by Klešč [13]. Some omitted proofs for the crossing numbers of $G + C_n$ for the graphs G of order four are given later also by Klešč [14]. Let us note that the exact values for the crossing numbers $G + P_n$ and $G + C_n$ have been investigated for some graphs G of order five and six in [2, 5, 6, 12, 17, 18, 20–22]. In all these cases, the graph G is usually connected and contains at least one cycle.

The methods in the paper mostly use the multiple combinatorial properties of the cyclic permutations. Yue *et al.* [25] introduced a new conjecture of the crossing number of $W_m + C_n$ that is equal to $Z(m+1)Z(n) + (Z(m)-1)\left\lfloor \frac{n}{2} \right\rfloor + n + \left\lceil \frac{m}{2} \right\rceil + 2$, for $m, n \geq 3$. To determine this conjecture the Zarankiewicz's number defined by $Z(n) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ is also used. This conjecture was proved for $W_3 + C_n$ by Klešč [14].

Our proof of Theorem 4.4 confirms the validity of this conjecture for $W_4 + C_n$. Based on the ability to generalize the optimal drawing for $W_4 + P_n$ onto the drawing of the graph $W_m + P_n$, we give a new conjecture of the crossing number of $W_m + P_n$, for all $m \geq 3$, $n \geq 2$. Also in this article, some parts of proofs can be simplified by utilizing the work of the software that generates all cyclic permutations due to Berežný and Buša [1]. The results in Theorem 3.3 and Theorem 4.4 have already been claimed by Su and Huang [23] and by Yue *et al.* [25], respectively. Since these papers do not seem to be available in English, we have not been able to verify the results. Clancy *et al.* [4] also placed an asterisk on a number of the results in their survey to essentially indicate that the mentioned results appeared in journals which do not have a sufficiently rigorous peer-review process.

2. POSSIBLE DRAWINGS OF W_4 AND PRELIMINARY RESULTS

Let W_4 be the wheel on five vertices. We consider the join product of W_4 with the discrete graph D_n on n vertices. Let T^i, $i = 1, \dots, n$, denote the subgraph which is uniquely induced by the five edges incident with the fixed vertex t_i. This means that the graph $T^1 \cup \ldots \cup T^n$ is isomorphic with the complete bipartite graph $K_{5,n}$ and therefore, we can write

$$W_4 + D_n = W_4 \cup K_{5,n} = W_4 \cup \left(\bigcup_{i=1}^{n} T^i \right). \tag{2.1}$$

The graph $W_4 + P_n$ contains $W_4 + D_n$ as a subgraph. For the subgraphs of the graph $W_4 + P_n$ which are also subgraphs of the graph $W_4 + D_n$ we use the same notations as above. Let P_n^* denote the path induced on n vertices of $W_4 + P_n$ not belonging to W_4. Hence, P_n^* consists of the vertices $t_1, t_2, \dots, t_n$ and of the edges $\{t_i, t_{i+1}\}$ for $i = 1, 2, \dots, n-1$. One can easily see that

$$W_4 + P_n = W_4 \cup K_{5,n} \cup P_n^* = W_4 \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup P_n^*. \tag{2.2}$$

Similarly, the graph $W_4 + C_n$ contains both $W_4 + D_n$ and $W_4 + P_n$ as subgraphs. Let C_n^* denote the subgraph of $W_4 + C_n$ induced on the vertices $t_1, t_2, \dots, t_n$. So,

$$W_4 + C_n = W_4 \cup K_{5,n} \cup C_n^* = W_4 \cup \left(\bigcup_{i=1}^{n} T^i \right) \cup C_n^*. \tag{2.3}$$

Let D be a good drawing of the graph $W_4 + D_n$. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D is the cyclic permutation that records the (cyclic) counterclockwise order in which the edges leave t_i, as defined by Hernández-Vélez *et al.* [8]. We use the notation (12345) if the counter-clockwise order the edges incident with the vertex t_i is t_iv_1, t_iv_2, t_iv_3, t_iv_4, and t_iv_5. We emphasize that a rotation is a cyclic permutation; that is, (12345), (23451), (34512), (45123), and (51234) denote the same rotation. Thus, $5!/5 = 24$ different $\mathrm{rot}_D(t_i)$ can appear in a drawing of the graph $W_4 + D_n$.

Since the complete bipartite graph $K_{5,n}$ is a subgraph of $W_4 + D_n$, let us discuss some properties of crossings among edges of its subgraph $K_{5,2}$. Assume, in general, D is a good drawing of the graph $K_{m,n}$ with the vertices $t_1, t_2, \dots, t_n$ of degree m. The rotation $\mathrm{rot}_D(t_i)$, $i = 1, 2, \dots, n$, is defined in the same way as above, i.e., as the cyclic permutation of m elements. Let $K_{m,2}$ be the subgraph of $K_{m,n}$ with the vertices t_i and t_j of degree m. Similarly as in the graph $W_4 + D_n$, we can use the symbol $\mathrm{cr}_D(T^i, T^j)$ for the number of crossings between the edges incident with t_i and the edges incident with t_j. Woodall [24] showed that if both vertices t_i and t_j have the same rotation in D, then $\mathrm{cr}_D(T^i, T^j) \geq \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. It is easy to see that $\mathrm{cr}_D(T^i, T^j) = 0$ only if $\mathrm{rot}_D(t_j)$ is inverse to $\mathrm{rot}_D(t_i)$. Moreover, if $Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$ denotes the minimum number of interchanges of adjacent elements of $\mathrm{rot}_D(t_i)$ required to produce the inverse cyclic permutation of $\mathrm{rot}_D(t_j)$ or, equivalently, from $\mathrm{rot}_D(t_j)$ to the inverse of $\mathrm{rot}_D(t_i)$, then $\mathrm{cr}_D(T^i, T^j) \geq Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j))$, and that

$$\mathrm{cr}_D(T^i, T^j) \equiv Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) \pmod 2 \quad \text{if } m \text{ is odd.} \tag{2.4}$$

This implies that $\mathrm{cr}_D(T^i, T^j) = 0$ only if $\mathrm{rot}_D(t_i)$ is inverse to $\mathrm{rot}_D(t_j)$, and $\mathrm{cr}_D(T^i, T^j) \geq 4$ if $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$.

In the given drawing D, we will separate all subgraphs T^i, $i = 1, 2, \dots, n$, of the graph $W_4 + D_n$ into four mutually-disjoint families of subgraphs depending on the number of times that T^i crosses the edges of W_4 in D. For $i = 1, 2, \dots, n$, let $R_D = \{T^i : \mathrm{cr}_D(W_4, T^i) = 0\}$, $S_D = \{T^i : \mathrm{cr}_D(W_4, T^i) = 1\}$, and $T_D = \{T^i : \mathrm{cr}_D(W_4, T^i) = 2\}$. Every other subgraph T^i crosses the edges of W_4 at least three times in D. For $T^i \in R_D \cup S_D \cup T_D$, let F^i denote the subgraph $W_4 \cup T^i$, $i \in \{1, 2, \dots, n\}$, of $W_4 + D_n$ and let $D(F^i)$ be its subdrawing induced by D. Clearly, the idea of dividing the subgraphs T^i into four mentioned families is also retained in all drawings of the graphs $W_4 + P_n$ and $W_4 + C_n$. Since the graph W_4 consists of one dominating vertex of degree 4 and of four vertices of degree 3 which form the subgraph isomorphic with the cycle C_4 (for brevity, we will write $C_4(W_4)$), we only need to consider possibilities of crossings between subdrawings of $C_4(W_4)$ and four edges incident with the dominating vertex which form the subgraph isomorphic with the star S_4 on five vertices (also for brevity, we will write $S_4(W_4)$).

Lemma 2.1. *Let $G \in \{D_n | n \geq 1\} \cup \{P_n | n \geq 2\} \cup \{C_n | n \geq 3\}$. In any optimal drawing of the join product $W_4 + G$, the edges of $C_4(W_4)$ do not cross each other. Moreover, the subdrawing of W_4 induced by D is isomorphic with one of the three drawings depicted in Figure 2.*

Proof. Assume an optimal drawing of the graph $W_4 + D_n$ in which two edges of $C_4(W_4)$ cross. Let x be the point of the plane in which two edges, say $\{c_i, c_{i+1}\}$ and $\{c_j, c_{j+1}\}$, of $C_4(W_4)$ cross. Since the plane is a normal space, in the plane there is an open set A_x such that A_x contains only x and parts of the crossing edges, that is, A_x does not contain a further vertex or part of another edge. Clearly, we can also assume that the dominating vertex of W_4 is not contained in A_x. Thus, all remaining edges of the drawing are disjoint with A_x, see Figure 1(a). Figure 1(b) shows that the edges $\{c_i, c_{i+1}\}$ and $\{c_j, c_{j+1}\}$ can be redrawn into new edges $\{c_i, c_j\}$ and

$\{c_{i+1}, c_{j+1}\}$ which do not cross. The vertices $c_i, c_j, c_{i+1}, c_{j+1}$ form the 4-cycle again. Since each vertex of the cycle $C_4(W_4)$ is adjacent to the dominating vertex of degree four of W_4, the new drawing of the graph $W_4 + D_n$ with less number of crossings is obtained. This contradiction completes the proof for the optimal drawings of $W_4 + D_n$, and the proof proceeds in the similar way also for the graph $G \in \{P_n | n \geq 2\} \cup \{C_n | n \geq 3\}$.

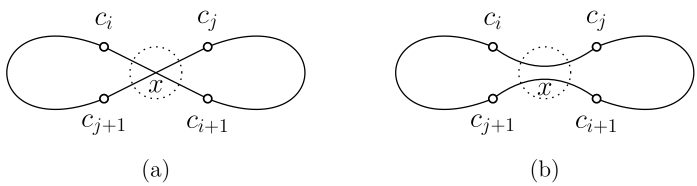

Fig. 1. Elimination of a crossing in $C_4(W_4)$

□

According to Lemma 2.1, suppose only three non isomorphic good drawings of the graph W_4 as shown in Figure 2, and where the vertex notation of W_4 will be justified later.

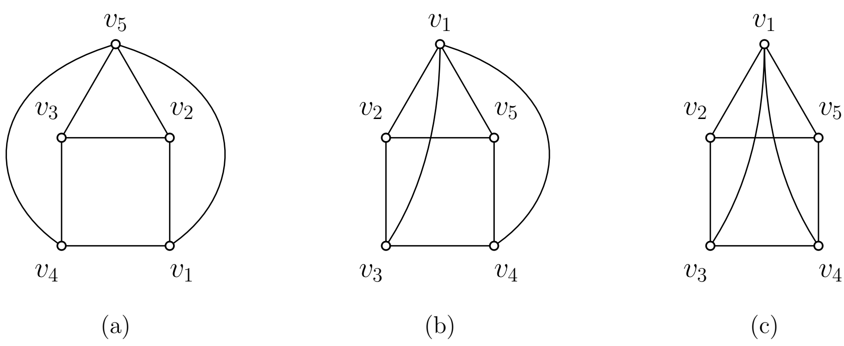

Fig. 2. Three possible non isomorphic good drawings of the graph W_4 with no crossing among edges of $C_4(W_4)$

3. THE CROSSING NUMBER OF $W_4 + P_n$

In the proofs of the paper, several parts are based on the previous Lemma 2.1 and on the following theorem presented in [19].

Theorem 3.1 ([19, Theorem 3.2])**.** $\mathrm{cr}(W_4 + D_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

Even though we are able to compute the exact values of crossing numbers of the graphs $W_4 + P_2$ and $W_4 + P_3$ using an algorithm located on the website http://crossings.uos.de/, due to the simplicity of these proofs, we prove the following Lemma 3.2.

Lemma 3.2. $\mathrm{cr}(W_4 + P_2) = 4$ *and* $\mathrm{cr}(W_4 + P_3) = 9$.

Proof. Notice that the graphs $W_4 + P_2$ and $W_4 + P_3$ are isomorphic with the join product of the cycle C_4 with the cycle C_3 and with the graph $K_4 \setminus e$ obtained by removing one edge from the complete graph K_4, respectively. In [13] and [14] were proved that $\mathrm{cr}(C_4+C_3) = 4$ and $\mathrm{cr}(C_4+K_4\setminus e) = 9$, respectively, and so $\mathrm{cr}(W_4+P_2) = 4$ and $\mathrm{cr}(W_4 + P_3) = 9$. □

Theorem 3.3. $\mathrm{cr}(W_4 + P_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ *for* $n \geq 2$.

Proof. In Figure 3, the edges of $K_{5,n}$ cross each other $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times, each subgraph T^i, $i = 1, \dots, \lceil \frac{n}{2} \rceil$ on the right side crosses the edges of $C_4(W_4)$ exactly once and each subgraph T^i, $i = \lceil \frac{n}{2} \rceil + 1, \dots, n$ on the left side crosses the edges of $S_4(W_4)$ exactly twice.

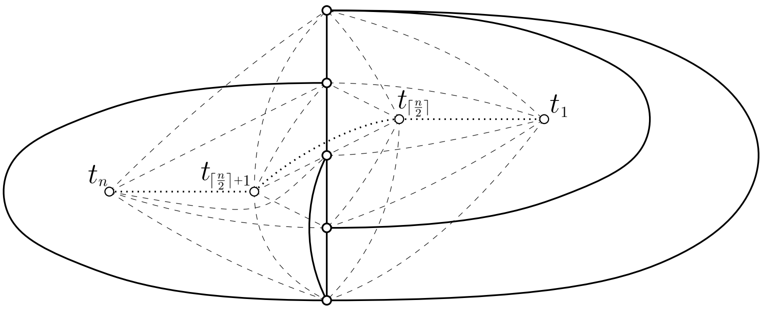

Fig. 3. The good drawing of $W_4 + P_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ crossings

The path P_n^* crosses W_4 once, and so $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$ crossings appear among the edges of the graph $W_4 + P_n$ in this drawing. Thus, $\mathrm{cr}(W_4 + P_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 1$. By Lemma 3.2, the result is true for $n = 2$ and $n = 3$. We prove the reverse inequality by induction on n. Suppose now that, for some $n \geq 4$, there is a drawing D with

$$\mathrm{cr}_D(W_4 + P_n) < 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 1, \tag{3.1}$$

and that

$$\mathrm{cr}(W_4 + P_m) = 4\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + m + \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ for any integer } m < n. \tag{3.2}$$

As the graph $W_4 + D_n$ is a subgraph of the graph $W_4 + P_n$, by Theorem 3.1, the edges of $W_4 + P_n$ are crossed exactly $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor$ times, and therefore, no edge of the path P_n^* is crossed in D. This also enforces that all vertices t_i of the path P_n^* must be placed in the same region of the considered good subdrawing of W_4. Our assumption on D, together with the well-known fact $\operatorname{cr}(K_{5,n}) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$, implies that

$$\operatorname{cr}_D(W_4) + \operatorname{cr}_D(W_4, K_{5,n}) \leq n + \left\lfloor \frac{n}{2} \right\rfloor,$$

that is,

$$\operatorname{cr}_D(W_4) + s + 2(n - r - s) \leq n + \left\lfloor \frac{n}{2} \right\rfloor, \tag{3.3}$$

if we use the notation $r = |R_D|$ and $s = |S_D|$. This forces that $2r + s \geq \lceil \frac{n}{2} \rceil + \operatorname{cr}_D(W_4)$, and if $r = 0$ then $s \geq \lceil \frac{n}{2} \rceil$. By Lemma 2.1, we can also suppose that there is no crossing among edges of $C_4(W_4)$ in all contemplated subdrawings of the graph W_4. Now, we will deal with the possibilities of obtaining a subgraph $T^i \in R_D \cup S_D$ in the drawing D and we show that in all cases a contradiction with the assumption (3.1) is obtained.

Case 1. $\operatorname{cr}_D(W_4) = 0$. In this case, without lost of generality, we can consider the drawing of W_4 with the vertex notation like that in Figure 2(a). Because no face is incident to all vertices in $D(W_4)$, there is no possibility to obtain a subdrawing of $W_4 \cup T^i$ for a $T^i \in R_D$. As $r = 0$, there are at least $\lceil \frac{n}{2} \rceil$ subgraphs T^i by which the edges of W_4 are crossed just once. For a subgraph $T^i \in S_D$, the vertex t_i must be placed in the region with four vertices v_1, v_2, v_3, and v_4 of the graph W_4 on its boundary. So, there are 4 different possible rotations systems with one crossing depending on which edge of $C_4(W_4)$ is crossed. These four possibilities under our consideration are denoted by $\mathcal{A}_p$, for $p = 1, 2, 3, 4$. For our purposes, it does not matter which of the regions is unbounded, and so we can assume the drawings shown in Figure 4. Thus the configurations $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ are represented by the cyclic permutations (12345), (12534), (15234), and (12354), respectively. They have been already introduced in [19]. Let $\mathcal{M}_D$ be the set of all configurations for the drawing D belonging to $\mathcal{M} = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4\}$.

For the rest of the proof, let us also assume that the number of subgraphs with the configuration $\mathcal{A}_p \in \mathcal{M}_D$ is at least as much as the number of subgraphs with the configuration $\mathcal{A}_q \in \mathcal{M}_D$, for each possible $p \neq q$, and let $T^i \in S_D$ be such a subgraph with the configuration $\mathcal{A}_p$ of F^i. Hence,

$$\sum_{l \neq i,\ T^l \in S_D} \operatorname{cr}_D(T^i, T^l) \geq 3(s-2) + 2,$$

that is,

$$\sum_{l \neq i,\ T^l \in S_D} \operatorname{cr}_D(W_4 \cup T^i, T^l) \geq 4(s-2) + 3,$$

where an idea of the arithmetic mean of the values four, three and two could be exploited (here, the lower bounds for the number of crossings of two configurations $\operatorname{cr}(\mathcal{A}_p, \mathcal{A}_q)$ were also established in Table 1 in [19]).

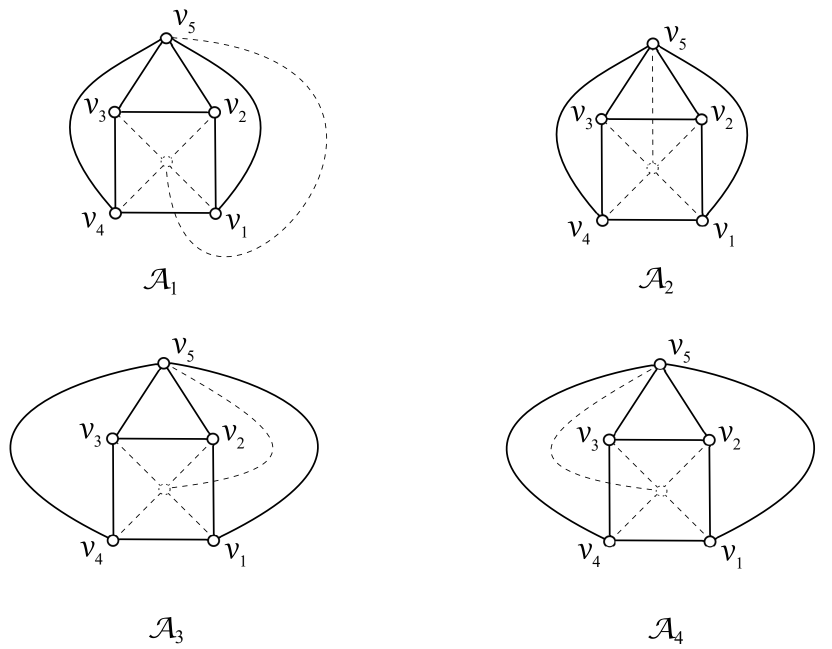

Fig. 4. Drawings of four possible configurations from $\mathcal{M}$ of the subgraph F^i

Table 1

The necessary number of crossings between T^k and T^l for the configurations $\mathcal{A}_p$, $\mathcal{A}_q$

$-$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
$\mathcal{A}_1$	4	2	3	3
$\mathcal{A}_2$	2	4	3	3
$\mathcal{A}_3$	3	4	4	2
$\mathcal{A}_4$	3	3	2	4

Moreover, it is not difficult to verify that $\mathrm{cr}_D(W_4 \cup T^i, T^l) \geq 4$ is fulfilling for each $T^l \notin S_D$ assuming that all vertices are placed in the same region of the considered good subdrawing of W_4. Thus, by fixing the graph $W_4 \cup T^i$, we have

$$\begin{aligned}
\mathrm{cr}_D(W_4 + P_n) &\geq \mathrm{cr}_D(K_{5,n-1}) + \mathrm{cr}_D(K_{5,n-1}, W_4 \cup T^i) + \mathrm{cr}_D(W_4 \cup T^i) \\
&\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(s-2) + 3 + 4(n-s) + 1 \\
&= 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n - 4 \\
&\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 1.
\end{aligned}$$

Case 2. $\mathrm{cr}_D(W_4) = 1$. Without lost of generality, we can choose the vertex notation of the graph W_4 given in Figure 2(b). The set R_D is also empty. Now, for a $T^i \in S_D$, the vertex t_i is placed in the region with four vertices v_1, v_2, v_3, and v_4 of the graph W_4 on its boundary. The edge t_iv_5 crosses either v_3v_4 or v_1v_4 of W_4, and these two possibilities under our consideration are denoted by $\mathcal{B}_1$ and $\mathcal{B}_2$. Again, as for our purposes, it does not matter which of the regions is unbounded, we can assume that the drawings are as shown in Figure 5. Let $\mathcal{N}_D$ be the set of all configurations for the drawing D belonging to $\mathcal{N} = \{\mathcal{B}_1, \mathcal{B}_2\}$.

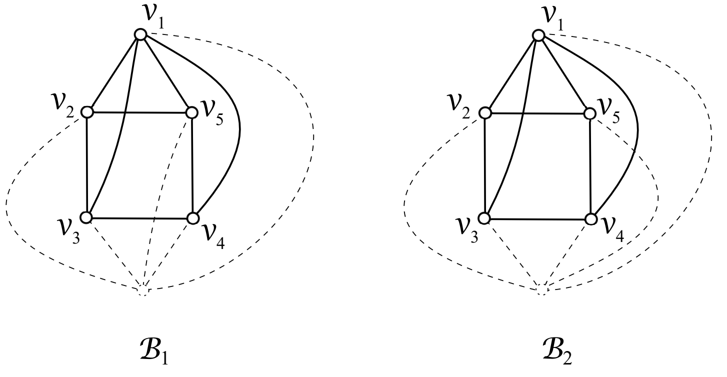

Fig. 5. Drawings of two possible configurations from $\mathcal{N}$ of the subgraph F^i

For any $T^i \in S_D$ with $\mathcal{B}_p \in \mathcal{N}_D$ of F^i, it is not difficult to verify that the edges of $W_4 \cup T^i$ are crossed at least four times by each subgraph T^l, $l \neq i$ using the subdrawing of F^i induced by D, see Figure 5. Thus, by fixing the subgraph $W_4 \cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(W_4 + P_n) &\geq 4\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 4(n-1) + 1 + 1\\ &\geq 4\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + n + \Big\lfloor\frac{n}{2}\Big\rfloor + 1.\end{aligned}$$

Case 3. $\mathrm{cr}_D(W_4) = 2$. Without lost of generality, we can consider the drawing of W_4 with the vertex notation like that in Figure 2(c). In this case, there is no possibility to obtain a subdrawing of $W_4 \cup T^i$ for a $T^i \in S_D$, that is, the set S_D must be empty. This fact, with the inequality (3.3), confirms that $r \geq 2$. So, we will discuss only about the subgraphs T^i whose edges do not cross the edges of W_4. For a $T^i \in R_D$, the reader can easily verify that the subgraph $F^i = W_4 \cup T^i$ is uniquely represented by $\mathrm{rot}_D(t_i) = (15432)$, and $\mathrm{cr}_D(T^i, T^j) \geq 4$ holds for any $T^j \in R_D$ with $j \neq i$ provided that $\mathrm{rot}_D(t_i) = \mathrm{rot}_D(t_j)$, for more see [24]. Moreover, it is not difficult to verify that $\mathrm{cr}_D(W_4 \cup T^i, T^l) \geq 4$ is true for any subgraph $T^l \notin R_D$ using the unique subdrawing $D(F^i)$. Thus, by fixing the graph $W_4 \cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(W_4 + P_n) &\geq 4\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 4(r-1) + 4(n-r) + 2\\ &= 4\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor + 4n - 2 \geq 4\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor + n + \Big\lfloor\frac{n}{2}\Big\rfloor + 1.\end{aligned}$$

We have shown, in all cases, that there is no good drawing D of the graph W_4+P_n with fewer than $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+\lfloor\frac{n}{2}\rfloor+1$ crossings. This completes the proof of Theorem 3.3. $\square$

4. THE CROSSING NUMBER OF W_4+C_n

In the proof of Theorem 4.4, the following lemma related to some restricted subdrawings of the graph W_4+C_n is going to be helpful.

Lemma 4.1. *Let D be a good drawing of W_4+C_n, $n\geq 3$. If the edges of $C_4(W_4)$ are crossed at least $\lceil\frac{n}{2}\rceil+2$ times, then there are at least $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+\lfloor\frac{n}{2}\rfloor+4$ crossings in D.*

Proof. As the wheel W_4 consists of two edge-disjoint subgraphs $C_4(W_4)$ and $S_4(W_4)$, then $\mathrm{cr}_D(C_4(W_4))+\mathrm{cr}_D(C_4(W_4),S_4(W_4)+C_n)\geq\lceil\frac{n}{2}\rceil+2$. The exact value for the crossing number of the graph S_4+C_n is given by Klešč *et al.* [15], i.e., $\mathrm{cr}(S_4+C_n)=4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor+2$. This enforces that the edges of $S_4(W_4)+C_n$ must be crossed at least $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor+2$ times in D. Consequently, we have

$$\begin{aligned}\mathrm{cr}_D(W_4+C_n)&=\mathrm{cr}_D(S_4(W_4)+C_n)+\mathrm{cr}_D(C_4(W_4))\\&\quad+\mathrm{cr}_D(C_4(W_4),S_4(W_4)+C_n)\\&\geq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lfloor\frac{n}{2}\right\rfloor+2+\left\lceil\frac{n}{2}\right\rceil+2\\&=4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lfloor\frac{n}{2}\right\rfloor+4.\end{aligned}$$ $\square$

Two vertices t_i and t_j of C_n^* are *antipodal* in a drawing of W_4+C_n if the subgraphs T^i and T^j do not cross. A drawing is *antipode-free* if it has no antipodal vertices.

Lemma 4.2. $\mathrm{cr}(W_4+C_3)=12$

Proof. Notice that the graph W_n+C_m is isomorphic with the graph W_m+C_n for all integers $m,n\geq 3$. By Klešč [14] was shown that $\mathrm{cr}(W_3+C_n)=2\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+4$, and so $\mathrm{cr}(W_4+C_3)=\mathrm{cr}(W_3+C_4)=12$. $\square$

Lemma 4.3. $\mathrm{cr}(W_4+C_4)=18$.

Proof. The possibility to add the edge t_1t_4 on the vertices of the path P_4^* into the subdrawing in Figure 3 with exactly three another crossings forces $\mathrm{cr}(W_4+C_4)\leq 18$. To prove the reverse inequality, suppose now that, there is a drawing D of the graph W_4+C_4 with at most 17 crossings. By Lemma 4.2, there are at most five crossings on edges of each subgraph T^i, otherwise, by deleting the edges of T^i with at least six crossings, a drawing of the graph homeomorphic to W_4+C_3 with at most eleven crossings is obtained. Consequently, using the pigeon principle, each subgraph T^i must be crossed by some subgraph T^j, $j\neq i$, no more than once.

Let us first show that the considered drawing D must be antipode-free. For a contradiction suppose, without loss of generality, that, T^1 does not cross T^2. If the edges of at least one of T^3 and T^4, say T^3, cross the edges of W_4 at least twice, then $\mathrm{cr}_D(W_4 \cup T^1 \cup T^2, T^3) \geq 2+4=6$ according to the well-known fact $\mathrm{cr}(K_{3,5})=4$. Three possible subdrawings of the graph W_4 induced by D are shown in Figure 2. They do not allow to obtain simultaneously the subgraphs T^3 and T^4 from two different sets R_D and S_D because just one of the sets R_D and S_D must be empty. Hence, if either $T^3, T^4 \in R_D$ or $T^3, T^4 \in S_D$, then $\mathrm{cr}_D(T^3, T^4) \geq 2$ holds provided that either $\mathrm{rot}_D(t_3) = \mathrm{rot}_D(t_4)$ or the lower bounds for the number of crossings of two configurations, respectively. Thus in all cases there are at least six crossings on edges of the subgraph T^3, a contradiction.

To finish the proof, let us assume that $\mathrm{cr}_D(T^i, T^j) \geq 1$ holds for every pair i, j, and let also $\mathrm{cr}_D(T^1, T^2) = 1$. If the edges of at least one of T^3 and T^4, say T^3, cross the edges of W_4 at least twice, then $\mathrm{cr}_D(W_4 \cup (T^1 \cup T^2) \cup T^4, T^3) \geq 2+3+1=6$ again according to $\mathrm{cr}(K_{3,5})=4$. If either $T^3, T^4 \in R_D$ or $T^3, T^4 \in S_D$, then the proof proceeds in the similar way as in the previous case. □

Theorem 4.4. $\mathrm{cr}(W_4+C_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ *for* $n \geq 3$.

Proof. Into the drawing in Figure 3, it is possible to add the edge t_1t_n which forms the cycle C_n^* on the vertices of the path P_n^* with exactly three another crossings. Thus, $\mathrm{cr}(W_4+C_n) \leq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$. By Lemma 4.2 and 4.3, the result holds for $n=3$ and $n=4$. We prove the reverse inequality by induction on n. Suppose now that, for some $n \geq 5$, there is a drawing D with

$$\mathrm{cr}_D(W_4+C_n) < 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 4, \tag{4.1}$$

and that

$$\mathrm{cr}(W_4+C_m) = 4\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor + m + \left\lfloor \frac{m}{2} \right\rfloor + 4 \text{ for any integer } m < n. \tag{4.2}$$

We claim that the considered drawing D must be antipode-free. For a contradiction suppose, without loss of generality, that $\mathrm{cr}_D(T^{n-1}, T^n) = 0$. If at least one of T^{n-1} and T^n, say T^n, does not cross W_4, it is not difficult to verify in Figure 2 that T^{n-1} must cross W_4 at least three times. Using the positive lower bounds for the number of crossings of two configurations in Table 3, the subgraphs T^{n-1} and T^n are not both from the set S_D, that is, $\mathrm{cr}_D(W_4, T^{n-1} \cup T^n) \geq 3$. By [11], we already know that $\mathrm{cr}(K_{5,3}) = 4$, which yields that each T^k, $k = 1, 2, \ldots, n-2$, crosses the edges of the subgraph $T^{n-1} \cup T^n$ at least four times. So, for the number of crossings in D we have

$$\begin{aligned}
&\mathrm{cr}_D\left(W_4+C_{n-2}\right) + \mathrm{cr}_D(T^{n-1} \cup T^n) \\
&+ \mathrm{cr}_D(W_4, T^{n-1} \cup T^n) + \mathrm{cr}_D(K_{5,n-2}, T^{n-1} \cup T^n) \\
&\quad \geq 4\left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor + n - 2 + 4 + 0 + 3 + 4(n-2) \\
&\quad = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor + 4.
\end{aligned}$$

This contradiction with the assumption (4.1) confirms that D is antipode-free. As the graph $W_4 + D_n$ is a subgraph of the graph $W_4 + C_n$, also by Theorem 3.1, the edges of $W_4 + C_n$ are crossed at least $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor$ times, and therefore, at most three edges of the cycle C_n^* can be crossed in D. This also enforces that the vertices t_i of the cycle C_n^* must be placed at most in three different regions in the considered good subdrawing of W_4. Our assumption on D, together with $\mathrm{cr}(K_{5,n}) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$, implies that

$$\mathrm{cr}_D(W_4) + \mathrm{cr}_D(W_4, K_{5,n}) \le \mathrm{cr}_D(W_4) + 0r + 1s + 2(n-r-s) \le n + \left\lfloor\frac{n}{2}\right\rfloor + 3, \quad (4.3)$$

if we use the notation $r = |R_D|$ and $s = |S_D|$ again. This forces that $2r + s \ge \left\lceil\frac{n}{2}\right\rceil + \mathrm{cr}_D(W_4) - 3$, and if $r = 0$ then $s \ge \left\lceil\frac{n}{2}\right\rceil - 3$. Of course, in this case, the set S_D is certainly nonempty for each $n \ge 7$. Again by Lemma 2.1, there is no crossing among edges of $C_4(W_4)$ in all contemplated subdrawings of the graph W_4. Now, we will deal with the possibilities of obtaining a subgraph $T^i \in R_D \cup S_D$ in the drawing D and we show that in all cases a contradiction with the assumption (4.1) is obtained.

Case 1. $\mathrm{cr}_D(W_4) = 0$. In this case, without lost of generality, we can choose the vertex notation of the graph W_4 as shown in Figure 2(a). The unique subdrawing of W_4 induced by D contains five different regions. Let us denote these five regions by $\omega_{1,2,3,4}$, $\omega_{1,2,5}$, $\omega_{1,4,5}$, $\omega_{2,3,5}$, and $\omega_{3,4,5}$ depending on which of vertices are located on the boundary of the corresponding region. Because no face is incident to all vertices in $D(W_4)$, there is no possibility to obtain a subdrawing of $W_4 \cup T^i$ for a $T^i \in R_D$, that is, $r = 0$. Since the vertices of C_n^* do not have to be placed in the same region in the considered subdrawing of W_4, three possible subcases may occur:

a) All vertices t_i of C_n^* are placed in the same region of subdrawing of W_4. If they are placed in the region $\omega_{1,2,3,4}$, then each subgraph T^i crosses some edge of $C_4(W_4)$ at least once. As $n \ge \left\lceil\frac{n}{2}\right\rceil + 2$, for $n \ge 5$, Lemma 4.1 forces a contradiction with (4.1) in D. Now, let us turn to the good drawing D of the graph $W_4 + C_n$ with the assumption that all vertices of C_n^* are placed in some region of subdrawing of W_4 with three vertices of the graph W_4 on its boundary. From their symmetry, we can suppose that $t_i \in \omega_{2,3,5}$ for each t_i, $i = 1, \ldots, n$. Let us denote by H_1 the subgraph of W_4 with the vertex set $V(W_4)$, and the edge set $E(W_4) \setminus \{v_2v_3, v_2v_5, v_3v_5\}$. Since the exact value for the crossing number of the graph $H_1 + D_n$ is given in [20], i.e., $\mathrm{cr}(H_1 + D_n) = 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$, the edges of $H_1 + D_n$ are crossed at least $4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$ times in D. As each subgraph T^i crosses edges of the cycle $v_2v_3v_5v_2$ at least twice, we obtain

$$\mathrm{cr}_D(W_4 + C_n) \ge 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 2n \ge 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + \left\lfloor\frac{n}{2}\right\rfloor + 4.$$

b) All vertices of C_n^* are placed in two regions of subdrawing of W_4. Clearly, the edges of C_n^* have to cross the edges of W_4 at least twice. Let $D^{'}$ be the subdrawing of $W_4 + D_n$ induced by D without the edges of C_n^*. Inequality (4.3) can be used to estimate the number of crossings on the edges of W_4 in $D^{'}$, that is, $s + 2(n-s) \le n + \left\lfloor\frac{n}{2}\right\rfloor + 1$. This enforces that $s \ge \left\lceil\frac{n}{2}\right\rceil - 1$. So, for $n \ge 5$, there is at least one subgraph T^i which crosses the edges of W_4 just once. All vertices t_i of subgraphs $T^i \in S_D$ must be in the region of subdrawing of W_4 with four vertices of the graph W_4 on its boundary. For the rest of the proof, we may therefore assume that $\omega_{2,3,5}$ is the second region. As the edges of C_n^* cross the edges of $C_4(W_4)$ twice, due to Lemma 4.1, we only consider the case for $s = \lceil \frac{n}{2} \rceil - 1$. Again, inequality (4.3) can be amplified by using the notion $t = |T_D|$, that is,

$$s + 2t + 3(n - s - t) \leq n + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

The resulting inequality $t \geq \lfloor \frac{n}{2} \rfloor + 1$ can be replaced by $t = \lfloor \frac{n}{2} \rfloor + 1$ provided that $s + t \leq n$ and $s = \lceil \frac{n}{2} \rceil - 1$. Since all vertices t_j of subgraphs $T^j \in T_D$ are placed in $\omega_{2,3,5}$, each subgraph $T^i \in S_D$ can only cross one from the edges v_1v_2, v_2v_3, and v_3v_4. In the rest of paper, based on their symmetry, let the edge v_1v_2 be crossed trough them at least as many times as the edge v_3v_4. Let us denote by H the subgraph of W_4 with the vertex set $V(W_4)$, and the edge set $E(W_4) \setminus \{v_1v_4, v_1v_5, v_3v_4, v_4v_5\}$. We are able to estimate the lower bound equal to $n+2$ for the number of crossings on edges of the graph H in D'. More precisely,

$$2t + \left\lceil \frac{s}{2} \right\rceil = 2\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) + \left\lceil \frac{\lceil \frac{n}{2} \rceil - 1}{2} \right\rceil \geq n + 2.$$

Let G be the graph difference of graphs W_4 and H, that is, $G = W_4 - H$. Since $\mathrm{cr}(G + D_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ by [20], the edges of $G + D_n$ are crossed at least $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ times in D'. Then

$$\begin{aligned}\mathrm{cr}_{D'}(W_4 + D_n) &= \mathrm{cr}_{D'}(G + D_n) + \mathrm{cr}_{D'}(H, G + D_n) \\ &\geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + n + 2.\end{aligned}$$

As the edges of C_n^* cross the edges of W_4 at least twice, we also obtain a contradiction with (4.1) in D

$$\mathrm{cr}_D(W_4 + C_n) \geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + n + 2 + 2.$$

c) All vertices of C_n^* are placed in three regions of subdrawing of W_4. Recall that at most three edges of the cycle C_n^* can be crossed in D. So, the edges of C_n^* cross the edges of W_4 three times, including the edges of $C_4(W_4)$ twice, which yields that the drawing D without the edges of C_n^* is an optimal drawing of the graph $W_4 + D_n$. This fact, with inequality (4.3), confirms that $s + 2(n - r - s) \leq n + \lfloor \frac{n}{2} \rfloor$. The last inequality implies that there are at least $\lceil \frac{n}{2} \rceil$ subgraphs T^i which cross the edges of $C_4(W_4)$ at least once. Consequently, Lemma 4.1 also contradicts the assumption of D.

Case 2. $\mathrm{cr}_D(W_4) = 1$. Without lost of generality, we can choose the vertex notation of the graph W_4 in such a way as shown in Figure 2(b). The set R_D is empty provided by no face is incident to all vertices in $D(W_4)$. This fact, with inequality (4.3) in the form $1 + 1s + 2(n - s) \leq n + \lfloor \frac{n}{2} \rfloor + 3$, confirms that $s \geq \lceil \frac{n}{2} \rceil - 2$, which yields that

$s \geq 1$. If we discuss about the subgraphs T^i whose edges cross the edges of W_4 exactly once, then the edge t_iv_5 crosses either v_3v_4 or v_1v_4 of W_4. These two possibilities under our consideration are denoted by $\mathcal{B}_1$ and $\mathcal{B}_2$, for more see Figure 5. In the rest of the paper, let us also assume that the number of subgraphs with the configuration $\mathcal{B}_1$ is at least as much as the number of subgraphs with the configuration $\mathcal{B}_2$, and let $T^i \in S_D$ be such a subgraph with the configuration $\mathcal{B}_1$ of F^i. Hence,

$$\sum_{l\neq i,\ T^l\in S_D} \mathrm{cr}_D(T^i, T^l) \geq \frac{7}{2}(s-2)+3,$$

that is,

$$\sum_{l\neq i,\ T^l\in S_D} \mathrm{cr}_D(W_4\cup T^i, T^l) \geq \frac{9}{2}(s-2)+4,$$

where an idea of the arithmetic mean of the values four and three could be exploited ($\mathrm{cr}(\mathcal{B}_1,\mathcal{B}_2) \geq 3$ and $\mathrm{cr}(\mathcal{B}_j,\mathcal{B}_j) \geq 4$ were also established in [19]). Let us denote by H the subgraph of W_4 with the vertex set $V(W_4)$, and the edge set $E(H) = \{v_1v_2, v_1v_3, v_1v_4, v_3v_4\}$. The graph H contains the cycle $v_1v_3v_4v_1$ as a subgraph by which the vertices v_2 and v_5 are separated in $D(H)$, that is, each T^k crosses the edges of H at least once. Let G be the graph difference of graphs W_4 and H, that is, $G = W_4 - H$. If there is a vertex t_k of the cycle C_n^* with the placement in the inner region of the cycle $v_1v_3v_4v_1$, then

$$\begin{aligned}\mathrm{cr}_D(W_4+C_n) &= \mathrm{cr}_D(G+C_n)+\mathrm{cr}_D(H)+\mathrm{cr}_D(H,G+C_n)\\ &\geq 4\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor+\Big\lfloor\frac{n}{2}\Big\rfloor+1+1+n+2,\end{aligned}$$

because the crossing number of $G+C_n$ is given in [18], i.e., $\mathrm{cr}(G+C_n) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor+1$ and the cycle C_n^* crosses H at least twice. This contradiction with the assumption (4.1) enforces that all vertices t_k of C_n^* are placed in the outer region of the cycle $v_1v_3v_4v_1$. Now, with respect to this restriction, we can verify that $\mathrm{cr}_D(W_4\cup T^i, T^l) \geq 4$ holds for each $T^l \notin S_D$. Thus, by fixing the graph $W_4\cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(W_4+C_n) &\geq 4\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor+\frac{9}{2}(s-2)+4+4(n-s)+2\\ &= 4\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor+4n+\frac{1}{2}s-3\\ &\geq 4\Big\lfloor\frac{n-1}{2}\Big\rfloor\Big\lfloor\frac{n-2}{2}\Big\rfloor+4n+\frac{1}{2}\bigg(\Big\lceil\frac{n}{2}\Big\rceil-2\bigg)-3\\ &\geq 4\Big\lfloor\frac{n}{2}\Big\rfloor\Big\lfloor\frac{n-1}{2}\Big\rfloor+\Big\lfloor\frac{n}{2}\Big\rfloor+n+4.\end{aligned}$$

The last inequality does not apply only to $n = 5$, thus we will discuss this case separately. We show that in all cases the contradiction with the number of crossings at most eight on edges of T^i is obtained, otherwise, by deleting the edges of T^i

with at least nine crossings, a drawing of the graph homeomorphic to $W_4 + C_4$ with fever than eighteen crossings is obtained.

If $s \geq 3$, say $T^i, T^j, T^k \in S_D$, then $\mathrm{cr}_D(T^i, T^j) \geq 3$ and $\mathrm{cr}_D(T^i, T^k) \geq 3$ hold according to the lower bounds for the number of crossings of two configurations. Since the drawing D is antipode-free, the subgraph $T^i \in S_D$ crosses the edges of the graph $W_4 \cup T^\ell \cup T^m$ at least three times. To finish the proof, let us suppose that $s \leq 2$ and let us also emphasize that $t \geq 6 - 2s$ provided by inequality (4.3) in the form $1 + 1s + 2t + 3(5 - s - t) \leq 5 + \lfloor \frac{5}{2} \rfloor + 3$. Further, for each $T^i \in S_D$, the edges of T^i are crossed by any $T^k \in T_D$ at least twice. If $s = 2$, say $T^i, T^j \in S_D$, then $t \geq 2$. For $T^k, T^\ell \in T_D$, we obtain $\mathrm{cr}_D(W_4 \cup T^j \cup T^k \cup T^l \cup T^m, T^i) \geq 1 + 3 + 2 + 2 + 1 = 9$. If $s = 1$, say $T^i \in S_D$, then $t \geq 4$. It means that T^i contains at least two crossings with all four $T^j, T^k, T^\ell, T^m \in T_D$ and one with the graph W_4. Thus in all cases the subgraph $T^i \in S_D$ contains at least nine crossings on its edges, a contradiction.

Case 3. $\mathrm{cr}_D(W_4) = 2$. Without lost of generality, we can consider the drawing of W_4 with the vertex notation like that in Figure 2(c). In this case, there is no possibility to obtain a subdrawing of $W_4 \cup T^i$ for a $T^i \in S_D$, that is, $s = 0$. This fact, with the inequality (4.3) in the form $2 + 2(n - r) \leq n + \lfloor \frac{n}{2} \rfloor + 3$, enforces $r \geq 1$. For any $T^i \in R_D$, the subgraph $F^i = W_4 \cup T^i$ is uniquely represented by $\mathrm{rot}_D(t_i) = (15432)$, and so we can verify that $\mathrm{cr}_D(W_4 \cup T^i, T^l) \geq 4$ is fulfilling for any subgraph $T^l \notin R_D$ using this unique subdrawing $D(F^i)$. Thus, by fixing the subgraph $W_4 \cup T^i$, we have

$$\begin{aligned}\mathrm{cr}_D(W_4 + C_n) &\geq 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4(r-1) + 4(n-r) + 2 \\ &= 4\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 4n - 2 \geq 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + n + 4.\end{aligned}$$

Although the last inequality does not apply only to $n = 5$, we are able to justify the existence of at least one other crossing and thus confirm its truth for all $n \geq 5$. Since each subgraph $T^k \in T_D$ crosses some edge of the cycle $C_4(W_4)$ at least once, by Lemma 4.1 for $n = 5$ and for $\mathrm{cr}_D(C_4(W_4)) = 2$, there are at most three subgraphs whose edges cross the edges of W_4 exactly twice. This fact, with inequality (4.3) in the form $2 + 2t + 3(5 - r - t) \leq 5 + \lfloor \frac{5}{2} \rfloor + 3$, implies $r \geq 2$. For $T^i \in R_D$, there are at least two different T^j and T^k such that $\mathrm{cr}_D(T^i, T^j) = 1$ and $\mathrm{cr}_D(T^i, T^k) = 1$, otherwise, by deleting the edges of T^i with more than eight crossings, a drawing of the graph homeomorphic to $W_4 + C_4$ with fewer than eighteen crossings is obtained. If any of the vertices t_j and t_k is not placed in the same region of subdrawing of W_4 as the vertex t_i, then there at least two new crossings on edges of C_5^* in D. If both are located in the same region as the vertex t_i, then each of them crosses the edges of $W_4 \cup T^i$ at least five times.

Thus, it was shown in all mentioned cases that there is no good drawing D of the graph $W_4 + C_n$ with fewer than $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ crossings, and the proof of Theorem 4.4 is complete. □

5. CONCLUSIONS

Let W_n and S_n denote the wheel and the star on $n+1$ vertices, respectively. In general, the graph $S_n + C_m$ is isomorphic with the graph $W_m + D_n$ for all integers $n \geq 1$ and $m \geq 3$. Based on this knowledge, Klešč *et al.* [15] also established the crossing numbers of the graphs $W_m + D_n$ for $3 \leq n \leq 5$ and $m \geq 3$. The crossing number of $W_4 + D_n$ was recently determined for any $n \geq 1$ by Staš [19]. Due to the possibility to generalize the optimal drawing for $W_4 + P_n$ in Figure 3 onto the drawings of the graphs $W_m + P_n$, we are able to postulate that

$$\operatorname{cr}(W_m + P_n) = Z(m+1)Z(n) + (Z(m) - 1)\left\lfloor \frac{n}{2} \right\rfloor + n + 1,$$

for all $m \geq 3$ and $n \geq 2$. Recently, this conjecture was proved for the graph $W_3 + P_n$ by Klešč and Schrötter [16]. Theorem 3.3 also confirms the validity of this conjecture for $W_4 + P_n$. On the other hand, the graphs $W_m + P_2$ and $W_m + P_3$ are isomorphic with the join product of the cycle C_m with the cycle C_3 and with the graph $K_4 \setminus e$ obtained by removing one edge from the complete graph K_4, respectively. The exact values for the crossing numbers of the graphs $C_m + C_n$ are given by Klešč [13], that is, $\operatorname{cr}(C_m + C_n) = Z(m)Z(n) + 2$ for any $m, n \geq 3$ with $\min\{m, n\} \leq 6$. The crossing numbers of $K_4 \setminus e + C_m$ equal to $2\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + 3$ were established also by Klešč [14]. These facts allow us to determine another results for the join product of the wheels W_m with the path on two and three vertices.

Theorem 5.1. $\operatorname{cr}(W_m + P_2) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + 2$ *for* $m \geq 3$.

Theorem 5.2. $\operatorname{cr}(W_m + P_3) = 2\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor + 3$ *for* $m \geq 3$.

One can easily verify that these results also confirm the validity of our conjecture for the graphs $W_m + P_2$ and $W_m + P_3$. Theorem 4.4 confirms the validity of the conjecture presented by Yue *et al.* [25] for the crossing number of $W_4 + C_n$. As we mentioned earlier, the graph $W_n + C_m$ is isomorphic with the graph $W_m + C_n$ for all integers $m, n \geq 3$. The crossing number of $W_3 + C_n$ obtained by Klešč in [14] and the crossing number of $W_4 + C_n$ by Theorem 4.4 force us further results for the join product of the wheels W_n with the cycles on three and four vertices.

Theorem 5.3. $\operatorname{cr}(W_n + C_3) = 2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + 4$ *for* $n \geq 3$.

Theorem 5.4. $\operatorname{cr}(W_n + C_4) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor + 4$ *for* $n \geq 3$.

Note that Berežný and Staš [3] have already established the conjecture that the crossing number of $W_m + D_n$ is equal to $Z(m+1)Z(n) + (Z(m) - 1)\lfloor \frac{n}{2} \rfloor + n$ for m at least three.

Acknowledgements

The authors would like to thank the referees for their careful readings and many constructive comments to improve the quality of the paper.

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Received: April 17, 2020.
Revised: October 30, 2020.
Accepted: November 23, 2020.