The Numerical Analysis of the Elementary, Fractional Order, Interval Transfer Function

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Abstract: In the paper the analysis of the impact of the interval uncertainty of parameters on the behaviour of the elementary Fractional Order (FO) transfer function is investigated. The fractional order and quasi time constant are defined as intervals describing deviation from nominal values. Such an analysis has not be considered yet. The proposed elementary, interval model can be applied in modeling of different, uncertain-parameters elements and physical phenomena. For the considered transfer function the methodology of its numerical analysis is proposed and illustrated by simulations. Results of numerical tests point that the best robustness of the model is achieved for relatively lower values of its parameters.

Keywords: fractional order transfer function, Caputo definition, interval parameters, sensitivity, time t_{90}

1. Introduction

A fractional order transfer function is a convenient tool to describe many different physical phenomena. This is mentioned by many books and papers, e.g. [1, 3, 10].

Simultaneously, it is well known, that each real measurement is disturbed by various external factors. This implies that a model of such a disturbed process should take into account this uncertainty. This can be done using different mathematical tools. For example the conference presentation [8] proposes models with parametric uncertainty, [13] deals with fractional order chaotic systems with uncertain parameters, article [7] consideres the two-norm bounded uncertainty the infinity-norm bounded uncertainty. Interval calculus is one of mathematical tools well describing different kinds of uncertainty.

Interval calculus is one of mathematical tools well describing different kinds of uncertainty. This approach in FO systems is presented e.g. in the paper [9], proposing the robust FOPID controller for plant described by an interval, fractional order transfer function

This paper proposes the methodology of numerical analysis of properties for the elementary, fractional order transfer function model. The parameters of the model: order and quasi-time constant are described by the interval numbers. For this plant the numerical algorithm of computing of the t_{90} time is given as well as the sensitivity of the step response to uncertainty of parameters is examined. The numerical approach is imposed by the fact that the explicit analytical form of the inverse Mittag-Leffler function is not known. Such an analysis has not been presented yet. Presented results are useful in analysis of a behaviour of phenomena and elements possible to describe by elementary FO transfer function, for example measurement sensors.

The paper is organized as follows. Preliminaries draw theoretical background to presenting of main results. Next the proposed interval transfer function is proposed and numerically analysed. Finally results are discussed.

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2. Preliminaries

2.1. Basics of fractional calculus

Basics of fractional calculus are given by many books, e.g. [2, 4, 11, 12]. Here only some definitions necessary to present of main results will be recalled.

First of all the fractional-order, integro-differential operator (see e.g. [2, 5, 12]) needs to be given. It is as follows:

Definition 1 (*The elementary fractional order operator*) *The fractional-order integro-differential operator is defined as follows*:

$$ {}_{t_s}D^{\alpha}_{t_f} f(t) = \begin{cases} \dfrac{d^{\alpha} f(t)}{dt^{\alpha}} & \alpha > 0 \\ f(t) & \alpha = 0 \\ \int\limits_{t_s}^{t_f} f(\tau)(d\tau)^{\alpha} & \alpha < 0 \end{cases} \tag{1}$$

where t_s and t_f denote time limits for operator calculation, $\alpha \in \mathbb{R}$ denotes the non-integer order of the operation.

Next remember the complete Gamma Euler function (see e.g. [5]):

Definition 2 (*The Gamma function*)

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \tag{2}$$

Mittag-Leffler function is a non-integer order generalization of exponential function $e^{\lambda t}$ and it plays crucial role in solution of FO state equation. The one parameter Mittag-Leffler function is defined as follows:

Definition 3 (*The one parameter Mittag-Leffler function*)

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)} \tag{3}$$

The two parameter Mittag-Leffler function is defined as follows:

Definition 4 (*The two parameters Mittag-Leffler function*)

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + \beta)} \tag{4}$$

For $\beta = 1$ the two parameter function (4) turns to one parameter function (3).

It is important to note that the analytical formula of the inverse Mittag-Leffler function is not known. This inverse function can be only computed numerically for particular values of α and x. Such an approach is presented e.g. in [6] and it will be applied in this paper.

The fractional-order, integro-differential operator can be described by different definitions, given by Grünvald and Letnikov (GL definition), Riemann and Liouville (RL definition) and Caputo (C definition). In the further consideration only C definition will be used. It is recalled below ([1]).

Definition 5 (*The Caputo definition of the FO operator*)

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\infty} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{5}$$

where $n - 1 < \alpha <$ denotes the non-integer order of operation and $\Gamma(..)$ is the complete Gamma function expressed by (2).

For the Caputo operator the Laplace transform can be given (see for example [4]):

Definition 6 (*The Laplace transform of the Caputo operator*)

$$\mathcal{L}\left({}_0^C D_t^\alpha f(t)\right) = s^\alpha F(s), \quad \alpha < 0$$
$$\mathcal{L}\left({}_0^C D_t^\alpha f(t)\right) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} {}_0D_t^k f(0) \tag{6}$$
$$\alpha > 0, \quad n-1 < \alpha \le n, \quad n \in N.$$

Consequently, the inverse Laplace transform for non-integer order function is expressed as follows ([5]):

$$\mathcal{L}^{-1}\left[s^\alpha F(s)\right] = {}_0D_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{t^{k-1}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+) \tag{7}$$
$$n-1 < \alpha < n, \quad n \in \mathbb{Z}.$$

2.2. Elementary FO transfer function

The elementary, scalar input-output differential equation using elementary fractional operator (1) takes the following form:

$$T\, {}_0D_t^\alpha y(t) = -y(t) + u(t). \tag{8}$$

where T is the quasi-time constant, expressed in [$second^\alpha$], $u(t)$ is the control signal and $y(t)$ is the output.

Assume homogenous initial condition. Applying (6) in (8) gives the elementary, fractional order transfer function:

$$G(s) = \frac{1}{Ts^\alpha + 1}. \tag{9}$$

For this transfer function its impulse and step responses are as beneath (see e.g. [1], p. 11):

$$g(t) = \frac{t^{\alpha-1}}{T} E_\alpha\left(-\frac{t^\alpha}{T}\right). \tag{10}$$

$$y(t) = 1(t) - E_\alpha\left(-\frac{t^\alpha}{T}\right). \tag{11}$$

In (10) and (11) $E_\alpha(..)$ is the one parameter Mittag-Leffler function (3).

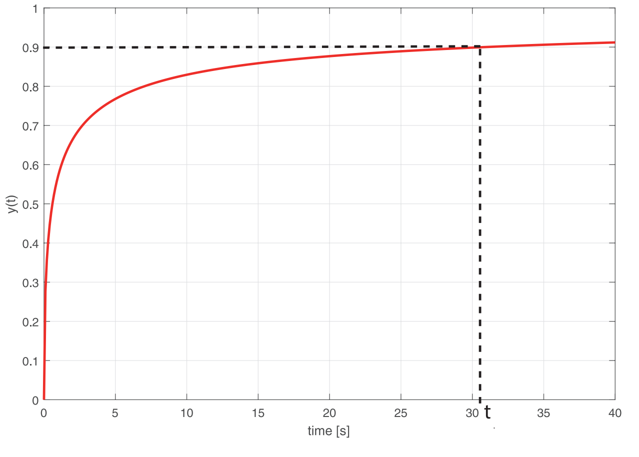

Fig. 1. The t_{90} time for step response (11) computed for: T = 1 [s^α] and α = 0.5
Rys. 1. Czas t_{90} dla odpowiedzi skokowej (11) wyznaczonej dla: T = 1 [s^α] oraz α = 0,5

For a plant or device described by a transfer function close to (9) an important parameter is the so called t_{90} time. This is the time for which the step response of a plant achieves 90 % of its steady-state response (see Figure 1). The implicit definition of the t_{90} time is as follows:

$$E_\alpha\left(-\frac{t_{90}^\alpha}{T}\right) = 0.1. \tag{12}$$

Computing of the t_{90} time from (12) requires to calculate the inverse Mittag-Leffler function. Unfortunately, its explicit analytical formula (in contrast to exponential and logarithm functions) is not known. Here only the numerical approach can be employed (see e.g. [6]). It consists in numerical solving of the equation (13) with respect to t. To do it the MATLAB function *fzero* can be employed.

$$E_\alpha\left(-\frac{t^\alpha}{T}\right)-0.1=0. \tag{13}$$

For $\alpha = 1.0$ the Mittag-Leffler function turns to the exponential function and the time t_{90} can be calculated analytically:

$$t_{90_1}=2.3026T. \tag{14}$$

3. Main results

3.1. Algorithm of calculating of the t_{90} time for FO transfer function with known parameters

The time instant t_t, when the step response $y(t_t)$ achieves predefined, threshold value $0.0 < y_t < 1.0$ is expressed by the following proposition.

Proposition 1 (*The time of achievement of the predefined value by the step response* $y(t)$)
Consider the fractional order transfer function (9) *and its step response* (11).
The step response achieves the predefined threshold value y_t *after time* t_t *equal:*

$$t_t=T^{\frac{1}{\alpha}}t_1^{\frac{1}{\alpha}}, \tag{15}$$

when $t_1^{\frac{1}{\alpha}}$ *is the numerical solution of the following equation*:

$$1-y_t-E_\alpha\left(-t_1^\alpha\right)=0. \tag{16}$$

Proof 1 *Threshold value* y_t *is expressed as follows*:

$$y_t=1-E_\alpha\left(-\frac{t_t^\alpha}{T}\right)\quad\Leftrightarrow\quad 1-y_t=E_\alpha\left(-\frac{t_t^\alpha}{T}\right). \tag{17}$$

Denote the inverse Mittag-Leffler function by $L_\alpha(..)$. *Consequently the time* t_t *can be expressed as follows*:

$$t_t=-T^{\frac{1}{\alpha}}\left(L_\alpha\left(1-y_t\right)\right)^{\frac{1}{\alpha}}. \tag{18}$$

In equation (18) *assume* $T = 1$. *By comparing* (18) *and* (15) *one obtains* (17) *and the proof is completed.*

To calculate the time t_{90} assume that $y_t = 0.9$. Exemplary values of t_{90} for $T = 1$ and selected α are presented in the tables 1 and 2.

Tab. 1. Values of t_{90} for T = 1 and 0.0 < α < 0.5
Tab. 1. Wartości czasu t_{90} dla T = 1 i 0,0 ≤ α < 0,5

α	0.10	0.20	0.25	0.30	0.40
t_1	> 1e+06	28831.46	3084.80	685.62	101.17

Tab. 2. Values of t_{90} for T = 1 and 0.5 ≤ α ≤ 1.0
Tab. 2. Wartości czasu t_{90} dla T = 1 i 0,5 ≤ α ≤ 1,0

α	0.50	0.60	0.70	0.75	0.80	0.90	0.95	1.00
t_1	30.85	13.48	7.22	5.57	4.43	3.10	2.61	2.30

3.2. The proposed, uncertain-parameter transfer function

Consider the transfer function (9) and assume that its both parameters are described by the following intervals:

$$\begin{aligned}&\alpha\in\left[\underline{\alpha};\overline{\alpha}\right]\subset\left(0;1\right)\\&T\in\left[\underline{T};\overline{T}\right]\in\mathbb{R}^+.\end{aligned} \tag{19}$$

where the borders of intervals describe the deviation of parameters from their nominal values denoted by index "n":

$$\begin{aligned}\underline{\alpha}&=\alpha_n-d\alpha,\\\overline{\alpha}&=\alpha_n+d\alpha.\end{aligned} \tag{20}$$

$$\begin{aligned}\underline{T}&=T_n-dT\\\overline{T}&=T_n+dT\end{aligned} \tag{21}$$

Each couple of parameters builds the vector of uncertain parameters q:

$$q=\left[\alpha;T\right]. \tag{22}$$

The whole space of uncertain parameters $Q = [\alpha \times T]$ can be interpreted as the rectangle in the $\mathrm{I}\left(\mathbb{R}^2\right)$ space. Its center is described by the nominal values α_n and T_n and its vertices are the border values of intervals α and T described by (19).

For interval parameters the step response (11) expands to the sector limited by the border values and consequently the time t_{90} also turns to an interval. The estimation of this interval is interesting from point of view of applications of the model we deal with.

3.3. The sensitivity of the model to its parameters uncertainty

The impact of the uncertainty of parameters to a behaviour of the proposed model can be estimated as the sensitivity of its step response. The difference between nominal and disturbed step responses equals to:

$$\Delta y\left(t,q\right)=E_\alpha\left(\frac{t^\alpha}{T}\right)-E_{\alpha_n}\left(\frac{t^{\alpha_n}}{T_n}\right). \tag{23}$$

In (23) index "n" denotes the nominal value and $\left[\alpha,T\right]\in q$ are the perturbed parameters.

Using $\Delta y(t)$ the following sensitivity functions can be proposed:

$$S_\infty\left(q\right)=\max_{0<t<t_f}\left|\Delta y\left(t\right)\right|. \tag{24}$$

$$S_2\left(q\right)=\int_0^{t_f}\left(\Delta y\left(t\right)\right)^2dt. \tag{25}$$

Third proposed sensitivity function describes the dependence of the t_{90} time on uncertain parameters q.

$$S_t(q) = \frac{\left| t_{90_n} - t_{90} \right|}{t_{90_n}} \cdot 100\ \%. \tag{26}$$

All the functions (24), (25) and (26) can be computed numerically for given interval set Q. An example of the numerical analysis is presented in the next section.

4. The numerical analysis

4.1. The analysis of the sensitivity of the step response

The tested intervals are given in the Table 3. In each case the deviation from the nominal value was equal 10 %, the nominal value is given in the bracket. The step responses for nominal and extreme values of vectors $q_{1,2,3}$ are shown in the Figure 2. Three-dimensional and contour plots of the sensitivity functions (25) and (24) for vectors $q_{1,2,3}$ are shown in the Figures 3–5.

Tab. 3. The tested vectors *q*
Tab. 3. Testowane wektory *q*

vector q	q_1	q_2	q_3
α (α_n)	[0.225;275], (0.25)	[0.45;0.55], (0.50)	[0.675;0.825], (0.75)
T (T_n)	[0.09;0.11], (0.10)	[0.9;1.1], (1.00)	[4.50;5.50], (5.00)

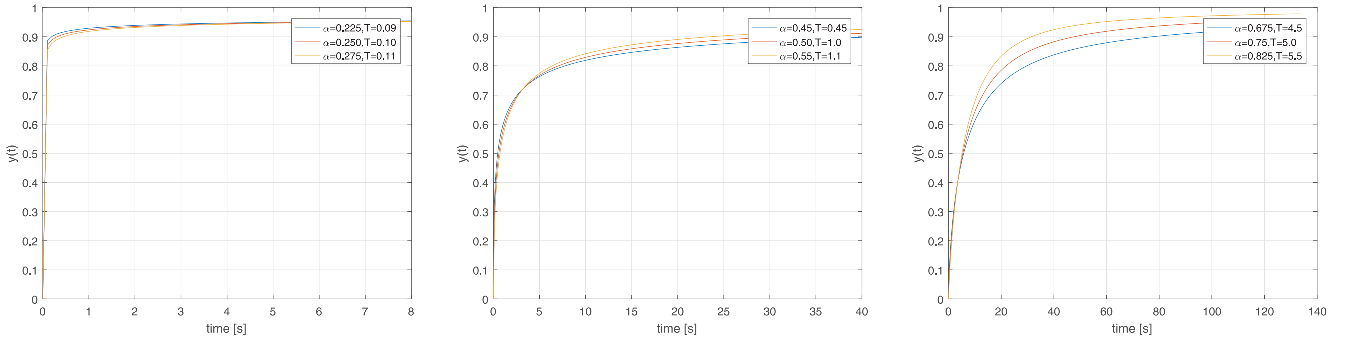

Fig. 2. The nominal and disturbed step responses (11) for vectors $q_{1,2,3}$
Rys. 2. Odpowiedzi skokowe przy nominalnych i zaburzonych parametrach dla wektorów $q_{1,2,3}$

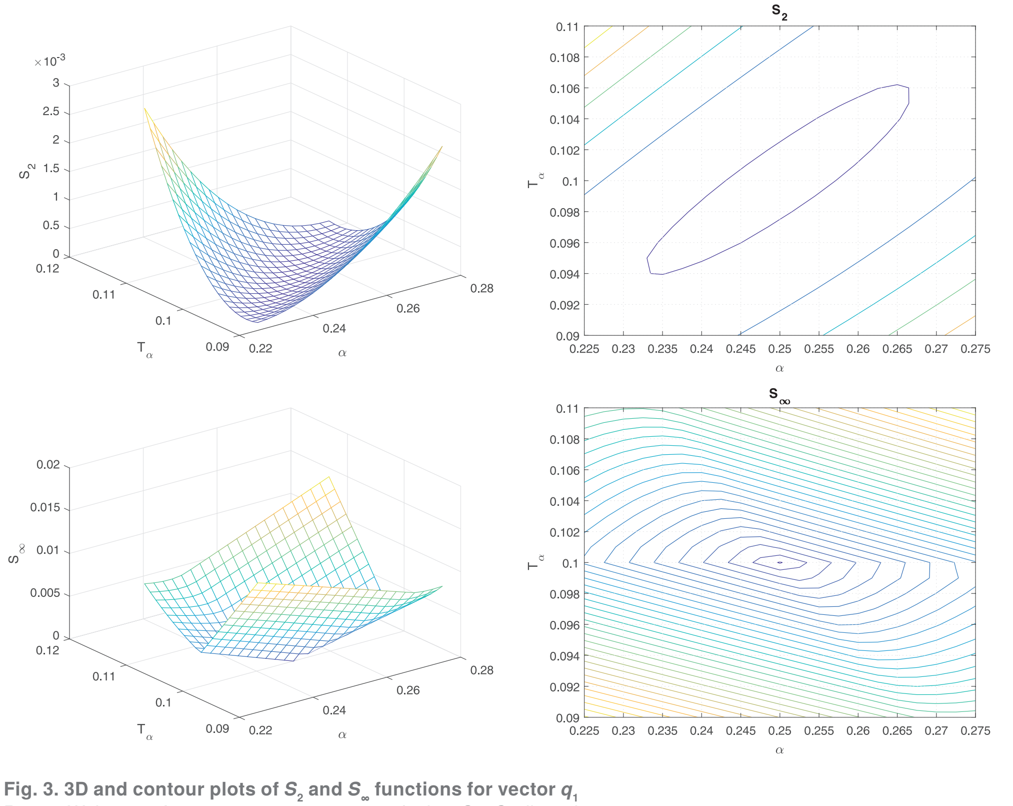

Fig. 3. 3D and contour plots of S_2 and S_∞ functions for vector q_1
Rys. 3. Wykresy trójwymiarowe i poziomicowe funkcji S_2 i S_∞ dla wektora q_1

From plots 3—5 it can be concluded that the sensitivity of the step response in the sense of functions S_∞ and S_2 strongly depends on the fractional order α and quasi-time constant T.

In particular (see Figure 3), for smaller order α and shorter T the sensitivity in the sense of S_2 is smallest in situation, when both parameters simultaneously increase. In the sense of the function S_∞ the lowest sensitivity is observed for stronger disturbed order α and slightly disturbed parameter T.

The behaviour of the function S_2 for order α close to 0.5 (Figure 4) is similar: the lowest sensitivity is achieved for simultaneous increasing of both parameters. The function S_∞ increases a little bit more slowly for changing α than for disturbation of both parameters.

The analysis of the vector q_3, illustrated by the Figure 5 allows to conclude that the lowest sensitivity in the sense of both functions is achieved for simultaneous change of both parameters.

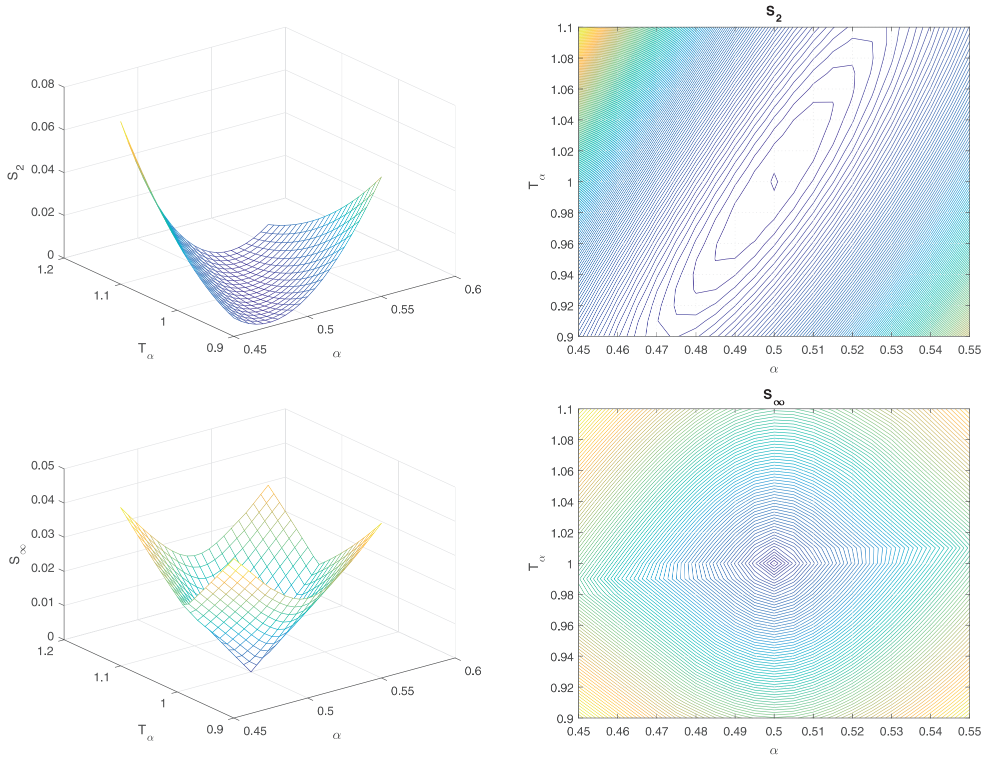

Fig. 4. 3D and contour plots of S_2 and S_∞ functions for vector q_2
Rys. 4. Wykresy trójwymiarowe i poziomicowe funkcji S_2 i S_∞ dla wektora q_2

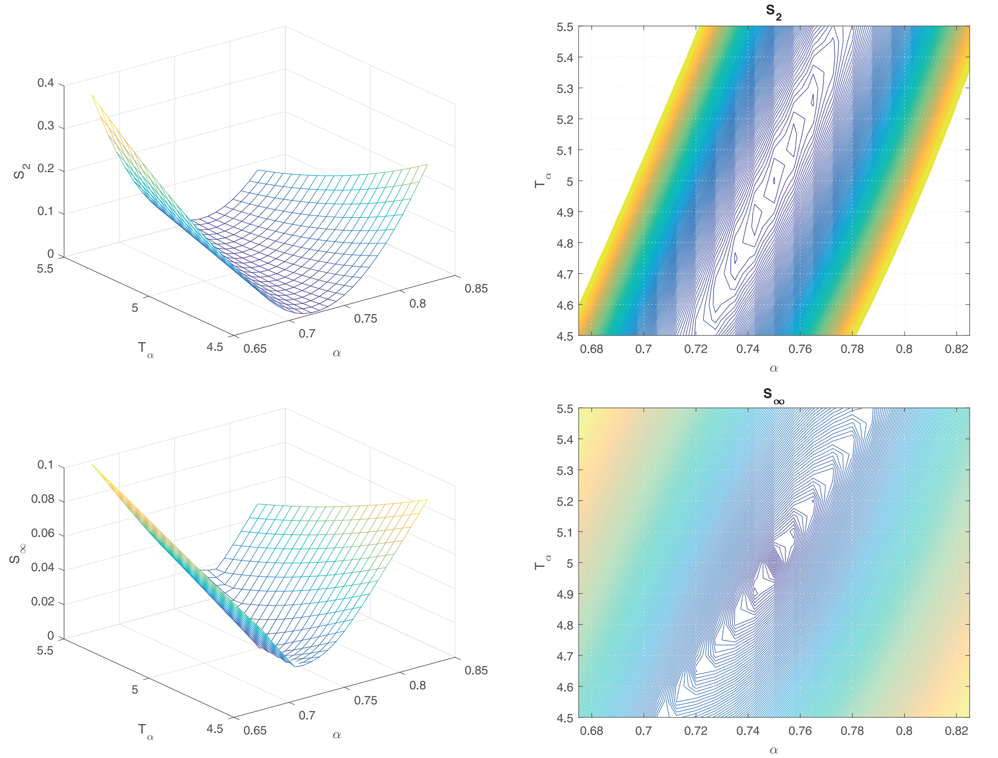

Fig. 5. 3D and contour plots of S_2 and S_∞ functions for vector q_3
Rys. 5. Wykresy trójwymiarowe i poziomicowe funkcji S_2 i S_∞ dla wektora q_3

4.2. The analysis of the time t_{90}

Next the time t_{90} was investigated. Firstly it was computed for nominal values of vectors $q_{1,2,3}$ from the Table 3 with the use of (15). Results are collected in the Table 4.

The Table 4 shows that for constant α the time t_{90} strongly increases with increasing of the quasi time constant T. This dependence is weak for $T = 0.1$ and varying α, however for longer T it can be observed decreasing of the time t_{90} for increasing order α.

Next the sensitivity function (26) was examined. Its values for vectors $q_{1,2,3}$ are presented in the Tables 5, 6 and 7. The value 0 in each table denotes the nominal parameters of tested interval. The Tables 5–7 show that the time t_{90} is most robust to disturbation of model parameters for vector q_1, describing relatively small values of α and T. For vector $q_{2,3}$ the time t_{90} is more sensitive to uncertainty of the parameters.

5. Final conclusions

The main final conclusion from the presented numerical results is that the considered interval transfer function is sensitive to uncertainty of its parameters and this sensitivity increases for order α going to 1.0 and longer quasi time constants T.

Next, the presented numerical results show that each use of the proposed transfer function should be preceded by its numerical analysis. The methodology of such an analysis was proposed in this paper.

The further investigation of the considered issue covers first of all its theoretical analysis. Here the main difficulty is caused by the fact that the analytical forms of the inverse Mittag-Leffler function as well as its derivatives along parameters α and T are not known. Here helpful can be the use of approximations: Oustaloup Recursive Approximation (ORA) or Power Series Expansion (PSE) instead of the analyzing of the solution (11). This idea is recently under consideration.

Tab. 4. The time t_{90} [s]
Tab. 4. Czas t_{90} [s]

$\alpha \mid T$	0.10	1.00	5.00	10.00	50.00
0.25	0.31	3084.80	> 10e+06	> 10e+06	> 10e+06
0.50	0.31	30.85	771.34	3085.30	7734.00
0.75	0.26	5.57	47.65	120.07	1026.70
0.95	0.23	2.61	14.19	29.44	160.21

Tab. 5. The function St in % for vector q_1
Tab. 5. Funkcja St w % dla wektora q_1

$\alpha \mid T$	0.09	0.10	0.11
0.225	39.12	2.76	48.56
0.250	34.39	0.00	46.39
0.275	30.53	1.91	44.12

Tab. 6. The function St in % for vector q_2
Tab. 6. Funkcja St w % dla wektora q_2

$\alpha \mid T$	0.90	1.00	1.10
0.45	34.91	70.66	110.72
0.50	19.00	0.00	21.00
0.55	47.07	36.01	23.91

Tab. 7. The function St in % for vector q_3
Tab. 7. Funkcja St w % dla wektora q_3

$\alpha \mid T$	0.90	1.00	1.10
0.675	62.14	89.54	118.28
0.750	13.10	0.00	13.55
0.825	48.22	41.17	33.97

An another issue is an investigation of the fractional order $\alpha > 1.0$ as well as the more complex form of the FO transfer function. Of course, is such a situation the only option is the use of an approximation.

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Analiza numeryczna elementarnej, przedziałowej transmitancji ułamkowego rzędu

Streszczenie: W pracy zaprezentowano analizę wpływu przedziałowej niepewności parametrów na zachowanie się elementarnej transmitancji niecałkowitego rzędu. Parametry modelu: rząd ułamkowy i pseudo-stała czasowa są zdefiniowane jako przedziały opisujące odchyłki od wartości nominalnych. Tego typu analiza nie była do tej pory rozważana. Proponowany elementarny model przedziałowy może znaleźć zastosowanie do opisu różnych elementów i zjawisk fizycznych, dla których wartości parametrów są opisane jedynie w sposób przybliżony. Dla rozważanej transmitancji zaproponowano metodologię jego analizy numerycznej i zilustrowano ją symulacjami. Wyniki testów numerycznych wskazują, że model jest najbardziej odporny na niepewność parametrów dla ich relatywnie niskich wartości.

Słowa kluczowe: transmitancja niecałkowitego rzędu, definicja Caputo, parametry przedziałowe, wrażliwość, czas t_{90}

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