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Some boundary value problems for a micropolar porous elastic body

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The paper reviews the static equilibrium of a micropolar porous elastic material. We assume that the body under consideration is an elastic Cosserat media with voids, however, it can also be considered as an elastic microstretch solid, since the basic differential equations and mathematical formulations of boundary value problems in these two cases are actually identical. As regards the three-dimensional case, the existence and uniqueness of a weak solution of some boundary value problems are proved. The two-dimensional system of equations corresponding to a plane deformation case is written in a complex form and its general solution is presented with the use of two analytic functions of a complex variable and two solutions of the Helmholtz equations. On the basis of the constructed general representation, specific boundary value problems are solved for a circle and an infinite plane with a circular hole.

Key words: materials with voids, Cosserat elastic media, theory of microstretch, boundary value problems, circle and infinite plane with a circular hole.

1. Introduction

Elastic materials with voids are very common in practice. These include hard tissues of animals and the humans; rocks, which are used as building materials; granular and some manufactured porous substances. In such a way, the determination of the elastic equilibrium of solids with voids under various boundary conditions facilitates practical applications. In many cases, it is also important to calculate porous materials based on the theory of micropolar elasticity [1–6]. This work makes it possible to apply effective methods of complex analysis to construct analytical solutions to the corresponding plane boundary value problems.

A linear model of elastic equilibrium of porous media with voids was originally constructed in [7]. In this theory, the volume fraction of voids at each point of an elastic body is considered as an independent function. Various questions of elastic equilibrium of porous bodies with empty pores in the case of a classical elastic medium are further discussed in [8–15]. The problems of porous elasticity for micropolar media with voids were considered in [16–19].

Noteworthy that the basic equations and mathematical formulations of boundary value problems in the cases of a micropolar media with voids and microstretch elastic solids coincide. The theory of microstretch has been introduced by Eringen in [20, 21]. The material points of microstretch solids can stretch and contract independently of their displacements and rotations and instead of the volume fraction function the microstretch function is introduced here. To the solution of various problems within the framework of theory of microstretch elastic solids was dedicated to the multitude of scientific publications from which we note the work [22–25].

The current work is dedicated to the study of a mixed three-dimensional boundary value problem and the analytical solution of some plane boundary value problems for a micropolar porous elastic body. In our opinion, such problems are along with theoretical and practical interest.

In Section 2, we prove the existence and uniqueness of a weak solution to the three-dimensional mixed boundary value problem. From this proof the existence and uniqueness of a weak solution of the Dirichlet-type problem follow. As for the Neumann-type problem, we have indicated the quotient space, where the only solution exists. In Section 3, we discuss the case of a plane deformed state and write the corresponding two-dimensional system of equilibrium equations in a complex form. In Section 4, we construct a general solution of the abovementioned system of equations using analytic functions of a complex variable and solutions of the Helmholtz equations. The resulting analogs of the Kolosov–Muskhelishvili formulas [26] make it possible to analytically solve plane boundary value problems of poroelastic equilibrium of Cosserat media with voids. In Section 5, the specific boundary value problem is solved analytically for a circle. In Section 6, the boundary value problem is solved analytically for an infinite domain with a circular hole.

2. Basic three-dimensional relations

Assume an elastic body with voids occupies the domain $\bar{\Omega} \in \mathbb{R}^3$. Denote by $\mathbf{x} = (x_1, x_2, x_3)$ a point of the domain $\bar{\Omega}$ in the Cartesian coordinate system. Assume the domain $\bar{\Omega}$ is filled with an elastic Cosserat media having voids. Denote the volume of the macro point of $\mathbf{x}$ by $V(\mathbf{x})$, and the volume of pores at this point by $V_p(\mathbf{x})$. The value $v(\mathbf{x})$, which is defined by the equality $v(\mathbf{x}) = V_p(\mathbf{x})/V(\mathbf{x})$, is called the relative volume of pores. In general, as a result of deformation of the body the relative volume of pores changes, too. The solid body is characterized by the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$, the rotation vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and the change in volume fraction from the reference volume fraction [7]

$$\phi(\mathbf{x}) = v(\mathbf{x}) - v_R(\mathbf{x})\,.$$

In this case, a system of static equilibrium equations is expressed as [5, 7, 21, 23]

$$\begin{aligned} &-\partial_i\sigma_{ij}(\mathbf{x}) = \rho F_j(\mathbf{x}), \\ &-\partial_i\mu_{ij}(\mathbf{x}) - \in_{jik}\sigma_{ik}(\mathbf{x}) = \rho G_j(\mathbf{x}), \qquad \text{in } \Omega, \\ &-\partial_i h_i(\mathbf{x}) - g(\mathbf{x}) = \rho l(\mathbf{x}), \end{aligned} \tag{2.1}$$

where σ_{ij} are stress tensor components; ρ is material density; F_j are the components of the mass force vectors; μ_{ij} are moment stress tensor components; $\in_{ijk}$ is the Levi–Civita symbol; G_j are the components of the mass moment vectors; h_i is the equilibrated stress vector; g is the intrinsic equilibrated body force; l is the extrinsic equilibrated body force; $\partial_i \equiv \partial/\partial x_i$.

In the above formulas, the Latin indices take the values 1, 2, 3 and it is assumed that summation is carried out over the repeated indices. The same is also assumed below.

Formulas that interrelate functions $\sigma_{ij}, \mu_{ij}, h_i, g$ to the functions u_j, ω_j and ϕ have the form [5, 7, 21, 23]

$$\begin{aligned} \sigma_{ij} &= (\lambda\,\mathrm{div}\,\mathbf{u} + \gamma\varphi)\delta_{ij} + (\mu+\alpha)\partial_i u_j + (\mu-\alpha)\partial_j u_i - 2\alpha \in_{ijk}\omega_k, \\ \mu_{ij} &= \sigma\,\mathrm{div}\,\boldsymbol{\omega}\delta_{ij} + (\nu+\beta)\partial_i\omega_j + (\nu-\beta)\partial_j\omega_i, \\ h_i &= \delta\partial_i\phi, \\ g &= -\xi\phi - \gamma\,\mathrm{div}\,\mathbf{u}, \end{aligned} \tag{2.2}$$

where λ and μ are the Lamé parameters; $\alpha, \beta, \nu, \sigma$ are the constants characterizing the microstructure of the discussed elastic media; δ, ξ, γ are the constants characterizing the body porosity; δ_{ij} is the Kronecker delta.

The three-dimensional system of Eqs. (2.1) and (2.2) describe the static equilibrium of a porous elastic Cosserat media with voids. Substituting (2.2) into (2.1), we obtain equilibrium equations with respect to the components of the displacement and rotation vectors and function ϕ

$$\begin{aligned} &(\mu+\alpha)\Delta u_j + (\lambda+\mu-\alpha)\partial_j(\partial_k u_k) - 2\alpha \in_{ijk}\partial_i\omega_k + \gamma\partial_j\phi = -\rho F_j, \\ &(\nu+\beta)\Delta\omega_j + (\sigma+\nu-\beta)\partial_j(\partial_k\omega_k) + 2\alpha \in_{jik}\partial_i u_k - 4\alpha\omega_j = -\rho G_j, \quad \text{in } \Omega, \\ &(\delta\Delta-\xi)\phi - \gamma\partial_k u_k = -\rho l, \end{aligned} \tag{2.3}$$

where $\Delta \equiv \partial_{11} + \partial_{22} + \partial_{33}$ is the three-dimensional Laplace operator.

Let Ω be a bounded open connected subset of $\mathbb{R}^3$ with a Lipshitz-continuous boundary; Γ_0 be a da-measurable subset of $\Gamma = \partial\Omega$ and area $\Gamma_0 > 0$; $\Gamma_1 = \Gamma - \Gamma_0$; $\mathbf{l}(l_1, l_2, l_3)$ is a unit outer normal along Γ_1.

We formulate a boundary value problem:

Let the functions $(F_1, F_2, F_3, G_1, G_2, G_3, l) \in (L^{6/5}(\Omega))^7$ and $(f_1, f_2, \ldots, f_7) \in (L^{4/3}(\Gamma_1))^7$ be given; Find a solution $\mathbf{w} = (u_1, u_2, u_3, \omega_1, \omega_2, \omega_3, \phi) \in \mathbf{V}(\Omega)$ of a system of Eqs. (2.3) which in the sense of a trace satisfies the following boundary condition

$$\mathbf{u} = \mathbf{0}, \ \boldsymbol{\omega} = \mathbf{0}, \ \phi = 0 \quad \text{on } \Gamma_0; \tag{2.4}$$

$$\left.\begin{aligned} \sigma_{ij}(\mathbf{w}) l_i &= f_j \\ \mu_{ij}(\mathbf{w}) l_i &= f_{j+3} \\ \delta h_i(\mathbf{w}) l_i &= f_7 \end{aligned}\right\} \quad \text{on } \Gamma_1, \tag{2.5}$$

where

$$\mathbf{V}(\Omega) = \{\mathbf{v} = (v_1, v_2, \ldots, v_7) \in (H^1(\Omega))^7, \ \mathbf{v} = \mathbf{0} \ da\text{-a.e. on } \Gamma_0\}.$$

Below, we investigate the question of the existence and uniqueness of the solution to the boundary value problem (2.3)–(2.5).

Using the Green formula and the Sobolev imbedding theorem and the continuity of the trace operator [27] we prove the following lemma.

Lemma. *Finding a solution* $\mathbf{w}$ *of the boundary value problem* (2.3)–(2.5) *is equivalent to finding a solution* $\mathbf{w} \in \mathbf{V}(\Omega)$ *of the equations*

$$B(\mathbf{w}, \mathbf{v}) = L(\mathbf{v}) \quad \textit{for all } \mathbf{v} \in \mathbf{V}(\Omega), \tag{2.6}$$

where

$$\begin{aligned} B(\mathbf{w}, \mathbf{v}) := \int_\Omega \{ & (\lambda \partial_i u_i + \gamma \phi) \partial_j v_j + 0.5 \mu (\partial_i u_j + \partial_j u_i)(\partial_i v_j + \partial_j v_i) \\ & + 0.5 \alpha (\partial_i u_j - \partial_j u_i + 2 \in_{kji} \omega_k)(\partial_i v_j - \partial_j v_i + 2 \in_{kji} v_{k+3}) + \sigma \partial_i \omega_i \partial_i v_{i+3} \\ & + 0.5 \nu (\partial_i \omega_j + \partial_j \omega_i)(\partial_i v_{j+3} + \partial_j v_{i+3}) + 0.5 \beta (\partial_i \omega_j - \partial_j \omega_i)(\partial_i v_{j+3} - \partial_j v_{i+3}) \\ & + \delta \partial_i \phi \partial_i v_7 + \xi \phi v_7 + \gamma \partial_i u_i v_7 \} \, dx, \end{aligned} \tag{2.7}$$

$$L(\mathbf{v}) := \rho \int_\Omega (F_j v_j + G_j v_{j+3} + l v_7) \, dx - \int_{\Gamma_1} \sum_{k=1}^{7} f_k v_k \, da. \tag{2.8}$$

Theorem. *Let the constants*

$$\begin{aligned} &\xi > 0, \quad (3\lambda + 2\mu)\xi > 3\gamma^2, \quad \mu > 0, \quad \alpha > 0, \\ &\delta > 0, \quad 3\sigma + 2\nu > 0, \quad \nu > 0, \quad \beta > 0 \end{aligned} \tag{2.8}$$

be given. Then there is one and only one solution $\mathbf{w} \in \mathbf{V}(\Omega)$ *of Eqs.* (2.6).

In addition

$$J(\mathbf{w}) = \inf_{\mathbf{v}\in\mathbf{V}} J(\mathbf{v}), \quad \textit{where} \quad J(\mathbf{v}) = \frac{1}{2}B(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}).$$

Proof. It follows from formula (2.7) that the bilinear form $B: \mathbf{V}\times\mathbf{V}\to\mathbb{R}$ is symmetric, i. e.

$$B(\mathbf{w},\mathbf{v}) = B(\mathbf{v},\mathbf{w}).$$

Continuity of the bilinear form $B(\mathbf{w},\mathbf{v})$ with respect to the norm

$$\|\mathbf{v}\|_{1,\Omega} = \left\{\int_\Omega \sum_{k=1}^{7}(v_k v_k + \partial_i v_k \partial_i v_k)\, dx\right\}^{1/2}$$

follows from the Cauchy–Schwarz inequality. Thus, there is a constant $C_1 > 0$ such that for all $\mathbf{w},\mathbf{v}\in\mathbf{V}$,

$$B(\mathbf{w},\, \mathbf{v}) \le C_1\|\mathbf{w}\|_{1,\Omega}\|\mathbf{v}\|_{1,\Omega}.$$

Taking into account that for all real number $k\neq 0$ the following inequality holds

$$2\gamma\partial_i v_i v_7 \ge -k^2(\partial_i v_i)^2 - \frac{\gamma^2}{k^2}v_7^2,$$

from formula (2.7), after obvious transformations we obtain

$$\begin{aligned}
(2.10)\quad B(\mathbf{v},\,\mathbf{v}) \ge \int_\Omega\Bigg\{&\left(\lambda+\frac{2}{3}\mu-k^2\right)\hat{e}_{ii}^2(\mathbf{v}) + \frac{4}{3}\mu\,[\hat{e}_{11}(\mathbf{v}) - 0.5(\hat{e}_{22}(\mathbf{v}) + \hat{e}_{33}(\mathbf{v}))]^2\\
&+ \mu(\hat{e}_{22}(\mathbf{v}) - \hat{e}_{33}(\mathbf{v}))^2 + 4\mu(\hat{e}_{12}^2(\mathbf{v}) + \hat{e}_{13}^2(\mathbf{v}) + \hat{e}_{23}^2(\mathbf{v}))\\
&+ \alpha[(\partial_1 v_2 - \partial_2 v_1 - 2v_6)^2 + (\partial_1 v_3 - \partial_3 v_1 + 2v_5)^2 + (\partial_2 v_3 - \partial_3 v_2 - 2v_4)]\\
&+ \left(\sigma + \frac{2}{3}\nu\right)\tilde{e}_{ii}^2(\mathbf{v}) + \frac{4}{3}\nu[\tilde{e}_{11}(\mathbf{v}) - 0.5(\tilde{e}_{22}(\mathbf{v}) + \tilde{e}_{33}(\mathbf{v}))]^2\\
&+ \nu(\tilde{e}_{22}(\mathbf{v}) - \tilde{e}_{33}(\mathbf{v}))^2 + 4\nu(\tilde{e}_{12}^2(\mathbf{v}) + \tilde{e}_{13}^2(\mathbf{v}) + \tilde{e}_{23}^2(\mathbf{v}))\\
&+ \beta[(\partial_1 v_5 - \partial_2 v_4)^2 + (\partial_1 v_6 - \partial_3 v_4)^2 + (\partial_2 v_6 - \partial_3 v_5)^2]\\
&+ \delta\partial_i v_7\partial_i v_7 + \left(\xi - \frac{\gamma^2}{k^2}\right)v_7^2\Bigg\}\,dx \quad \text{for all } \mathbf{v}\in\mathbf{V}(\Omega),
\end{aligned}$$

where

$$\hat{e}_{ij}(\mathbf{v}) = 0.5(\partial_i v_j + \partial_j v_i), \quad \tilde{e}_{ij}(\mathbf{v}) = 0.5(\partial_i v_{j+3} + \partial_j v_{i+3}).$$

Let $\xi > 0$ and $k^2 = \gamma^2/\xi$, then $\xi - \gamma^2/k^2 = 0$. If in addition $\alpha > 0$ and $\beta > 0$, then from (2.10) it follows

$$B(\mathbf{v},\mathbf{v}) \geq \int\limits_{\Omega}\bigg\{\bigg(\lambda+\frac{2}{3}\mu-\frac{\gamma^2}{\xi}\bigg)\hat{e}_{ii}^2(\mathbf{v})+\frac{4}{3}\mu\,[\hat{e}_{11}(\mathbf{v})-0.5(\hat{e}_{22}(\mathbf{v})+\hat{e}_{33}(\mathbf{v}))]^2$$
$$+\mu(\hat{e}_{22}(\mathbf{v})-\hat{e}_{33}(\mathbf{v}))^2+4\mu(\hat{e}_{12}^2(\mathbf{v})+\hat{e}_{13}^2(\mathbf{v})+\hat{e}_{23}^2(\mathbf{v}))$$
$$+\bigg(\sigma+\frac{2}{3}\nu\bigg)\tilde{e}_{ii}^2(\mathbf{v})+\frac{4}{3}\nu[\tilde{e}_{11}(\mathbf{v})-0.5(\tilde{e}_{22}(\mathbf{v})+\tilde{e}_{33}(\mathbf{v}))]^2$$
$$+\nu(\tilde{e}_{22}(\mathbf{v})-\tilde{e}_{33}(\mathbf{v}))^2+4\nu(\tilde{e}_{12}^2(\mathbf{v})+\tilde{e}_{13}^2(\mathbf{v})+\tilde{e}_{23}^2(\mathbf{v}))$$
$$+\delta\partial_i v_7\partial_i v_7\bigg\}\,dx \quad \text{for all } \mathbf{v}\in\mathbf{V}(\Omega). \tag{2.11}$$

Further, let

$$\lambda+\frac{2}{3}\mu-\frac{\gamma^2}{\xi}>0,\quad \mu>0,\quad \sigma+\frac{2}{3}\nu>0,\quad \nu>0,\quad \delta>0$$

and we note that

$$\begin{pmatrix} e_{11}+e_{22}+e_{33} \\ e_{11}-0.5(e_{22}+e_{33}) \\ e_{22}-e_{33}\end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -0.5 & -0.5 \\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix} e_{11} \\ e_{22} \\ e_{33}\end{pmatrix},$$

where transformation determinant is

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & -0.5 & -0.5 \\ 0 & 1 & -1\end{pmatrix}=3.$$

Then the above inequality (2.11) implies that there exists a constant $C_2>0$ such that

$$B(\mathbf{v},\mathbf{v})\geq C_2\bigg\{\int\limits_{\Omega}[\hat{e}_{ij}(\mathbf{v})\hat{e}_{ij}(\mathbf{v})+\tilde{e}_{ij}(\mathbf{v})\tilde{e}_{ij}(\mathbf{v})+\partial_i v_7\partial_i v_7]\,dx\bigg\} \quad \text{for all } \mathbf{v}\in\mathbf{V}(\Omega). \tag{2.12}$$

In formula (2.12) we use the Korn inequality [27]

$$\|\mathbf{v}\|_{1,\Omega}\leq c_0\bigg\{\int\limits_{\Omega}[\hat{e}_{ij}(\mathbf{v})\hat{e}_{ij}(\mathbf{v})+\tilde{e}_{ij}(\mathbf{v})\tilde{e}_{ij}(\mathbf{v})]\,dx\bigg\}^{1/2} \quad \text{for all } \mathbf{v}\in\mathbf{V}(\Omega)$$

and the Friedrichs inequality

$$\|v_7\|_{1,\Omega}\leq c_1\bigg\{\int\limits_{\Omega}\partial_i v_7\partial_i v_7\,dx\bigg\}^{1/2} \quad \text{for all } v_7\in H^1(\Omega),\ v_7=0\ da\text{-a.e. on } \Gamma_0.$$

Then there exists a constant $c > 0$ such that

$$B(\mathbf{v},\,\mathbf{v}) \geq c\|\mathbf{v}\|^2_{1,\Omega}.$$

Thus, following the conditions (2.9) satisfied, then the symmetric and continuous bilinear form $B : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ is $\mathbf{V}$-elliptic.

In addition, the Sobolev imbedding theorem and the continuity of the trace operator imply that the linear form L is continuous on the space $(H^1(\Omega))^7$.

Thus the theorem is proved (please refer to Theorem 6.3-2 in [27] and Lax–Milgram Lemma).

If $\Gamma = \Gamma_0$, the weak solution found in Theorem possesses additional regularity if the data also possess regularity (Theorem 6.3-6) in [27]).

Now we find all real solutions of the equation $B(\mathbf{v},\,\mathbf{v}) = 0$. By formula (2.7), this equation is equivalent to the following system of equations

$$\begin{gathered}
\partial_i v_j + \partial_j v_i = 0, \quad \partial_i v_{j+3} + \partial_j v_{i+3} = 0,\\
\partial_1 v_2 - \partial_2 v_1 - 2v_6 = 0, \quad \partial_1 v_3 - \partial_3 v_1 + 2v_5 = 0, \quad \partial_2 v_3 - \partial_3 v_2 - 2v_4 = 0,\\
\partial_1 v_5 - \partial_2 v_4 = 0, \quad \partial_1 v_6 - \partial_3 v_4 + 2v_5, \quad \partial_2 v_6 - \partial_3 v_5 = 0, \quad \partial_i v_7 = 0, \quad v_7 = 0.
\end{gathered}$$

The general solution of this system of equations has the following form

$$\begin{aligned}
v_1 &= -a_3 x_2 + a_2 x_3 + b_1,\\
v_2 &= a_3 x_1 - a_1 x_3 + b_2,\\
v_3 &= -a_2 x_1 + a_1 x_2 + b_3,\\
v_4 &= b_1, \quad v_5 = b_2, \quad v_6 = b_3, \quad v_7 = 0,
\end{aligned} \tag{2.13}$$

where a_j, b_j are arbitrary real numbers.

If the space of all vectors $(v_1, v_2, \ldots, v_7)$, whose components are expressed using formulas (2.13) is denoted by $\mathbf{W}$ and instead of $\mathbf{V}$ we take the quotient space $(H^1(\Omega))^7 \setminus \mathbf{W}$, then the above proved theorem is applicable to the case $\Gamma_1 = \Gamma$ [28].

3. The plane deformation case

From the basic three-dimensional equations, we obtain the basic equations for the case of plane deformation. Let Ω be a sufficiently long cylindrical body with generatrix parallel to the Ox_3-axis. Denote by D the cross-section of this cylindrical body, thus $D \in \mathbb{R}^2$. In the case of plane deformation $u_3 = 0$, $\omega_1 = 0$, $\omega_2 = 0$, while the functions u_1, u_2, ω_3 and ϕ do not depend on the coordinate x_3 [26]. We also assume

$$u_1, u_2, \omega_3, \phi \in C^2(D) \cap C^1(\bar{D}).$$

As follows from formula (2.2), in the case of plane deformation

$$\sigma_{\alpha 3}=0,\ \sigma_{3\alpha}=0,\ \mu_{\alpha\beta}=0,\ \mu_{33}=0,\ h_3=0;\ \alpha=1,2,\ \beta=1,2.$$

Therefore the homogeneous system ($F_\alpha=0$, $G_\alpha=0$, $l=0$) of equilibrium Eqs. (2.1) takes the form

$$\begin{aligned}&\partial_1\sigma_{11}+\partial_2\sigma_{21}=0,\\ &\partial_1\sigma_{12}+\partial_2\sigma_{22}=0,\\ &\partial_1\mu_{13}+\partial_2\mu_{23}+(\sigma_{12}-\sigma_{21})=0,\\ &\partial_1 h_1+\partial_2 h_2+g=0,\end{aligned}\quad \text{in } D, \tag{3.1}$$

where $\Delta_2\equiv\partial_{11}+\partial_{22}$ is the two-dimensional Laplace operator.

Relations (2.2) are rewritten as

$$\begin{aligned}&\sigma_{11}=\gamma\phi+\lambda\theta+2\mu\partial_1 u_1,\\ &\sigma_{22}=\gamma\phi+\lambda\theta+2\mu\partial_2 u_2,\\ &\sigma_{12}=(\mu+\alpha)\partial_1 u_2+(\mu-\alpha)\partial_2 u_1-2\alpha\omega_3,\\ &\sigma_{21}=(\mu+\alpha)\partial_2 u_1+(\mu-\alpha)\partial_1 u_2+2\alpha\omega_3,\\ &\sigma_{33}=\gamma\phi+\lambda\theta,\quad \mu_{13}=(\nu+\beta)\partial_1\omega_3,\quad \mu_{23}=(\nu+\beta)\partial_2\omega_3,\\ &\mu_{31}=(\nu-\beta)\partial_1\omega_3,\quad \mu_{32}=(\nu-\beta)\partial_2\omega_3,\\ &h_1=\delta\partial_1\phi,\quad h_2=\delta\partial_2\phi,\quad g=-\xi\phi-\gamma\theta,\end{aligned} \tag{3.2}$$

where $\theta:=\partial_1 u_1+\partial_2 u_2$.

If relations (3.2) are substituted into the system (3.1) then we obtain the following system of equilibrium equations with respect to the functions u_1,u_2,ω_3 and ϕ

$$\begin{aligned}&(\mu+\alpha)\Delta_2 u_1+(\lambda+\mu-\alpha)\partial_1\theta+2\alpha\partial_2\omega_3+\gamma\partial_1\phi=0,\\ &(\mu+\alpha)\Delta_2 u_2+(\lambda+\mu-\alpha)\partial_2\theta-2\alpha\partial_1\omega_3+\gamma\partial_2\phi=0,\\ &(\nu+\beta)\Delta_2\omega_3+2\alpha(\partial_1 u_2-\partial_2 u_1)-4\alpha\omega_3=0,\\ &(\delta\Delta_2-\xi)\phi-\gamma\theta=0,\end{aligned}\quad \text{in } D. \tag{3.3}$$

On the plane Ox_1x_2, we introduce the complex variable $z=x_1+ix_2=re^{i\alpha}$ ($i^2=-1$) and the operators $\partial_z=0.5(\partial_1-i\partial_2)$, $\partial_{\bar z}=0.5(\partial_1+i\partial_2)$, $\bar z=x_1-ix_2$; $\Delta_2=4\partial_z\partial_{\bar z}$.

To write system (3.1) in the complex form, the second equation of this system is multiplied by i and summed up with the first equation

$$\begin{aligned}&\partial_z(\sigma_{11}-\sigma_{22}+i(\sigma_{12}+\sigma_{21}))+\partial_{\bar z}(\sigma_{11}+\sigma_{22}+i(\sigma_{12}-\sigma_{21}))=0,\\ &\partial_z(\mu_{13}+i\mu_{23})+\partial_{\bar z}(\mu_{13}-i\mu_{23})+\sigma_{12}-\sigma_{21}=0,\\ &\partial_z(h_1+ih_2)+\partial_{\bar z}(h_1-ih_2)+g=0,\end{aligned}\quad \text{in } D, \tag{3.4}$$

where by formulas (3.2)

$$
\begin{aligned}
&\sigma_{11}-\sigma_{22}+i(\sigma_{12}+\sigma_{21})=4\mu\partial_{\bar z}u_+,\\
&\sigma_{11}+\sigma_{22}+i(\sigma_{12}-\sigma_{21})=2(\lambda+\mu-\alpha)\theta-4\alpha i\omega_3+2\gamma\phi+4\alpha\partial_z u_+,\\
&\mu_{13}+i\mu_{23}=2(\nu+\beta)\partial_{\bar z}\omega_3,\\
&\mu_{31}+i\mu_{32}=2(\nu-\beta)\partial_{\bar z}\omega_3,\\
&h_1+ih_2=2\delta\partial_{\bar z}\phi;\\
&u_+:=u_1+iu_2,\ \bar u_+:=u_1-iu_2,\ \theta=\partial_z u_+ +\partial_{\bar z}\bar u_+.
\end{aligned}
\tag{3.5}
$$

If relations (3.5) are substituted into system (3.4), then system (3.3) is written in the complex form

$$
\begin{aligned}
&2(\mu+\alpha)\partial_z\partial_{\bar z}u_+ +(\lambda+\mu-\alpha)\partial_{\bar z}\theta-2\alpha i\partial_{\bar z}\omega_3+\gamma\partial_{\bar z}\phi=0,\\
&2(\nu+\beta)\partial_z\partial_{\bar z}\omega_3+\alpha i(\theta-2\partial_z u_+)-2\alpha\omega_3=0, \qquad \text{in } D.\\
&(4\delta\partial_z\partial_{\bar z}-\xi)\phi-\gamma\theta=0,
\end{aligned}
\tag{3.6}
$$

If we take $\gamma=0$ in the system of Eqs. (3.6), then it splits into the system of equations of the plane moment theory of elasticity and the Helmholtz equation with respect to the function ϕ.

4. The general solution of system of equations (3.6)

In this section, we construct the analogues of the Kolosov–Muskhelishvili formulas [26] (see also [29–31]) for the system (3.6).

We take the operator $\partial_{\bar z}$ out of the brackets in the left-hand part of the first equation of the system (3.6)

$$
\partial_{\bar z}\left[2(\mu+\alpha)\partial_z u_+ +(\lambda+\mu-\alpha)\theta-2\alpha i\omega_3+\gamma\phi\right]=0. \tag{4.1}
$$

Since (4.1) is a system of Cauchy-Riemann equations, we have

$$
2(\mu+\alpha)\partial_z u_+ +(\lambda+\mu-\alpha)\theta-2\alpha i\omega_3+\gamma\phi=(\kappa+1)\varphi'(z), \tag{4.2}
$$

where $\varphi(z)$ is an arbitrary analytic function of z;

$$
\kappa=\frac{\lambda+3\mu}{\lambda+\mu}.
$$

A conjugate equation to (4.2) has the form

$$
2(\mu+\alpha)\partial_{\bar z}\bar u_+ +(\lambda+\mu-\alpha)\theta+2\alpha i\omega_3+\gamma\phi=(\kappa+1)\overline{\varphi'(z)}. \tag{4.3}
$$

Summing up Eqs. (4.2) and (4.3) and taking into account that

$$\theta = \partial_z u_+ + \partial_{\bar z} \bar u_+,$$

we obtain

$$\theta = -\frac{\gamma}{\lambda+2\mu}\phi + \frac{1}{\lambda+\mu}(\varphi'(z) + \overline{\varphi'(z)}). \tag{4.4}$$

If from Eq. (4.2) we subtract Eq. (4.3) and write the expression $i(\partial_z u_+ - \partial_{\bar z}\bar u_+)$ then we have

$$i(\partial_z u_+ - \partial_{\bar z}\bar u_+) = \frac{\kappa+1}{2(\mu+\alpha)} i(\varphi'(z) - \overline{\varphi'(z)}) - \frac{2\alpha}{\mu+\alpha}\omega_3. \tag{4.5}$$

The second equation of the system (3.6) is written as

$$4\partial_z\partial_{\bar z}\omega_3 - \frac{2\alpha}{\nu+\beta} i(\partial_z u_+ - \partial_{\bar z}\bar u_+) - \frac{4\alpha}{\nu+\beta}\omega_3 = 0. \tag{4.6}$$

Substituting formula (4.5) into formula (4.6) we obtain the equation

$$\Delta_2\omega_3 - \varsigma_1^2\omega_3 = \frac{\alpha(\kappa+1)}{(\nu+\beta)(\mu+\alpha)} i(\varphi'(z) - \overline{\varphi'(z)}), \tag{4.7}$$

where

$$\varsigma_1^2 = \frac{4\mu\alpha}{(\nu+\beta)(\mu+\alpha)} > 0.$$

The general solution of Eq. (4.7) is written in the form

$$2\mu\omega_3 = \frac{4\mu}{\nu+\beta}\chi(z,\bar z) - \frac{\kappa+1}{2} i(\varphi'(z) - \overline{\varphi'(z)}), \tag{4.8}$$

where $\chi(z,\bar z)$ is a general solution of the Helmholtz equation

$$\Delta_2\chi - \varsigma_1^2\chi = 0. \tag{4.9}$$

The multiplier $4\mu/\nu+\beta$ has been introduced for convenience in writing our subsequent formulas.

Substituting formulas (4.4) into the last equation of the system (3.6) we have

$$\Delta_2\phi - \varsigma_2^2\phi = \frac{\gamma}{(\lambda+\mu)\delta}(\varphi'(z) + \overline{\varphi'(z)}), \tag{4.10}$$

where

$$\varsigma_2^2 = \frac{(\lambda+2\mu)\xi - \gamma^2}{(\lambda+2\mu)\delta} > 0.$$

The general solution of Eq. (4.10) is written in the form

$$\phi = \frac{(\lambda+2\mu)\xi-\gamma^2}{\mu\delta}\eta(z,\bar{z}) - \frac{\gamma}{(\lambda+\mu)\delta\varsigma_2^2}(\varphi'(z)+\overline{\varphi'(z)}), \tag{4.11}$$

where $\eta(z,\bar{z})$ is a general solution of the Helmholtz equation

$$\Delta_2\eta - \varsigma_2^2\eta = 0. \tag{4.12}$$

Substituting formulas (4.8) and (4.11) into Eq. (4.2) we obtain

$$2\mu\partial_z u_+ = (\kappa+\kappa_0)\varphi'(z) - (1-\kappa_0)\overline{\varphi'(z)} + 4i\partial_z\partial_{\bar{z}}\chi(z,\bar{z}) - 4\gamma\partial_z\partial_{\bar{z}}\eta(z,\bar{z}), \tag{4.13}$$

where

$$\kappa_0 = \frac{\gamma^2\mu}{(\lambda+\mu)((\lambda+2\mu)\xi-\gamma^2)}.$$

As a result of integration (4.13), we obtain

$$2\mu u_+ = (\kappa+\kappa_0)\varphi(z) - (1-\kappa_0)z\overline{\varphi'(z)} - \overline{\psi(z)} + 4\partial_{\bar{z}}(i\chi(z,\bar{z}) - \gamma\eta(z,\bar{z})), \tag{4.14}$$

where $\psi(z)$ is an arbitrary analytic function of z.

Thus, the general solution of the system of Eqs. (3.6) is represented using formulas (4.8), (4.11), (4.14). These formulas include two arbitrary analytic functions of a complex variable $\varphi(z), \psi(z)$ and two arbitrary solutions of the Helmholtz Eqs. (4.9) and (4.12), respectively $\chi(z,\bar{z})$ and $\eta(z,\bar{z})$.

From (4.1) we have

$$2(\lambda+\mu-\alpha)\theta - 4\alpha i\omega_3 + 2\gamma\phi = 2(\kappa+1)\varphi'(z) - 4(\mu+\alpha)\partial_z u_+.$$

Substituting these formulas into the second equation of (3.5) we obtain

$$\sigma_{11}+\sigma_{22}+i(\sigma_{12}-\sigma_{21}) = 2(\kappa+1)\varphi'(z) - 4\mu\partial_z u_+.$$

From the last formula, taking into account (4.13), we deduce

$$\begin{aligned}\sigma_{11}+\sigma_{22}&+i(\sigma_{12}-\sigma_{21})\\ &= 2(1-\kappa_0)(\varphi'(z)+\overline{\varphi'(z)}) - 2\varsigma_1^2 i\chi(z,\bar{z}) + 2\gamma\varsigma_2^2\eta(z,\bar{z}).\end{aligned} \tag{4.15}$$

Substituting the formula (4.14) into the first equation of (3.5) we obtain

$$\begin{aligned}\sigma_{11}-\sigma_{22}&+i(\sigma_{12}+\sigma_{21})\\ &= -2(1-\kappa_0)z\overline{\varphi''(z)} - 2\overline{\psi'(z)} + 8\partial_{\bar{z}}\partial_{\bar{z}}(i\chi(z,\bar{z}) - \gamma\eta(z,\bar{z})).\end{aligned} \tag{4.16}$$

Substituting the formula (4.8) into the third and fourth equations of (3.5), we obtain

$$\mu_{13}+i\mu_{23}=4\partial_{\bar z}\chi(z,\ \bar z)+\frac{(\kappa+1)(\nu+\beta)}{2\mu}i\overline{\varphi''(z)}, \tag{4.17}$$

$$\mu_{31}+i\mu_{32}=\frac{4(\nu-\beta)}{\nu+\beta}\partial_{\bar z}\chi(z,\ \bar z)+\frac{(\kappa+1)(\nu-\beta)}{2\mu}i\overline{\varphi''(z)}. \tag{4.18}$$

Substituting the formulas (4.4) and (4.11) into the fifth equation of (3.2) we obtain the formula for σ_{33}

$$\sigma_{33}=\frac{\lambda(\lambda+2\mu)\delta\varsigma_2^2-2\mu\gamma^2}{(\lambda+\mu)(\lambda+2\mu)\delta\varsigma_2^2}(\varphi'(z)+\overline{\varphi'(z)})+\frac{2(\lambda+2\mu)\xi-2\gamma^2}{(\lambda+2\mu)\delta}\gamma\eta(z,\bar z). \tag{4.19}$$

From the last formula (3.5), taking into account (4.11), we have

$$h_1+ih_2=-\frac{2\gamma}{(\lambda+\mu)\varsigma_2^2}\overline{\varphi''(z)}+\frac{2(\lambda+2\mu)\xi-2\gamma^2}{\mu}\partial_{\bar z}\eta(z,\bar z). \tag{4.20}$$

Thus, the general solution of a two-dimensional system of differential equations that describes the static equilibrium of a Cosserat elastic media with voids is represented by means of two analytic functions of a complex variable and two solutions of the Helmholtz equation. By appropriate choice of these functions, we can satisfy four independent classical boundary conditions.

Assume that mutually perpendicular unit vectors $\mathbf{l}$ and $\mathbf{s}$ be such that

$$\mathbf{l}\times\mathbf{s}=\mathbf{e}_3,$$

where $\mathbf{e}_3$ is the unit vector directed along the x_3-axis. The vector $\mathbf{l}$ forms the angle α with the positive direction of the x_1-axis. Then the displacement components $u_l=\mathbf{u}\cdot\mathbf{l}$, $u_s=\mathbf{u}\cdot\mathbf{s}$, as well as the stress and moment stress components acting on an area of arbitrary orientation are expressed by the formulas

$$\begin{aligned}
&u_l+iu_s=e^{-i\alpha}u_+,\\
&\sigma_{ll}+\sigma_{ss}+i(\sigma_{ls}-\sigma_{sl})=\sigma_{11}+\sigma_{22}+i(\sigma_{12}-\sigma_{21}),\\
&\sigma_{ll}-\sigma_{ss}+i(\sigma_{ls}+\sigma_{sl})=[\sigma_{11}-\sigma_{22}+i(\sigma_{12}+\sigma_{21})]e^{-2i\alpha},\\
&\sigma_{ll}+i\sigma_{ls}=0.5[\sigma_{11}+\sigma_{22}+i(\sigma_{12}-\sigma_{21})+(\sigma_{11}-\sigma_{22}+i(\sigma_{12}+\sigma_{21}))e^{-2i\alpha}],\\
&\mu_{l3}=0.5[(\mu_{13}+i\mu_{23})e^{-i\alpha}+(\mu_{13}-i\mu_{23})e^{i\alpha}],\\
&h_l=0.5[(h_1+ih_2)e^{-i\alpha}+(h_1-ih_2)e^{i\alpha}].
\end{aligned} \tag{4.21}$$

Using formulas obtained in this section, we can analytically solve the class of plane boundary value problems for both, finite and infinite, domains.

5. Boundary value problem for a circle

Let us consider the elastic circle, consisting of Cosserat media with voids bounded by the circumference of radius R (Fig. 1). The origin of coordinates is at the center of the circle.

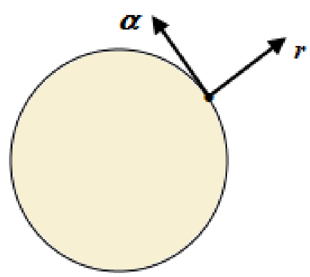

Fig. 1. The elastic circle.

On the circumference, we consider the following boundary value problem

$$\sigma_{rr} - i\sigma_{r\alpha} = N_0 - iT_0,\ \mu_{r3} = M_0,\ \phi = F_0 \quad \text{on } r = R, \tag{5.1}$$

where N_0, T_0, M_0 and F_0 are constants.

Substituting the formulas (4.15)–(4.17) into (4.21) we have

$$\begin{aligned}\sigma_{rr} - i\sigma_{r\alpha} &= (1-\kappa_0)(\varphi'(z) + \overline{\varphi'(z)}) + \varsigma_1^2 i\chi(z,\bar z) + \gamma\varsigma_2^2\eta(z,\bar z)\\ &\quad - e^{2i\alpha}\big[(1-\kappa_0)\bar z\varphi''(z) + \psi'(z) + 4\partial_z\partial_z(i\chi(z,\bar z) + \gamma\eta(z,\bar z))\big],\end{aligned} \tag{5.2}$$

$$\mu_{r3} = \mathrm{Re}\bigg(\frac{(\kappa+1)(\nu+\beta)}{2\mu} i\overline{\varphi''(z)}e^{-i\alpha} + 4\partial_{\bar z}\chi(z,\bar z)e^{-i\alpha}\bigg). \tag{5.3}$$

The analytic functions $\varphi'(z)$, $\psi'(z)$ and the metaharmonic functions $\chi(z,\bar z)$, $\eta(z,\bar z)$ are represented as the following series

$$\varphi'(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad \psi'(z) = \sum_{n=0}^{\infty} b_n z^n; \tag{5.4}$$

$$\chi(z,\bar z) = \sum_{-\infty}^{+\infty} \alpha_n I_n(\varsigma_1 r)e^{in\alpha}, \qquad \eta(z,\bar z) = \sum_{-\infty}^{+\infty} \beta_n I_n(\varsigma_2 r)e^{in\alpha}, \tag{5.5}$$

where $I_n(\varsigma_1 r)$ and $I_n(\varsigma_2 r)$ are the modified Bessel function of the first kind of n-th order.

Substituting (5.4), (5.5) in (5.2), (5.3), (4.11), taking into account the boundary conditions (5.1) and assuming that the series converge on the circumference $r = R$, one finds

$$(5.6)\qquad (1-\kappa_0)\sum_{n=0}^{+\infty} R^n((1-n)a_n e^{in\alpha} + \bar{a}_n e^{-in\alpha}) - \sum_{n=0}^{+\infty} R^n b_n e^{i(n+2)\alpha}$$

$$- \frac{2}{R}\sum_{-\infty}^{+\infty}(n-1)(\varsigma_1 I_{n-1}(\varsigma_1 R) i\alpha_n + \varsigma_2\gamma I_{n-1}(\varsigma_2 R)\beta_n)e^{in\alpha} = N_0 - iT_0,$$

$$(5.7)\qquad 2\varsigma_1\sum_{-\infty}^{+\infty} I_n'(\varsigma_1 R)\alpha_n e^{in\alpha}$$

$$+ \frac{(\kappa+1)(\nu+\beta)}{4\mu}\sum_{n=0}^{+\infty} R^{n-1} ni(\bar{a}_n e^{-in\alpha} - a_n e^{in\alpha}) = M_0,$$

$$(5.8)\qquad \frac{(\lambda+2\mu)\xi - \gamma^2}{\mu\delta}\sum_{-\infty}^{+\infty} I_n(\varsigma_2 R)\beta_n e^{in\alpha}$$

$$- \frac{\gamma}{(\lambda+\mu)\delta\varsigma_2^2}\sum_{n=0}^{+\infty} R^n(a_n e^{in\alpha} + \bar{a}_n e^{-in\alpha}) = F_0.$$

As a conclusion of the previous relations, we used the following well-known formulas

$$I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x), \quad I_{n-1}(x) + I_{n+1}(x) = 2I_n'(x).$$

Comparing in (5.6)–(5.8) the coefficients of $e^{0i\alpha}$ we have (it is also assumed that a_0 is a real value [26])

$$(5.9)\qquad 2(1-\kappa_0)a_0 + \frac{2\gamma}{R}\varsigma_2 I_1(\varsigma_2 R)\beta_0 = N_0,$$

$$(5.10)\qquad -\frac{2\gamma}{(\lambda+\mu)\delta\varsigma_2^2}a_0 + \frac{(\lambda+2\mu)\xi-\gamma^2}{\mu\delta} I_0(\varsigma_2 R)\beta_0 = F_0,$$

$$(5.11)\qquad \alpha_0 = -\frac{R}{2\varsigma_1 I_1(\varsigma_1 R)}T_0 = \frac{1}{2\varsigma_1 I_0'(\varsigma_1 R)}M_0.$$

In order for the problem to have a solution, the following condition must be met

$$M_0 = -\frac{R I_0'(\varsigma_1 R)}{I_1(\varsigma_1 R)}T_0.$$

From Eqs. (5.9), (5.10) we determine the coefficients a_0 and β_0

$$(5.12)\qquad a_0 = \frac{\Delta_1}{\Delta}, \quad \beta_0 = \frac{\Delta_2}{\Delta},$$

where

$$\Delta = \frac{2(\lambda+2\mu)((\lambda+\mu)\xi-\gamma^2)I_0(\varsigma_2 R)}{(\lambda+\mu)\mu\delta} + \frac{4\gamma^2 I_1(\varsigma_2 R)}{(\lambda+\mu)\delta\varsigma_2 R},$$
$$\Delta_1 = \frac{(\lambda+2\mu)\xi-\gamma^2}{\mu\delta} I_0(\varsigma_2 R) N_0 - \frac{2\gamma}{R}\varsigma_2 I_1(\varsigma_2 R) F_0,$$
$$\Delta_2 = \frac{2\gamma}{(\lambda+\mu)\delta\varsigma_2^2} N_0 + 2(1-\kappa_0)F_0.$$

All other coefficients in series (5.4), (5.5) are equal to zero. Thus

$$\varphi'(z) = a_0, \quad \psi'(z) = 0, \quad \chi(z,\bar z) = \alpha_0 I_0(\varsigma_1 r), \quad \eta(z,\bar z) = \beta_0 I_0(\varsigma_2 r), \tag{5.13}$$

where α_0, a_0, β_0 are determined using formulas (5.11), (5.12).

Substituting (5.13) into (4.15)–(4.18), (4.11) and using formulas (4.21), we find the polar components of the stresses and moment stresses, as well as the function ϕ

$$\begin{aligned}
\sigma_{rr} &= 2(1-\kappa_0)a_0 + \varsigma_2^2\gamma\beta_0[I_0(\varsigma_2 r) - I_2(\varsigma_2 r)],\\
\sigma_{\alpha\alpha} &= 2(1-\kappa_0)a_0 + \varsigma_2^2\gamma\beta_0[I_0(\varsigma_2 r) + I_2(\varsigma_2 r)],\\
\sigma_{r\alpha} &= \varsigma_1^2\alpha_0[I_2(\varsigma_1 r) - I_0(\varsigma_1 r)],\\
\sigma_{\alpha r} &= \varsigma_1^2\alpha_0[I_2(\varsigma_1 r) + I_0(\varsigma_1 r)],\\
\mu_{r3} &= 2\varsigma_1\alpha_0 I_1(\varsigma_1 r),\\
\mu_{3r} &= \frac{2(\nu-\beta)}{\nu+\beta}\varsigma_1\alpha_0 I_1(\varsigma_1 r),\\
\phi &= \frac{(\lambda+2\mu)\xi-\gamma^2}{\mu\delta}\beta_0 I_0(\varsigma_2 r) - \frac{2\gamma a_0}{(\lambda+\mu)\delta\varsigma_2^2}.
\end{aligned} \tag{5.14}$$

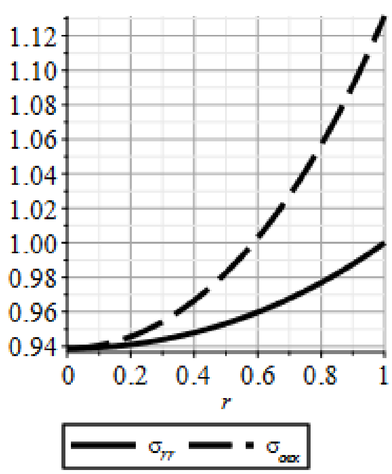

FIG. 2. P.1. Graphs of functions σ_{rr} and $\sigma_{\alpha\alpha}$.

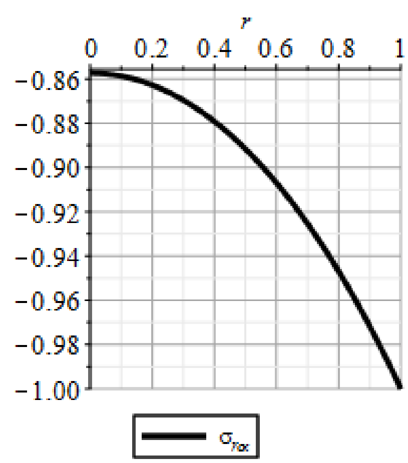

FIG. 3. P.1. Graph of function $\sigma_{r\alpha}$.

Since the problem under consideration is axisymmetric, the solution depends only on r. As it can be seen from the obtained formulas, the stresses $\sigma_{rr}, \sigma_{\alpha\alpha}$ and function ϕ depend on the elastic constants and constants characterizing the porosity of the body. Stresses $\sigma_{r\alpha}, \sigma_{\alpha r}$ and moment stresses μ_{r3}, μ_{3r} depend only on elastic constants. When $\gamma = 0$, then $\kappa_0 = 0$ and $\sigma_{rr} = \sigma_{\alpha\alpha} = N_0$.

Figures 2–6 show graphs of the mentioned functions σ_{rr}, $\sigma_{\alpha\alpha}$, $\sigma_{r\alpha}$, $\sigma_{\alpha r}$, μ_{r3}, μ_{3r} and ϕ, that are designed on the basis of the obtained solutions (5.14). The radius of the circle is taken equal to $R = 1$ unit, and the constants N_0, T_0, M_0

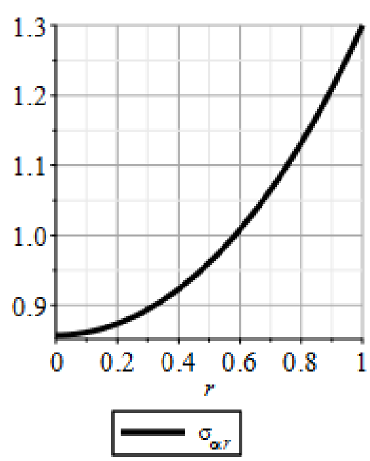

Fig. 4. P.1. Graph of function $\sigma_{\alpha r}$.

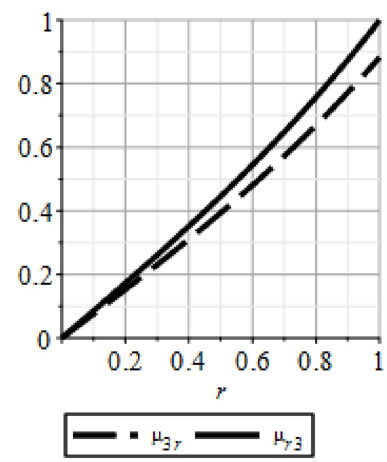

Fig. 5. P.1. Graphs of functions μ_{r3} and μ_{3r}.

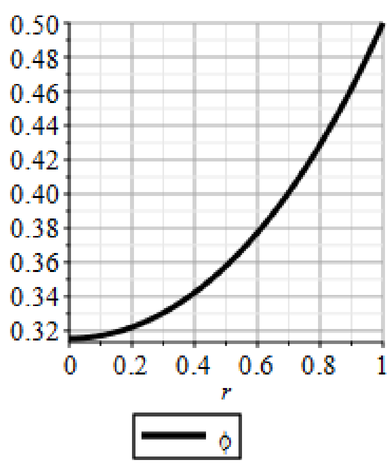

Fig. 6. P.1. Graph of function ϕ.

and F_0 in relative units are accepted as follows

$$N_0 = 1, \quad T_0 = -1, \quad M_0 = 1, \quad F_0 = 0.5.$$

Elastic constants and constants characterizing the porosity of the body were taken in such a way as to satisfy conditions (2.9).

As it can be seen from the graphs, the solution of the problem (5.14) satisfies the boundary conditions (5.1). Since the mentioned solution was obtained on the basis of the general solution that we constructed in Section 3, it also satisfies the equilibrium equations inside the domain under consideration.

The procedure of solving a boundary value problem remains the same when stresses, moment stresses, and change in volume fraction on the domain boundary are given arbitrarily, but the condition that the principal vector and the principal moment of external forces are equal to zero is fulfilled.

6. The problem for the infinite plane with a circular hole

Now let we have an infinite plane with a circular hole (Fig. 7). Assume that the origin of coordinates is at the center of the hole of radius R.

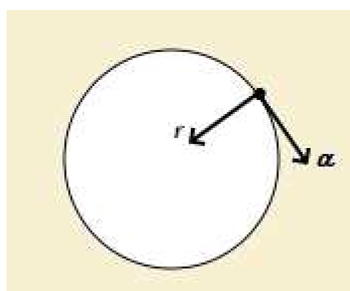

Fig. 7. The infinite plane with a circular hole.

On the circle we consider the following boundary value problem

$$\sigma_{rr} - i\sigma_{r\alpha} = N - iT, \quad \mu_{r3} = M, \quad \phi = F \quad \text{on } r = R, \tag{6.1}$$

where N, T, M and F are sufficiently smooth functions.

Conditions at infinity

$$\begin{aligned} &\sigma_{11}^{(\infty)} = S_1, \quad \sigma_{22}^{(\infty)} = S_2, \quad \sigma_{12}^{(\infty)} = \sigma_{21}^{(\infty)} = S_3, \\ &\mu_{13}^{(\infty)} = \mu_{23}^{(\infty)} = 0; \quad \phi = S_4, \end{aligned} \tag{6.2}$$

where S_1, S_2, S_3, S_4 are the constants.

In this case the analytic functions $\varphi'(z)$, $\psi'(z)$ and the metaharmonic functions $\chi(z,\bar{z})$, $\eta(z,\bar{z})$ are represented as a series [26, 30]

$$\varphi'(z)=\sum_{n=0}^{\infty}a_n z^{-n},\quad \psi'(z)=\sum_{n=0}^{\infty}b_n z^{-n};\tag{6.3}$$

$$\chi(z,\bar{z})=\sum_{-\infty}^{+\infty}\alpha_n K_n(\varsigma_1 r)e^{in\alpha},\quad \eta(z,\ \bar{z})=\sum_{-\infty}^{+\infty}\beta_n K_n(\varsigma_2 r)e^{in\alpha},\tag{6.4}$$

where $K_n(\varsigma_1 r)$ and $K_n(\varsigma_2 r)$ are the modified Bessel function of the second kind of n-th order.

Substituting (6.3), (6.4) in (5.2), (5.3), (4.11), taking into account the boundary conditions (6.1) and assuming that the series to converge on the circumference $r=R$, one finds

$$\begin{aligned}(1-\kappa_0)\sum_{n=0}^{+\infty}\frac{1}{R^n}((1+n)a_n e^{-in\alpha}+\bar{a}_n e^{in\alpha})-b_0e^{2i\alpha}-\frac{b_1}{R}e^{i\alpha}\\ -\sum_{n=0}^{+\infty}\frac{b_{n+2}}{R^{n+2}}e^{-in\alpha}+\frac{2}{R}\sum_{-\infty}^{+\infty}(n-1)(\varsigma_1K_{n-1}(\varsigma_1R)i\alpha_n+\varsigma_2\gamma K_{n-1}(\varsigma_2R)\beta_n)e^{in\alpha}\\ =N-iT,\end{aligned}\tag{6.5}$$

$$\begin{aligned}2\varsigma_1\sum_{-\infty}^{+\infty}K_n'(\varsigma_1R)\alpha_n e^{in\alpha}\\ +\frac{(\kappa+1)(\nu+\beta)}{4\mu}\sum_{n=1}^{+\infty}\frac{ni}{R^{n+1}}(-\bar{a}_n e^{in\alpha}+a_n e^{-in\alpha})=M,\end{aligned}\tag{6.6}$$

$$\begin{aligned}\frac{(\lambda+2\mu)\xi-\gamma^2}{\mu\delta}\sum_{-\infty}^{+\infty}K_n(\varsigma_2R)\beta_n e^{in\alpha}\\ -\frac{\gamma}{(\lambda+\mu)\delta\varsigma_2^2}\sum_{n=0}^{+\infty}\frac{1}{R^n}(a_n e^{-in\alpha}+\bar{a}_n e^{in\alpha})=F.\end{aligned}\tag{6.7}$$

As a conclusion of the previous relations, we used the following well-known formulas

$$K_{n-1}(x)-K_{n+1}(x)=-\frac{2n}{x}K_n(x),\quad K_{n-1}(x)+K_{n+1}(x)=-2K_n'(x).\tag{6.8}$$

Expand the function $N-iT$, M and F, given on $r=R$, in a complex Fourier series

$$N-iT=\sum_{-\infty}^{+\infty}A_n e^{in\alpha},\quad M=\sum_{-\infty}^{+\infty}B_n e^{in\alpha},\quad F=\sum_{-\infty}^{+\infty}C_n e^{in\alpha}.\tag{6.9}$$

Due to the fact that χ, η, M and F are real functions, we have

$$\alpha_n = \bar{\alpha}_{-n}, \quad \beta_n = \bar{\beta}_{-n}, \quad B_n = \bar{B}_{-n}, \quad C_n = \bar{C}_{-n}.$$

It is known that [29]

$$a_0 = \Gamma, \quad b_0 = \Gamma^{/}, \tag{6.10}$$

where Γ, $\Gamma^{/}$ are known quantities, specifying the stress distribution at infinity (It is also assumed that a_0 is a real value [26]). As follows from formulas (4.11), (4.15), (4.16) and conditions (6.2)

$$\operatorname{Re}\Gamma = \frac{S_1 + S_2}{4(1-\kappa_0)} = \frac{(\lambda+\mu)\delta\varsigma_2^2 S_4}{2}, \quad \operatorname{Re}\Gamma^{/} = \frac{S_2 - S_1}{2}, \quad \operatorname{Im}\Gamma^{/} = S_3.$$

We use the condition of single-valuedness of the displacements which in the present case is expressed as

$$\kappa a_1 + \bar{b}_1 = 0. \tag{6.11}$$

After introducing (6.8) into (6.5)–(6.7), and comparing the coefficients of $e^{in\alpha}$, we have

$$\begin{cases} 2(1-\kappa_0)a_0 - \dfrac{b_2}{R^2} - \dfrac{2i\varsigma_1}{R}K_1(\varsigma_1 R)\alpha_0 - \dfrac{2\gamma\varsigma_2}{R}K_1(\varsigma_2 R)\beta_0 = A_0, \\ 2\varsigma_1 K_0'(\varsigma_1 R)\alpha_0 = B_0; \end{cases} \tag{6.12}$$

$$\frac{1-\kappa_0}{R}\bar{a}_1 - \frac{1}{R}b_1 = A_1; \tag{6.13}$$

$$\frac{1-\kappa_0}{R^2}\bar{a}_2 - b_0 + \frac{2i\varsigma_1}{R}K_1(\varsigma_1 R)\alpha_2 + \frac{2\gamma\varsigma_2}{R}K_1(\varsigma_2 R)\beta_2 = A_2; \tag{6.14}$$

$$\begin{cases} \dfrac{1-\kappa_0}{R^2}\bar{a}_n + \dfrac{2i\varsigma_1}{R}(n-1)K_{n-1}(\varsigma_1 R)\alpha_n \\ \qquad + \dfrac{2\gamma\varsigma_2}{R}(n-1)K_{n-1}(\varsigma_2 R)\beta_n = A_n, \quad n \geq 3, \\ 2\varsigma_1 K_n'(\varsigma_1 R)\alpha_n - \dfrac{(\kappa+1)(\nu+\beta)ni}{4\mu R^{n+1}}\bar{a}_n = B_n, \quad n \geq 1, \\ \dfrac{(\lambda+2\mu)\xi - \gamma^2}{\mu\delta}K_n(\varsigma_2 R)\beta_n - \dfrac{\gamma}{(\lambda+\mu)\delta\varsigma_2^2 R^n}\bar{a}_n = C_n, \quad n \geq 0. \end{cases} \tag{6.15}$$

For $e^{-in\alpha}$ $(n > 1)$ gives

$$\begin{aligned} \frac{(1-\kappa_0)(1+n)}{R^n}a_n - \frac{1}{R^{n+2}}b_{n+2} - \frac{2i\varsigma_1}{R}(n+1)K_{n+1}(\varsigma_1 R)\alpha_{-n} \\ - \frac{2\gamma\varsigma_2}{R}(n+1)K_{n-1}(\varsigma_2 R)\beta_{-n} = A_{-n}. \end{aligned} \tag{6.16}$$

From (6.10) and (6.12) one finds

$$\begin{aligned} \alpha_0 &= \frac{B_0}{2\varsigma_1 K_0'(\varsigma_1 R)}, \\ \beta_0 &= \frac{\mu\delta C_0}{(\lambda+2\mu)\xi-\gamma^2}\,\frac{1}{K_0(\varsigma_2 R)} + \frac{2\gamma\mu\Gamma}{(\lambda+\mu)((\lambda+2\mu)\xi-\gamma^2)\varsigma_2^2}, \end{aligned} \tag{6.17}$$

$$b_2 = 2(1-\kappa_0)R^2\Gamma - 2i\varsigma_1 R K_1(\varsigma_1 R)\alpha_0 - 2\gamma\varsigma_2 R K_1(\varsigma_2 R)\beta_0 - R^2 A_0. \tag{6.18}$$

From (6.11) and (6.13) one finds

$$\begin{aligned} a_1 &= \frac{R\bar{A}_1}{\kappa-\kappa_0+1}, \quad b_1 = \frac{\kappa R A_1}{\kappa-\kappa_0+1}, \\ \alpha_1 &= \frac{B_1}{2\varsigma_1 K_1'(\varsigma_1 R)} + \frac{(\kappa+1)(\nu+\beta)iA_1}{8\mu R\varsigma_1 K_1'(\varsigma_1 R)}, \\ \beta_1 &= \frac{\mu\delta}{((\lambda+2\mu)\xi-\gamma^2)K_1(\varsigma_2 R)}\left(C_1 + \frac{\gamma A_1}{(\lambda+\mu)(\kappa-\kappa_0+1)\delta\varsigma_2^2}\right). \end{aligned}$$

From (6.10) and (6.14)–(6.16) one finds

$$\begin{aligned} \bar{a}_2 &= \frac{A_2 + \Gamma^{/} - k_{12}B_2 - k_{22}C_2}{(1-\kappa_0)R^{-2} - k_{32} + k_{42}}, \\ \bar{a}_n &= \frac{A_n - k_{1n}B_n - k_{2n}C_n}{(1-\kappa_0)R^{-n} - k_{3n} + k_{4n}}, \quad n > 2; \end{aligned} \tag{6.19}$$

$$\alpha_n = \frac{1}{2\zeta_1 K_n'(\zeta_1 R)}\left(B_n + \frac{(\kappa+1)(\nu+\beta)ni}{4\mu R^{n+1}}\bar{a}_n\right), \quad n \ge 2; \tag{6.20}$$

$$\beta_n = \frac{\mu\delta}{((\lambda+2\mu)\xi-\gamma^2)K_n(\varsigma_2 R)}\left(C_n + \frac{\gamma\bar{a}_n}{(\lambda+\mu)\delta\varsigma_2^2}\right), \quad n \ge 2; \tag{6.21}$$

$$\begin{aligned} b_{n+2} = {}& (1-\kappa_0)(n+1)R^2 a_n - 2(n+1)i\varsigma_1 R^{n+1} K_{n+1}(\varsigma_1 R)\alpha_n \\ & - 2(n+1)\gamma\varsigma_2 R^{n+1} K_{n+1}(\varsigma_2 R)\beta_n - R^{n+2}A_n, \quad n > 0, \end{aligned} \tag{6.22}$$

where

$$\begin{aligned} k_{1n} &= \frac{(n-1)iK_{n-1}(\varsigma_1 R)}{RK_n'(\varsigma_1 R)}, \quad k_{2n} = \frac{2(n-1)\varsigma_2\gamma\mu\delta K_{n-1}(\varsigma_2 R)}{R((\lambda+2\mu)\xi-\gamma^2)K_n(\varsigma_2 R)}, \\ k_{3n} &= \frac{n(n-1)(\kappa+1)(\nu+\beta)K_{n-1}(\varsigma_1 R)}{4\mu R^{n+2}K_n'(\varsigma_1 R)}, \\ k_{4n} &= \frac{2(n-1)\gamma^2\mu K_{n-1}(\varsigma_2 R)}{\varsigma_2 R^{n+1}(\lambda+\mu)((\lambda+2\mu)\xi-\gamma^2)K_n(\varsigma_2 R)}. \end{aligned}$$

It is easy to prove the absolute and uniform convergence of the series obtained in the infinite plane with a circular hole (including the contours) when the functions set on the boundaries have sufficient smoothness. Let the functions N, T, M and F have second order derivatives satyisfing the Dirichlet condition. In this case [26],

$$(|A_n|, |B_n|, |C_n|) < \frac{C}{n^3}, \quad (|A_{-n}|, |B_{-n}|, |C_{-n}|) < \frac{C}{n^3} \quad (n = 1, 2, \ldots), \tag{6.23}$$

where C is some positive constant.

We show that under the above condition the series (6.3) and (6.4), as well as the series for $\varphi''(z)$

$$\varphi''(z) = -\sum_{n=1}^{\infty} n a_n z^{-n-1}$$

will have absolutely and uniformly convergence on circle $r = R$, and hence also off circle $r \leq R$.

To prove the convergence of these series on the circle $r = R$, consider the series, formed by the moduli of the terms of the former when $|z| = R$

$$\begin{gathered} \sum |a_n| R^{-n}, \quad \sum n|a_n| R^{-n-1}, \quad \sum |b_n| R^{-n}, \\ \sum |\alpha_n| K_n(\zeta_1 R), \quad \sum |\beta_n| K_n(\zeta_2 R). \end{gathered} \tag{6.24}$$

Using inequality (6.23), on the basis of formulas (6.19)–(6.22) and taking into acaunt formulas (6.8), we conclude that

$$\begin{aligned} &|a_n| R^{-n} < \frac{C^/}{n^3}, \quad n|a_n| R^{-n-1} < \frac{C^{//}}{n^2}, \quad |b_n| R^{-n} < \frac{C^{///}}{n^2}, \\ &|\alpha_n| K_n(\zeta_1 R) < \frac{\tilde{C}}{n^3}, \quad |\beta_n| K_n(\zeta_2 R) < \frac{\hat{C}}{n^3}, \end{aligned}$$

where $C^/$, $C^{//}$, $C^{///}$, $\tilde{C}$, $\hat{C}$ are some other constants. This directly implies the convergence of the series (6.24), and therefore the uniform and absolute convergence of the series for φ', φ'', ψ', χ, η. The series for $\partial_z\chi$, $\partial_z\eta$, $\partial_z\partial_z\chi$ and $\partial_z\partial_z\eta$ will also be absolutely and uniformly convergent

$$\begin{aligned} &\partial_z\chi = -\frac{\varsigma_1}{2} \sum_{-\infty}^{+\infty} \alpha_n K_{n-1}(\varsigma_1 r) e^{i(n-1)\alpha}, \quad \partial_z\eta = -\frac{\varsigma_2}{2} \sum_{-\infty}^{+\infty} \beta_n K_{n-1}(\varsigma_2 r) e^{i(n-1)\alpha}, \\ &\partial_z\partial_z\chi = \frac{\zeta_1^2}{4} \sum_{-\infty}^{+\infty} \alpha_n K_{n-2}(\varsigma_1 r) e^{i(n-2)\alpha}, \quad \partial_z\partial_z\eta = \frac{\zeta_2^2}{4} \sum_{-\infty}^{+\infty} \beta_n K_{n-2}(\varsigma_2 r) e^{i(n-2)\alpha}. \end{aligned}$$

Thus, the problem is solved. Substituting the found functions in formulas (4.15)–(4.19), we can find all components of stresses and moment stresses at each point of the considered domain.

Let us now consider the simplest particular case when the edge of the hole is subject only to uniform normal pressure P. The constant value ϕ_0 of the function ϕ is also set on the hole contour. At infinity, the stresses, moment stresses, and also the value of the function ϕ are equal to zero. Then we have

$$N=-P,\quad T=0,\quad M=0,\quad F=\phi_0,\quad \Gamma=\Gamma^{/}=0.$$

In formulas (6.9) we have

$$A_0=-P,\quad C_0=\phi_0,\quad A_k=C_k=0,\quad k\neq 0.$$

In accordance with this on the basis of formulas (6.17), (6.18), we obtain then

$$\beta_0=\frac{\mu\delta\phi_0}{(\lambda+2\mu)\xi-\gamma^2}\frac{1}{K_0(\varsigma_2 R)},\quad b_2=-\frac{2\gamma\mu\delta\phi_0\varsigma_2 RK_1(\varsigma_2 R)}{(\lambda+2\mu)\xi-\gamma^2}+PR^2$$

and that all other coefficients of the expansions for φ', ψ', χ and η are equal to zero. Thus

$$\varphi(z)=0,\quad \psi'(z)=\frac{b_2}{z^2},\quad \psi(z)=-\frac{b_2}{z},\quad \chi(z,\bar z)=0,\quad \eta(z,\bar z)=\beta_0 K_0(\varsigma_2 r)\,.$$

Afterwards on the basis of formulas (4.14)-(4.21), we obtain

$$\begin{aligned}
\sigma_{rr}&=-\frac{b_2}{r^2}-\gamma\varsigma_2^2\beta_0(K_2(\varsigma_2 r)-K_0(\varsigma_2 r)),\\
\sigma_{\alpha\alpha}&=\frac{b_2}{r^2}+\gamma\varsigma_2^2\beta_0(K_2(\varsigma_2 r)+K_0(\varsigma_2 r)),\\
\sigma_{r\alpha}&=\sigma_{\alpha r}=0,\quad \mu_{r3}=\mu_{3r}=0,\\
\phi&=\frac{\phi_0}{K_0(\varsigma_2 R)}K_0(\varsigma_2 r),\\
u_r&=\frac{b_2}{2\mu r}+\frac{\gamma}{\mu}\varsigma_2\beta_0 K_1(\zeta_2 r),\quad u_\alpha=0,\quad \omega_3=0.
\end{aligned}\tag{6.25}$$

Stress component σ_{33} is also present in the body

$$\sigma_{33}=\frac{2\mu\gamma\phi_0}{\lambda+2\mu}K_0(\varsigma_2 r),$$

which is necessary to maintain a plane deformed state. As it can be seen from formulas (6.25), the solution depends only on r. The corresponding graphs are shown in Figs. 8–11. The following values were taken as $R=1$, $P=1$, $\phi_0=0.5$.

The obtained analytical solutions of boundary value problems can be used as a test for numerical algorithms [32, 33].

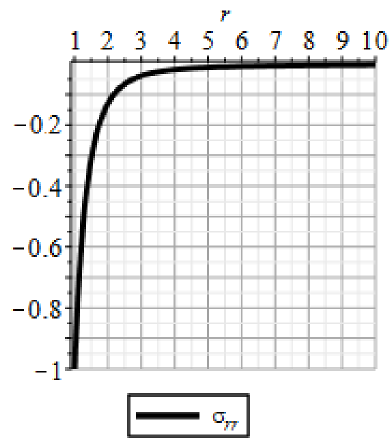

FIG. 8. P.2. Graph of function σ_{rr}.

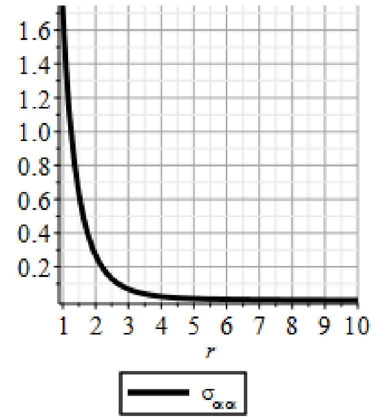

FIG. 9. P.2. Graph of function $\sigma_{\alpha\alpha}$.

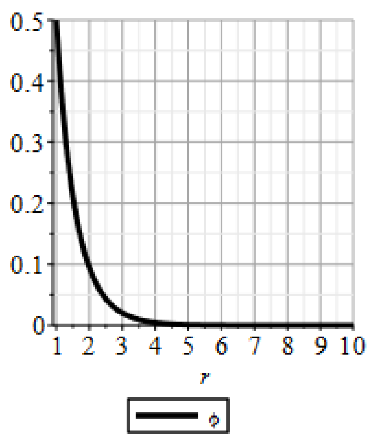

FIG. 10. P.2. Graph of function ϕ.

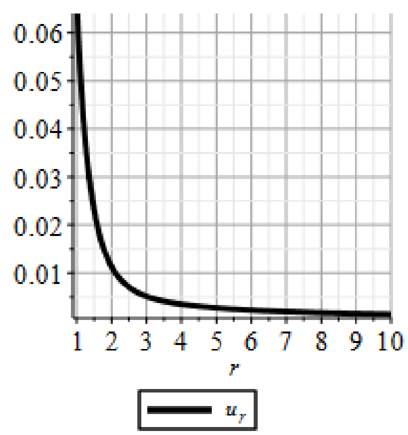

FIG. 11. P.2. Graph of function u_r.

7. Conclusion

As it is well known, to study the elastic properties of materials, it is very important to construct explicit solutions of the corresponding boundary value problems. These solutions are especially valuable when the elastic media has a complex internal structure and is characterized by porosity. In this paper, we discussed the static equilibrium of elastic materials with voids in the case of an asymmetric Cosserat media. As regards the three-dimensional case, the existence and uniqueness of a weak solution of the mixed boundary value problem is proved. For the case of plane deformation, the method of the theory of functions of a complex variable is first applied to the corresponding system of differential equations. The obtained analogs of the Kolosov–Muskhelishvili formulas allow one to obtain an analytical solution of a rather wide class of the corresponding boundary value problems. There analytical solutions of problems are also important in that they can be used as a test for numerical algorithms. In our opinion, the problems of poroelasticity for an asymmetric elastic media with voids have theoretical and practical interest.

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