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Mean time to failure for periodic failure rate

Keywords

reliability, mean time to failure

Abstract

The paper is concerned with the determination of the Mean Time To Failure (MTTF) in configurations where the failure rate is periodical. After solving two configurations exactly, we show that when the period of the failure rate oscillations is small with respect to the average failure rate, the MTTF is essentially given by the inverse of the average failure rate, give or take corrections that can be expressed analytically. This could be helpful in the description of systems the environment of which is subject to changes.

1. Introduction

The occurrence of failures in systems is often described by using well-known distributions (see for instance [6], [8],[10]) such as exponential, Weibull, etc. However, most of these distributions have associated failure rates which are constant or monotonous. This cannot realistically describe many real-life situations. To quite but a few examples, the probability of hurricanes is highest during the ``right'' seasons, computing systems exhibit different level activities during the day (there is also a weekly dependence because of week-ends). The efficiency of cooling units for electronic equipments in telecommunication networks depends on the ambient temperature; problems may arise in summer.

Quite naturally, the possibility of the periodicity of the failure rate has been raised. Castillo and Sieworek [3] have considered the reliability of computing systems, and presented several data, clearly showing that hard disk failures seem to follow the workload. The influence of this workload can be taken into account quite satisfactorily by the addition of a (periodical) failure rate. A few fundamental, mathematical studies have also been devoted to the issue of periodic random environment [4], [9], the emphasis being laid on time distributions, nonstationary Poisson processes and other probability properties such as the ``almost lack of memory''. More practical consideration emerge again, as witnessed by recent work on high-performance computing systems such as grids [5], [11]. To quote reference [5],

> *Accurate failure prediction in Grids is critical for reasoning about QoS guarantees such as job completion time and availability.*

Another recent practical paper [1] considers a problem that could (somehow) ring a bell to all of us: what is the life expectancy of our mobile phones? In these electronic devices, the temperature of specific part of the circuits may substantially increase during operations such as finding the next antenna, working in conditions of huge traffic. It has been recognized for many decades that some processes ultimately responsible for hardware failures in electronic components have a temperature dependence which obeys the Arrhenius law [2], used in many acceleration life test procedures. While the universality of this law is to be considered very carefully, there is no doubt that even a small increase in temperature may lead to surges in the failure rate. Should we consider the worst-case (meaning: temperature) scenario, or the most-of-the-time situation, knowing that these two hypotheses lead to mean times to failure (MTTF) differing by orders of magnitude? A review of the potential problems linked to temperature can be found in [7].

For this reason, we have tried to answer the following question: is there some way to perform a quick and not so dirty evaluation of the MTTF of a system subject to periodic failure? What are the important parameters?

Our paper is organized as follows: in section 2, we recall the well-known general expressions for the reliability and the MTTF, and compute the latter in two exactly solvable cases: in the first one, the failure rate takes two possible (constant) values; in the second one, we add an sinusoidal contribution to an otherwise constant failure rate. We show that when the oscillation period T of the added failure process is small compared with the otherwise expected lifetime, what really matters is merely the averaged failure rate $\overline{\lambda}$ over one period T (see equation (3) below). We confirm in section 3 this assertion in the general case, and provide the corrections to this asymptotic result in equation (4). We conclude by a brief discussion of possible extensions of this work.

2. Two exactly solvable cases

2.1. Reliability of large two-state series systems

The reliability may be written quite generally as

$$R(t) = e^{-\int_0^t \lambda(\tau)\,d\tau}, \qquad (1)$$

and the MTTF is given by

$$MTTF = \int_0^\infty t\,(-R'(t))\,dt = \int_0^\infty R(t)\,dt \quad . \qquad (2)$$

Let us now turn to two cases where the MTTF may be exactly computed.

2.2. Bimodal failure rate

We assume that the failure rate λ takes two values: λ_+ if $0 < t < \alpha T$, and λ_- if $\alpha T < t < T$ (see *Figure 1*).

After considering the successive intervals [n T, (n+α) T] and [(n+α) T, (n+1) T], and summing the easy to integrate exponentials, we eventually get

$$MTTF = \frac{1}{\lambda_-} + \left(\frac{1}{\lambda_+} - \frac{1}{\lambda_-}\right) \frac{1-e^{-\alpha\lambda_+ T}}{1-e^{-(\alpha\lambda_+ + (1-\alpha)\lambda_-)T}}$$

This relatively cumbersome expression of λ_+, λ_-, and T is actually very simple when considered in the $T \to 0$ limit, that is when the period of the oscillation is small compared with λ_+ and λ_-. We obtain

$$MTTF \to \frac{1}{\lambda_-} + \left(\frac{1}{\lambda_+} - \frac{1}{\lambda_-}\right) \frac{\alpha\lambda_+}{(\alpha\lambda_+ + (1-\alpha)\lambda_-)}$$
$$= \frac{1}{\alpha\lambda_+ + (1-\alpha)\lambda_-}$$

which is nothing but the inverse of the average of the failure rate in the time period [0,*T*].

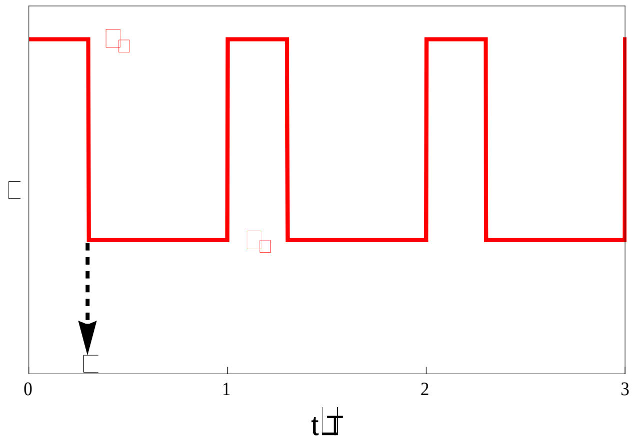

Figure 1. Simple variation of the failure rate.

2.3. Sinusoidal failure rate

We now assume that the failure rate is given by $\lambda(t) = \lambda_0 + \lambda_1 \cos\omega t$ (see *Figure 2*), so that

$$R(t) = \exp\left(-\lambda_0\, t - \frac{\lambda_1}{\omega}\sin\omega t\right)$$

($T = \frac{2\pi}{\omega}$ is the period of the failure rate oscillations). The gist of the MTTF calculation is to expand the factor $\exp\left(-\frac{\lambda_1}{\omega}\sin\omega t\right)$ in the integral as a power series in λ_1. Each contribution is then (somewhat tediously) assessed. After some work, it is possible to show that even powers of λ_1 contribute to an hypergeometric function ${}_1F_2$ defined by

$${}_1F_2(\alpha;\beta,\gamma;z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(n+\beta)\Gamma(n+\gamma)} \frac{z^n}{n!}$$

where $\Gamma(z)$ is the Euler gamma function. A similar conclusion is reached for the odd powers of λ_1.

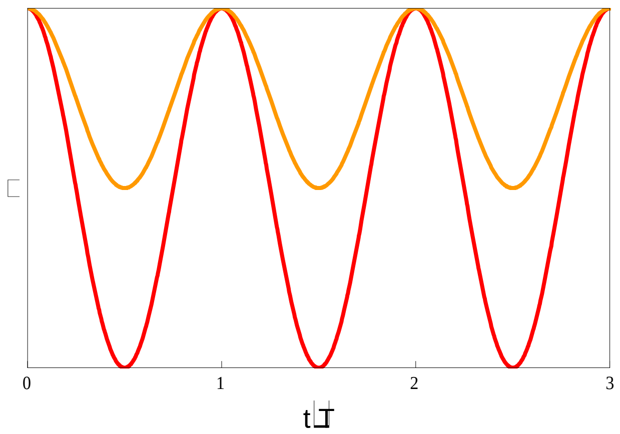

Figure 2. Sinusoidal variations of the failure rate: $\lambda_1 = \lambda_0$ (red) and $\lambda_1 = \lambda_0 / 3$ (orange).

Finally, we obtain (i is such that $i^2 = -1$)

$$MTTF = \frac{1}{\lambda_0}\Bigg({}_1F_2\left(1;1-\frac{i\lambda_0}{2\omega},1+\frac{i\lambda_0}{2\omega},\frac{\lambda_1^2}{4\omega^2}\right) - \frac{\lambda_0\lambda_1}{\lambda_0^2+\omega^2}\,{}_1F_2\left(1;\frac{3}{2}-\frac{i\lambda_0}{2\omega},\frac{3}{2}+\frac{i\lambda_0}{2\omega},\frac{\lambda_1^2}{4\omega^2}\right)\Bigg)$$

The prefactor $1/\lambda_0$ indicates that the MTTF will be linked to the inverse of the ``average" failure rate. It is indeed the exact result when $\lambda_1 = 0$, as expected. However, when $\lambda_1 > 0$, there are corrections to the simple result $1/\lambda_0$. If we assume that the two failure rates λ_0 and λ_1 are small compared with respect to ω, keeping the first two orders of the expansion, we find, expanding the hypergeometric functions ${}_1F_2$

$$MTTF \approx \frac{1}{\lambda_0}\left(1+\frac{\lambda_1^2}{\lambda_0^2+4\omega^2}+\ldots-\frac{\lambda_0\lambda_1}{\lambda_0^2+\omega^2}-\ldots\right)$$

or

$$MTTF \approx \frac{1}{\lambda_0}\left(1-\frac{\lambda_0\lambda_1}{\omega^2}+\frac{\lambda_1^2}{4\omega^2}\right)$$

We have displayed in *Figure 3* the value of the MTTF as a function of ω, for two different values of (λ_0, λ_1), using λ_0 as a scaling parameter. We see that both curves are monotonous and that the asymptotic limits are quickly reached after initial, steep increases.

When the period of the oscillations is large, we would expect the MTTF to be the inverse of the ``initial" failure rate, i.e., $\frac{1}{\lambda_0+\lambda_1}$. This is indeed observed in *Figure 3* when $\omega \to 0$.

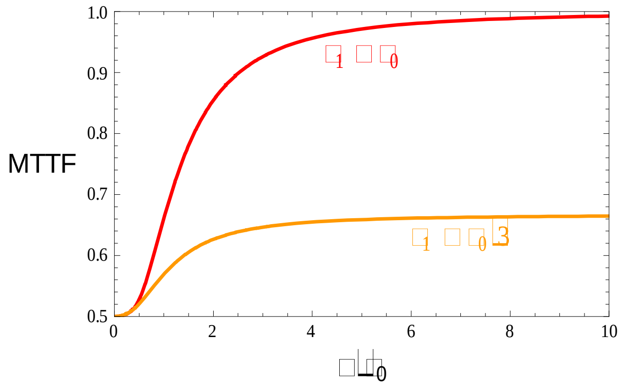

Figure 3. MTTF as a function of $\omega = \frac{2\pi}{T}$ for the configuration of *Figure2* ($\lambda_0 + \lambda_1 = 2$).

Similar curves could be drawn for higher moments of the failure time distribution. It should be noted, however, that the variation of the average of t^2 is not necessarily monotonous anymore.

3. General case when the oscillation period is small

It may be satisfying to obtain an analytical solution to a few configurations, but this, unfortunately, is not true in general. The question is now to establish whether the MTTF may be evaluated by averaging a few quantities, and if so, the result is not too inaccurate. Recall that in many real situations, the period of the oscillations may be one day or one week — one year or more in the context of climatologic studies — and therefore much shorter than expected failure times.

Let us now consider the general case when the oscillation period is T. We can define an average failure rate

$$\bar{\lambda} = \frac{1}{T}\int_0^T \lambda(t)\,dt \qquad (3)$$

Going back to the expression of the MTTF, we see that

$$\int_0^t \lambda(\tau)\,d\tau = \overline{\lambda} T \left\lfloor \frac{t}{T} \right\rfloor + \int_{T\left\lfloor \frac{t}{T}\right\rfloor}^{t} \lambda(\tau)\,d\tau$$
$$= \overline{\lambda} t + \int_{T\left\lfloor \frac{t}{T}\right\rfloor}^{t} (\lambda(\tau) - \overline{\lambda})\,d\tau$$

where $\lfloor x \rfloor$ is the integer part of *x*. Turning now to the MTTF expression (see (1) and (2)), we have

$$MTTF = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} \exp\left(-\overline{\lambda}\, nT - \int_{nT}^{t} \lambda(\tau)\,d\tau \right) dt$$
$$= \sum_{n=0}^{\infty} \exp\left(-n\overline{\lambda} T\right) \int_0^T \exp\left(-\int_0^t \lambda(\tau)\,d\tau \right) dt$$
$$= \frac{\int_0^T \exp\left(-\int_0^t \lambda(\tau)\,d\tau \right) dt}{1 - \exp\left(-\overline{\lambda} T\right)} = \frac{N}{D}$$

When the oscillation period is small ($T \to 0$, or the failure rate is assumed to be too small to matter during *T*), the expressions of the numerator *N* and denominator *D* may be expanded. Up to second order, we get

$$N = \int_0^T \left(1 - \int_0^t \lambda(\tau)\,d\tau + \frac{1}{2} \left(\int_0^t \lambda(\tau)\,d\tau \right)^2 - \cdots \right) dt$$
$$= T \left(1 - \frac{1}{T} \int_0^T \int_0^t \lambda(\tau)\,d\tau\,dt + \frac{1}{2T} \int_0^T \left(\int_0^t \lambda(\tau) d\tau \right)^2 dt + \cdots \right),$$

while

$$D = \overline{\lambda} T \left(1 - \frac{1}{2} \overline{\lambda} T + \frac{1}{6} \left(\overline{\lambda} T\right)^2 + \cdots \right).$$

From the expressions of *N* and *D*, the leading order for the MTTF's expansion gives $MTTF \approx \frac{1}{\overline{\lambda}}$.

Using integrations by parts and $\tilde{\lambda}(t) = \lambda(t) - \overline{\lambda}$, we can easily obtain the corrections to the $T \to 0$ limit. After simplification, they give the main result of this paper

$$MTTF \approx \frac{1}{\overline{\lambda}} \left(1 + \frac{1}{T} \int_0^T t \tilde{\lambda}(t)\,dt + \frac{1}{2T} \int_0^T \left(\int_0^t \tilde{\lambda}(\tau) d\tau \right)^2 dt - \frac{\overline{\lambda}}{2T} \int_0^T \tilde{\lambda}(t) \left(t - \frac{T}{2} \right)^2 dt + \cdots \right) \quad (4)$$

Depending on the actual form of $\tilde{\lambda}(t)$, the first-order correction $\int_0^T t\tilde{\lambda}(t)\,dt$ may cancel (this is actually the case in Example 2.3, where the corrections to 1 are 0, $\frac{\lambda_1^2}{4\omega^2}$, and $-\frac{\lambda_0\,\lambda_1}{\omega^2}$, respectively) or be finite (in Example 2.2., it is $-\frac{\alpha(1-\alpha)}{2}(\lambda_+ - \lambda_-)T$, in agreement with the expansion of the exact result).

4. What about temperature effects?

We mentioned in the Introduction that temperature is an important issue in the reliability of electronic components. Some data on the failure rates may be found in hardware catalogs, in operating condition (at a given temperature, mainly 20 or 25 °C). In some cases, estimates of the failure rate at *higher* temperatures are also given. It might therefore be more suitable to express the instantaneous failure rate as $\lambda(\mathrm{T}(t))$. Assuming that the Arrhenius law is valid *for a given physical process* we would have

$$\lambda(T) \propto e^{-\frac{E_a}{k\mathrm{T}}}$$

where E_a is the activation energy of the process, *k* the Boltzmann constant and T the temperature. The calculations of the preceding sections would have to take this further cause of variation into account. We would expect the MTTF to be weighted by the times spent in the higher temperature regimes.

5. Conclusion and outlook

We have provided simple analytical results for the MTTF with a periodical failure rate, which may prove helpful when evaluating the lifetime of various kinds of components operating in environments for which the workload may induce failures to occur in a periodic manner.

Generalizations of the present results would of course include the assessment of the variation of the

MTTF, when the initial distribution is not exponential, but a more realistic one.

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