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CALCULATING STEADY-STATE PROBABILITIES OF SINGLE-CHANNEL QUEUEING SYSTEMS WITH CHANGES OF SERVICE TIMES DEPENDING ON THE QUEUE LENGTH

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Abstract. In this paper, we propose a method for calculating steady-state probabilities of the $G/G/1/m$ and $M/G/1/m$ queueing systems with service times changes depending of the number of customers in the system. The method is based on the use of fictitious phases and hyperexponential approximations with parameters of the paradoxical and complex type. A change in the service mode can only occur at the moment the service is started. We verified the obtained numerical results using the potential method and simulation models, constructed by means of GPSS World.

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1. Introduction

Models of queueing systems in which the intensity of service is purposefully changing along with the queue length, are often used to study telecommunication processes, in particular, processes of data transmission in ATM networks using multiplexing technologies [1, 2]. Let us study the $G/G/1/m$ system, at which the service time of each customer is determined by the rule: if at the moment of service start of this customer the number of customers in the system is n, its service time has the distribution function $F_n(x)$. We denote such a system by $G/G(n)/1/m$ where m is the restriction on the queue length. A special case of the $G/G(n)/1/m$ system is a system denoted by $G/G(h)/1/m$ with a threshold change of service times, for which

$$F_n(x) = F(x),\ 1 \le n \le h-1;\quad F_n(x) = \tilde{F}(x),\ h \le n \le m;\quad h \in \{2,3,\ldots,m\}. \tag{1}$$

A review of the results obtained for queueing systems with state dependent parameters can be found in [3]. In the work [4] the potentials method is proposed, which allows one to obtain formulas, convenient for numerical realization, for steady-state distribution of the number of customers in the $M/G(n)/1/m$ and $M/G(h)/1/m$ systems with group arrivals of customers.

Papers [5-7] suggest the use of hyperexponential approximation (denoted by H_k) for calculating of steady-state probabilities of the non-Markovian queueing systems. If variation coefficients V_1 and V_2 of distributions of the interarrival time between two consecutive customers and the service times satisfy the conditions $V_1 + V_2 > 0.6$ and $\max\{V_1, V_2\} < 2$ then we are able to calculate steady-state probabilities with high accuracy (higher than in the case of using simulation models); see [7]. These probabilities are determined using solutions of a system of linear algebraic equations obtained by the method of fictitious phases. To find parameters of the H_k-approximation of a certain distribution, it is sufficient to solve the system of equations of the moments method. For the values $V < 1$ of the variation coefficient, the roots of this system are complex-valued or paradoxical (i.e., negative or with probabilities that exceed the boundaries of the interval $[0, 1]$) but in most cases as a result of summation of probabilities of microstates, their complex-valued and paradoxical parts are annihilated.

The purpose of the paper is to use of the hyperexponential approximation method for calculating steady-state probabilities of the $G/G(n)/1/m$ and $G/G(h)/1/m$ systems. We also consider the $M/G(n)/1/m$ and $M/G(h)/1/m$ systems for which results can be checked using the potentials method.

2. Equations for steady-state probabilities of the $H_l/H_r(n)/1/m$, $H_l/H_r(h)/1/m$, $M/H_r(n)/1/m$ and $M/H_r(h)/1/m$ systems

The hyperexponential distribution of order k is a phase-type distribution and provides for choosing one of k alternative phases by a random process. With probability y_i, the process is at the i-th phase and is in it during an exponentially distributed time with a parameter θ_i.

Suppose that the times elapsed between two consecutive arrivals are independent random variables distributed according to the hyperexponential law H_l $(l \geq 2)$ with probabilities α_s and parameters $\lambda_s (1 \leq s \leq l)$, and the service time of each customer is distributed according to the hyperexponential law $H_r(n)$ $(r \geq 2)$ with probabilities β_{ni} and parameters $\mu_{ni} (1 \leq n \leq m, 1 \leq i \leq r)$, depending on the number n of customers in the system. A change in the service mode can only occur at the moment the service is started. The system under consideration is denoted by $H_l/H_r(n)/1/m$ and will be used for an approximate calculation of the steady-state probabilities of the $G/G(n)/1/m$ system.

Let us enumerate the $H_l/H_r(n)/1/m$ system's states as follows: $x_{0(s)}$ corresponds to the empty system (that is $n = 0$) and the time interval until the arrival of the first customer is in the phase s $(1 \leq s \leq l)$; $x_{1(s\,j)}$ is the state, when $n = 1$, the time interval until the arrival of the next customer is in the phase s $(1 \le s \le l)$ and j is the phase number of service time, distributed according to the $H_r(1)$ law; $x_{k(sij)}$ is the state, when $n = k$ $(2 \le k \le m+1)$, the time interval until the arrival of the next customer is in the phase s $(1 \le s \le l)$, i is the number of the $H_r(i)$ law of service time distribution $(1 \le i \le k \le m)$ and j is the phase number of service time, distributed according to the $H_r(i)$ law. We denote by $p_{0(s)}$, $p_{1(sj)}$ and $p_{k(sij)}$ respectively, steady-state probabilities that the system is in the each of these states. To calculate $p_{0(s)}$, $p_{1(sj)}$ and $p_{k(sij)}$ we obtain the system of linear equations:

$$
\begin{aligned}
&-\lambda_s p_{0(s)} + \sum_{i=1}^{r} \mu_{1j} p_{1(sj)} = 0, \ \ 1 \le s \le l;\\
&-(\lambda_s + \mu_{1j}) p_{1(sj)} + \beta_{1j} \alpha_s \sum_{u=1}^{l} \lambda_u p_{0(u)} + \beta_{1j} \sum_{u=1}^{2} \sum_{i=1}^{r} \mu_{ui} p_{2(sui)} = 0, 1 \le s \le l, \ 1 \le j \le r;\\
&-(\lambda_s + \mu_{1j}) p_{2(s1j)} + \alpha_s \sum_{u=1}^{l} \lambda_u p_{1(uj)} = 0, \ \ 1 \le s \le l, \ 1 \le j \le r;\\
&-(\lambda_s + \mu_{kj}) p_{k(skj)} + \beta_{kj} \sum_{u=1}^{k+1} \sum_{i=1}^{r} \mu_{ui} p_{k+1(sui)} = 0, \ 2 \le k \le m-1, \ 1 \le s \le l, \ 1 \le j \le r;\\
&-(\lambda_s + \mu_{ij}) p_{k(sij)} + \alpha_s \sum_{u=1}^{l} \lambda_u p_{k-1(uij)} = 0,\\
&\qquad\qquad\qquad 3 \le k \le m, \ 1 \le s \le l, \ 1 \le i \le k-1, \ 1 \le j \le r;\\
&-(\lambda_s + \mu_{mj}) p_{m(smj)} + \beta_{mj} \sum_{u=1}^{m} \sum_{i=1}^{r} \mu_{ui} p_{m+1(sui)} = 0, \ 1 \le s \le l, \ 1 \le j \le r;\\
&-(\lambda_s + \mu_{ij}) p_{m+1(sij)} + \alpha_s \sum_{u=1}^{l} \lambda_u \left(p_{m(uij)} + p_{m+1(uij)} \right) = 0,\\
&\qquad\qquad\qquad 1 \le s \le l, \ 1 \le i \le m, \ 1 \le j \le r;\\
&\sum_{s=1}^{l} p_{0(s)} + \sum_{s=1}^{l} \sum_{j=1}^{r} p_{1(sj)} + \sum_{k=2}^{m} \sum_{s=1}^{l} \sum_{j=1}^{r} \left(\sum_{i=1}^{k-1} p_{k(sij)} + p_{k(skj)} \right) + \sum_{s=1}^{l} \sum_{i=1}^{m} \sum_{j=1}^{r} p_{m+1(sij)} = 1.
\end{aligned}
\tag{2}
$$

Solving the system (2), we find the steady-state probabilities p_k of the presence in the queueing system of k customers using the formulas

$$
\begin{aligned}
&p_0 = \sum_{s=1}^{l} p_{0(s)}, \ \ p_1 = \sum_{s=1}^{l} \sum_{j=1}^{r} p_{1(sj)}, \ \ p_{m+1} = \sum_{s=1}^{l} \sum_{i=1}^{m} \sum_{j=1}^{r} p_{m+1(sij)};\\
&p_k = \sum_{s=1}^{l} \sum_{j=1}^{r} \left(\sum_{i=1}^{k-1} p_{k(sij)} + p_{k(skj)} \right), \ \ 2 \le k \le m.
\end{aligned}
\tag{3}
$$

Let us consider the $M/H_r(n)/1/m$ system that differs from the $H_l/H_r(n)/1/m$ system in that the times elapsed between two consecutive arrivals are independent random variables exponentially distributed with parameter λ.

Let us enumerate the $M/H_r(n)/1/m$ system's states as follows: x_0 corresponds to the empty system (that is $n=0$); $x_{1(j)}$ is the state, when $n=1$ j is the phase number of service time, distributed according to the $H_r(1)$ law; $x_{k(ij)}$ is the state, when $n=k$ $(2\le k\le m+1)$, i is the number of the $H_r(i)$ law of service time distribution $(1\le i\le k\le m)$ and j is the phase number of service time, distributed according to the $H_r(i)$ law. We denote by p_0, $p_{1(j)}$ and $p_{k(ij)}$ respectively, steady-state probabilities that the system is in the each of these states. To calculate p_0, $p_{1(j)}$ and $p_{k(ij)}$ we obtain the system of linear equations:

$$\begin{aligned}
&-\lambda p_0+\sum_{i=1}^{r}\mu_{1j}p_{1(j)}=0;\\
&-(\lambda+\mu_{1j})p_{1(j)}+\beta_{1j}\lambda p_0+\beta_{1j}\sum_{u=1}^{2}\sum_{i=1}^{r}\mu_{ui}p_{2(ui)}=0,\ 1\le j\le r;\\
&-(\lambda+\mu_{1j})p_{2(1j)}+\lambda p_{1(j)}=0,\quad 1\le j\le r;\\
&-(\lambda+\mu_{kj})p_{k(kj)}+\beta_{kj}\sum_{u=1}^{k+1}\sum_{i=1}^{r}\mu_{ui}p_{k+1(ui)}=0,\ 2\le k\le m-1,\ 1\le j\le r;\\
&-(\lambda+\mu_{ij})p_{k(ij)}+\lambda p_{k-1(ij)}=0,\ 3\le k\le m,\ 1\le i\le k-1,\ 1\le j\le r;\\
&-(\lambda+\mu_{mj})p_{m(mj)}+\beta_{mj}\sum_{u=1}^{m}\sum_{i=1}^{r}\mu_{ui}p_{m+1(ui)}=0,\ 1\le j\le r;\\
&-\mu_{ij}p_{m+1(ij)}+\lambda p_{m(ij)}=0,\ 1\le i\le m,\ 1\le j\le r;\\
&p_0+\sum_{j=1}^{r}p_{1(j)}+\sum_{k=2}^{m}\sum_{j=1}^{r}\left(\sum_{i=1}^{k-1}p_{k(ij)}+p_{k(kj)}\right)+\sum_{i=1}^{m}\sum_{j=1}^{r}p_{m+1(ij)}=1.
\end{aligned}\tag{4}$$

Solving the system (4), we find the steady-state probabilities p_k by the formulas

$$p_1=\sum_{j=1}^{r}p_{1(j)};\ \ p_k=\sum_{j=1}^{r}\left(\sum_{i=1}^{k-1}p_{k(ij)}+p_{k(kj)}\right),\ 2\le k\le m;\ \ p_{m+1}=\sum_{i=1}^{m}\sum_{j=1}^{r}p_{m+1(ij)}.\tag{5}$$

Let us consider the $H_l/H_r(h)/1/m$ system in which the expedited service mode is used if, at the start of service of the customer, the number of customers in the system is not less than h. The system states are numbered as follows: $x_{0(s)}$ corresponds to the empty system and the time interval until the arrival of the first customer is in the phase s $(1\le s\le l)$; $x_{k(si)}$ is the state, when $n=k$ $(1\le k\le m+1)$, the time interval until the arrival of the next customer is in the phase s $(1 \le s \le l)$ and i is the phase number of service time, distributed according to the H_r law with probability β_i and intensity μ_i $(1 \le i \le r)$; $\tilde{x}_{k(si)}$ is the state, when $n = k$ $(h \le k \le m+1)$, the time interval until the arrival of the next customer is in the phase s $(1 \le s \le l)$ and i is the phase number of service time, distributed according to the H_r law with probability $\tilde{\beta}_i$ and intensity $\tilde{\mu}_i$ $(1 \le i \le r)$. We denote by $p_{0(s)}$, $p_{k(si)}$ and $\tilde{p}_{k(si)}$ respectively, steady-state probabilities that the system is in the each of these states. To calculate $p_{0(s)}$, $p_{k(si)}$ and $\tilde{p}_{k(si)}$, we obtain the system of linear equations:

$$
\begin{aligned}
&-\lambda_s p_{0(s)} + \sum_{i=1}^{r} \mu_i p_{1(si)} = 0, \ \ 1 \le s \le l; \\
&-(\lambda_s + \mu_i) p_{1(si)} + \beta_i \alpha_s \sum_{u=1}^{l} \lambda_u p_{0(u)} + \beta_i \sum_{u=1}^{r} \mu_u p_{2(su)} = 0, \ 1 \le s \le l, \ 1 \le i \le r; \\
&-(\lambda_s + \mu_i) p_{k(si)} + \alpha_s \sum_{u=1}^{l} \lambda_u p_{k-1(ui)} + \beta_i \sum_{u=1}^{r} \mu_u p_{k+1(su)} = 0, \\
&\qquad\qquad 2 \le k \le h-2, \ \ 1 \le s \le l, \ 1 \le i \le r; \\
&-(\lambda_s + \mu_i) p_{k(si)} + \alpha_s \sum_{u=1}^{l} \lambda_u p_{k-1(ui)} = 0, \ h \le k \le m, \ \ 1 \le s \le l, \ 1 \le i \le r; \\
&-(\lambda_s + \mu_i) p_{h-1(si)} + \alpha_s \sum_{u=1}^{l} \lambda_u p_{h-2(ui)} + \beta_i \sum_{u=1}^{r} \left(\mu_u p_{h(su)} + \tilde{\mu}_u \tilde{p}_{h(su)} \right) = 0, \\
&\qquad\qquad 1 \le s \le l, \ 1 \le i \le r; \qquad (6) \\
&-(\lambda_s + \tilde{\mu}_i) \tilde{p}_{h(si)} + \tilde{\beta}_i \sum_{u=1}^{r} \left(\mu_u p_{h+1(su)} + \tilde{\mu}_u \tilde{p}_{h+1(su)} \right) = 0, \ 1 \le s \le l, \ 1 \le i \le r; \\
&-(\lambda_s + \tilde{\mu}_i) \tilde{p}_{k(si)} + \alpha_s \sum_{u=1}^{l} \lambda_u \tilde{p}_{k-1(ui)} + \tilde{\beta}_i \sum_{u=1}^{r} \left(\mu_u p_{k+1(su)} + \tilde{\mu}_u \tilde{p}_{k+1(su)} \right) = 0, \\
&\qquad\qquad h+1 \le k \le m, \ 1 \le s \le l, \ 1 \le i \le r; \\
&-(\lambda_s + \mu_i) p_{m+1(si)} + \alpha_s \sum_{u=1}^{l} \lambda_u \left(p_{m(ui)} + p_{m+1(ui)} \right) = 0, \ 1 \le s \le l, \ 1 \le i \le r; \\
&-(\lambda_s + \tilde{\mu}_i) \tilde{p}_{m+1(si)} + \alpha_s \sum_{u=1}^{l} \lambda_u \left(\tilde{p}_{m(ui)} + \tilde{p}_{m+1(ui)} \right) = 0, \ 1 \le s \le l, \ 1 \le i \le r; \\
&\sum_{s=1}^{l} p_{0(s)} + \sum_{k=1}^{m+1} \sum_{s=1}^{l} \sum_{i=1}^{r} p_{k(si)} + \sum_{k=h}^{m+1} \sum_{s=1}^{l} \sum_{i=1}^{r} \tilde{p}_{k(si)} = 1.
\end{aligned}
$$

Solving the system (6), we find the steady-state probabilities p_k by the formulas

$$p_0=\sum_{s=1}^{l}p_{0(s)};\ \ p_k=\sum_{s=1}^{l}\sum_{i=1}^{r}p_{k(si)},\ \ 1\le k\le h-1;$$
$$p_k=\sum_{s=1}^{l}\sum_{i=1}^{r}\left(p_{k(si)}+\tilde{p}_{k(si)}\right),\ \ h\le k\le m+1. \tag{7}$$

Let us consider the $M/H_r(h)/1/m$ system that differs from the $H_l/H_r(h)/1/m$ system in that the times elapsed between two consecutive arrivals are independent random variables exponentially distributed with parameter λ. The system states are numbered as follows: x_0 corresponds to the empty system; $x_{k(i)}$ is the state, when $n=k$ $(1\le k\le m+1)$ and i is the phase number of service time, distributed according to the H_r law with probability β_i and intensity μ_i $(1\le i\le r)$; $\tilde{x}_{k(i)}$ is the state, when $n=k$ $(h\le k\le m+1)$ and i is the phase number of service time, distributed according to the H_r law with probability $\tilde{\beta}_i$ and intensity $\tilde{\mu}_i$ $(1\le i\le r)$. We denote by p_0, $p_{k(i)}$ and $\tilde{p}_{k(i)}$ respectively, steady-state probabilities that the system is in the each of these states. To calculate p_0, $p_{k(i)}$ and $\tilde{p}_{k(i)}$ we obtain the system of linear equations:

$$-\lambda p_0+\sum_{i=1}^{r}\mu_i p_{1(i)}=0;$$
$$-(\lambda+\mu_i)p_{1(i)}+\lambda\beta_i p_0+\beta_i\sum_{j=1}^{r}\mu_j p_{2(j)}=0,\ \ 1\le i\le r;$$
$$-(\lambda+\mu_i)p_{k(i)}+\lambda p_{k-1(i)}+\beta_i\sum_{u=1}^{r}\mu_u p_{k+1(u)}=0,\ \ 2\le k\le h-2,\ 1\le i\le r;$$
$$-(\lambda+\mu_i)p_{k(i)}+\lambda p_{k-1(i)}=0,\ \ h\le k\le m,\ \ 1\le i\le r;$$
$$-(\lambda+\mu_i)p_{h-1(i)}+\lambda p_{h-2(i)}+\beta_i\sum_{u=1}^{r}\left(\mu_u p_{h(u)}+\tilde{\mu}_u\tilde{p}_{h(u)}\right)=0,\ \ 1\le i\le r;$$
$$-(\lambda+\tilde{\mu}_i)\tilde{p}_{h(i)}+\tilde{\beta}_i\sum_{u=1}^{r}\left(\mu_u p_{h+1(u)}+\tilde{\mu}_u\tilde{p}_{h+1(u)}\right)=0,\ \ 1\le i\le r; \tag{8}$$
$$-(\lambda+\mu_i)p_{k(i)}+\lambda p_{k-1(i)}+\beta_i\sum_{u=1}^{r}\mu_u p_{k+1(u)}=0,\ \ 2\le k\le m,\ \ 1\le i\le r;$$
$$-(\lambda+\tilde{\mu}_i)\tilde{p}_{k(i)}+\lambda\tilde{p}_{k-1(i)}+\tilde{\beta}_i\sum_{u=1}^{r}\left(\mu_u p_{k+1(u)}+\tilde{\mu}_u\tilde{p}_{k+1(u)}\right)=0,\ h+1\le k\le m, 1\le i\le r;$$
$$-\mu_i p_{m+1(i)}+\lambda p_{m(i)}=0,\quad -\tilde{\mu}_i\tilde{p}_{m+1(i)}+\lambda\tilde{p}_{m(i)}=0,\ \ 1\le i\le r;$$
$$p_0+\sum_{k=1}^{m+1}\sum_{i=1}^{r}p_{k(i)}+\sum_{k=h}^{m+1}\sum_{i=1}^{r}\tilde{p}_{k(i)}=1.$$

Solving the system (8), we find the steady-state probabilities p_k by the formulas

$$p_k = \sum_{i=1}^{r} p_{k(i)},\ 1 \le k \le h-1;\ p_k = \sum_{i=1}^{r} \left(p_{k(i)} + \tilde{p}_{k(i)} \right),\ h \le k \le m+1. \tag{9}$$

3. Numerical results

Let us present the results of calculating steady-state probabilities on examples of $G/G(n)/1/10$, $M/G(n)/1/10$, $G/G(h)/1/15$ and $M/G(h)/1/15$ systems. For times elapsed between two consecutive arrivals and service times we consider uniform distributions, labeled as U, and gamma distributions with coefficients of variation $V = 0.7$, labeled as Γ. For calculations, we use the systems of linear equations for steady-state probabilities written for $H_k/H_k(n)/1/10$, $M/H_k(n)/1/10$, $H_k/H_k(h)/1/15$ and $M/H_k(h)/1/15$ systems respectively, using the order of approximation k from 2 to 6 for uniform and gamma distributions.

Let $E(T_\lambda)$ and $E(T_\mu(n))$ or $E(T_\mu(h_-))$ and $E(T_\mu(h_+))$ denote the mean of the times elapsed between two consecutive arrivals and the service times, respectively. Here $T_\mu(h_-)$ and $T_\mu(h_+)$ denote the service times for a usual and expedited service mode respectively.

For the $U/U(n)/1/10$, $\Gamma/\Gamma(n)/1/10$, $M/U(n)/1/10$ and $M/\Gamma(n)/1/10$ systems we take

$$E(T_\lambda) = 0.5,\ E(T_\mu(n)) = (11-n)/10,\ 1 \le n \le 10; \tag{10}$$

the interval $[0, 1]$ for the uniform distribution of T_λ, named U, and the intervals $[0, (11-n)/5]$ for the uniform distributions of the service times $T_\mu(n)$, named $U(n)$.

For the $U/U(h)/1/15$, $\Gamma/\Gamma(h)/1/15$, $M/U(h)/1/15$ and $M/\Gamma(h)/1/15$ systems we take

$$E(T_\lambda) = 0.6,\ E(T_\mu(h_-)) = 1,\ E(T_\mu(h_+)) = 0.5,\ h = 7; \tag{11}$$

the interval $[0, 1.2]$ for the uniform distribution of T_λ, named U, and the intervals $[0,2]$ and $[0,1]$ for the uniform distributions of $T_\mu(h_-)$ and $T_\mu(h_+)$, respectively, named $U(n)$.

To find parameters of the H_k-approximation of a certain distribution with a given coefficient of variation, it is sufficient to solve the system of equations of the moments method only for the case of any one given mean value of this distribution since the roots of the equations of the moments method are invariant with respect to the scale transformation. As an example we give the parameters of H_6-approximation of the uniform distribution on the interval $[0, 2]$

$$\beta_{1,2} = 0.319835 \pm 1.179031i, \quad \mu_{1,2} = 2.515932 \pm 4.492673i,$$
$$\beta_{3,4} = -3.409262 \mp 12.719777i, \quad \mu_{3,4} = 3.735708 \pm 2.626272i, \qquad (12)$$
$$\beta_{5,6} = 3.589427 \pm 36.226047i, \quad \mu_{5,6} = 4.248359 \pm 0.867510i;$$

and the gamma distributions with coefficient of variation $V = 0.7$ and the mean 1:

$$\beta_1 = 0.000809, \quad \mu_1 = 19.563118, \quad \beta_2 = 0.006791, \quad \mu_2 = 6.410937,$$
$$\beta_3 = 0.037377, \quad \mu_3 = 3.493911, \quad \beta_4 = 0.269433, \quad \mu_4 = 2.456711, \qquad (13)$$
$$\beta_{5,6} = 0.342795 \pm 69.581446i, \quad \mu_{5,6} = 2.037657 \pm 0.016175i.$$

The obtained results are verified using simulation models constructed with the help of the GPSS World tools [8]. The results obtained using GPSS World slightly differ from one another for different numbers of library random-number generators used for simulating random variables, i.e., times elapsed between two consecutive arrivals and service times. Therefore, we use averaged results obtained using simulation models with different values of random-numbers generators that take on values of natural numbers from 6 to 10. Simulation time is equal to $t = 5 \cdot 10^6$.

Let us introduce the designation: N is the average number of customers in the queueing system, and

$$\Delta_{k(Pot)} = \sum_{j=0}^{m+1} | p_{j(k)} - p_{j(Pot)} |, \quad \Delta_{(k,k-1)} = \sum_{j=0}^{m+1} | p_{j(k)} - p_{j(k-1)} |, \quad \Delta_{(6,k)} = \sum_{j=0}^{m+1} | p_{j(6)} - p_{j(k)} |,$$
$$\Delta_{k(sim)} = \sum_{j=0}^{m+1} | p_{j(k)} - p_{j(sim)} |, \quad p_{j(sim)} = \frac{1}{5} \sum_{i=6}^{10} p_{j(sim,i)}, \quad 0 \le j \le m+1, \quad 2 \le k \le 6. \qquad (14)$$

Here $p_{j(Pot)}$ and $p_{j(k)}$ are values of probabilities p_j obtained using the potential method and H_k-approximation respectively ($p_{j(Pot)} = p_j$); $p_{j(sim)}$ is the average value of probabilities $p_{j(sim,i)}$, obtained by means of the simulation model using the number i of random-numbers generator for $6 \le i \le 10$. Thus, the quantities $\Delta_{k(Pot)}$ and $\Delta_{k(sim)}$ are measures of deviations of the distributions $\{p_{j(k)}\}$ from distributions $\{p_{j(Pot)}\}$ and $\{p_{j(sim)}\}$ respectively, and the quantities $\Delta_{(k,k-1)}$ and $\Delta_{(6,k)}$ give an opportunity to estimate the deviation of distributions $\{p_{j(k)}\}$ from distributions $\{p_{j(k-1)}\}$ and $\{p_{j(6)}\}$ respectively.

In Tables 1 and 2 in the Appendix we present the results of the calculation of steady-state characteristics of the $M/G(n)/1/10$, $M/G(h)/1/15$ and $G/G(n)/1/10$, $G/G(h)/1/15$ systems respectively, with the considered uniform and gamma distributions. The values of $\Delta_{k(Pot)}$ and $\Delta_{(6,k)}$ $(2 \le k \le 5)$ in Table 1 are either identical or at least are numbers of the same order. This means that we can use values $\Delta_{(6,k)}$ to evaluate the accuracy of the approximation of the distribution $\{p_{j(k)}\}$ to the true $\{p_j\}$ for $2 \le k \le 5$. The values of deviations $\Delta_{k(Pot)}$ and $\Delta_{(6,k)}$ decrease with increasing order of H_k-distributions in approximations, as well as the values of $\Delta_{(k,k-1)}$, which decrease with an increase of k means that the values of distribution $\{p_{j(k)}\}$ with each step getting closer to a true distribution $\{p_j\}$.

Taking into account the values of deviations $\Delta_{6(Pot)}$ and $\Delta_{(6,5)}$, we can state the high accuracy of the approach of steady-state distributions $\{p_{j(k)}\}$ $(k = 5, 6)$ to the true distribution for the considered systems. The order of values of deviations $\Delta_{6(Pot)}$ and $\Delta_{(6,5)}$ varies from 10^{-11} to 10^{-6} and from 10^{-8} to 10^{-4} (in Table 2), respectively. For gamma distributions accuracy is higher than for uniform distributions. It is also higher for systems with the simplest input flow and for systems with queue length limit of $m = 10$ compared to considered systems with alternative values of the specified parameters.

Let V_1 and V_2 denote the variation coefficients of the times elapsed between two consecutive arrivals and the service times of a queueing system. For the queueing systems $U/U(n)/1/10$ and $U/U(h)/1/15$ with distributions having small coefficients of variation, namely, for the considered uniform distributions the condition $V_1+V_2<1.2$ is fulfilled, a part of the probabilities of distribution $\{p_j\}$ are less than 10^{-4}, that is the distribution has a "tail". As a consequence, the distributions $\{p_{j(2)}\}$ contain the pseudo-probabilities with negative values. Therefore, the results for these systems in the case $k = 2$ are not displayed in the tables.

Comparison of distributions $\{p_{j(k)}\}$ with distributions $\{p_{j(sim)}\}$ obtained using the simulation model, for $k \ge 3$ does not provide objective information about the approach of $\{p_{j(k)}\}$ to the true distribution $\{p_j\}$, since the minimum values of the deviation of the distribution $\{p_{j(sim)}\}$ from the true distribution have the order 10^{-4} or 10^{-3} and in most cases the results obtained by the method of H_k-approximation have higher accuracy for specified values of k.

4. Conclusions

This paper shows that the application of hyperexponential approximation of distributions of the interarrival time between two consecutive customers and the service times allow us to calculate steady-state probabilities of the non-Markovian single-channel queueing systems with service times changes depending of the number of customers in the system, with high accuracy (higher than in the case of using simulation models). We find these probabilities using solutions of a system of linear algebraic equations obtained by the method of fictitious phases.

To obtain parameters of H_k-approximation of a certain distribution, it is necessary to solve the system of equations of the moments method. For the values $V<1$ of the variation coefficient, some of the roots of this system are complex-valued or, having a sense of probabilities, go beyond the interval $[0, 1]$, but in most cases the final result is close to the desired distribution $\{p_j\}$.

Computing deviations $\Delta_{(k,k-1)}$ and $\Delta_{(6,k)}$ allows us to track the accuracy of approaching distributions $\{p_{j(k)}\}$ to the true distribution $\{p_j\}$ without the need to use simulation models.

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Appendix

Table 1. Results of the calculation of steady-state characteristics of the $M/G(n)/1/10$ and $M/G(h)/1/15$ systems with different G-distributions

G-distribution name	Characteristic name	Method of calculation and values of characteristics					
		Approximation using H_k					Potential method
		$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	
$U(n)$	N	6.0125	6.0137	6.0137	6.0137	6.0137	6.0137
	$\Delta_{k(Pot)}$	$8.83\cdot10^{-3}$	$4.14\cdot10^{-4}$	$2.26\cdot10^{-5}$	$1.53\cdot10^{-6}$	$1.17\cdot10^{-7}$	–
	$\Delta_{(k,k-1)}$	–	$8.46\cdot10^{-3}$	$4.02\cdot10^{-4}$	$2.23\cdot10^{-5}$	$1.60\cdot10^{-6}$	–
	$\Delta_{(6,k)}$	$8.83\cdot10^{-3}$	$4.14\cdot10^{-4}$	$2.27\cdot10^{-5}$	$1.60\cdot10^{-6}$	–	–
$\Gamma(n)$	N	6.0170	6.0171	6.0171	6.0171	6.0171	6.0171
	$\Delta_{k(Pot)}$	$6.81\cdot10^{-5}$	$7.50\cdot10^{-7}$	$2.02\cdot10^{-8}$	$9.11\cdot10^{-10}$	$5.05\cdot10^{-11}$	–
	$\Delta_{(k,k-1)}$	–	$6.84\cdot10^{-5}$	$7.50\cdot10^{-7}$	$2.01\cdot10^{-8}$	$8.92\cdot10^{-10}$	–
	$\Delta_{(6,k)}$	$6.81\cdot10^{-5}$	$7.50\cdot10^{-7}$	$2.02\cdot10^{-8}$	$8.92\cdot10^{-10}$	–	–
$U(h)$	N	8.8570	8.8322	8.8325	8.8325	8.8325	8.8325
	$\Delta_{k(Pot)}$	0.0123	$8.78\cdot10^{-4}$	$6.94\cdot10^{-5}$	$5.23\cdot10^{-6}$	$3.11\cdot10^{-7}$	–
	$\Delta_{(k,k-1)}$	–	0.0123	$8.92\cdot10^{-4}$	$7.18\cdot10^{-5}$	$5.44\cdot10^{-6}$	–
	$\Delta_{(6,k)}$	0.0123	$8.78\cdot10^{-4}$	$6.93\cdot10^{-5}$	$5.44\cdot10^{-6}$	–	–
$\Gamma(h)$	N	8.9647	8.9645	8.9645	8.9645	8.9645	8.9645
	$\Delta_{k(Pot)}$	$9.98\cdot10^{-5}$	$1.59\cdot10^{-6}$	$5.32\cdot10^{-8}$	$2.30\cdot10^{-9}$	$1.33\cdot10^{-10}$	–
	$\Delta_{(k,k-1)}$	–	$9.98\cdot10^{-5}$	$1.57\cdot10^{-6}$	$5.16\cdot10^{-8}$	$2.24\cdot10^{-9}$	–
	$\Delta_{(6,k)}$	$9.98\cdot10^{-5}$	$1.59\cdot10^{-6}$	$5.32\cdot10^{-8}$	$2.24\cdot10^{-9}$	–	–

Table 2. Results of the calculation of steady-state characteristics of the $G/G(n)/1/10$ and $G/G(h)/1/15$ systems with different G-distributions

G/G-distribution name	Characteristic name	Method of calculation and values of characteristics					
		Approximation using H_k					GPSS World
		$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	
$U/U(n)$	N	–	6.3314	6.3312	6.3312	6.3311	6.3310
	$\Delta_{k(sim)}$	–	0.0037	0.0007	0.0004	0.0004	–
	$\Delta_{(k,k-1)}$	–	–	$3.11\cdot10^{-3}$	$4.35\cdot10^{-4}$	$1.72\cdot10^{-4}$	–
	$\Delta_{(6,k)}$	–	$3.53\cdot10^{-3}$	$5.71\cdot10^{-4}$	$1.72\cdot10^{-4}$	–	–
$\Gamma/\Gamma(n)$	N	6.2536	6.2536	6.2536	6.2536	6.2536	6.2532
	$\Delta_{k(sim)}$	0.0003	0.0004	0.0004	0.0004	0.0004	–
	$\Delta_{(k,k-1)}$	–	$2.38\cdot10^{-4}$	$2.68\cdot10^{-6}$	$3.75\cdot10^{-8}$	$1.11\cdot10^{-8}$	–
	$\Delta_{(6,k)}$	$2.35\cdot10^{-4}$	$2.68\cdot10^{-6}$	$3.96\cdot10^{-8}$	$1.11\cdot10^{-8}$	–	–
$U/U(h)$	N	–	8.4555	8.4520	8.4516	8.4519	8.8325
	$\Delta_{k(sim)}$	–	0.0043	0.0013	0.0010	0.0009	–
	$\Delta_{(k,k-1)}$	–	–	$3.82\cdot10^{-3}$	$6.65\cdot10^{-4}$	$2.10\cdot10^{-4}$	–
	$\Delta_{(6,k)}$	–	$4.48\cdot10^{-3}$	$8.27\cdot10^{-4}$	$2.10\cdot10^{-4}$	–	–
$\Gamma/\Gamma(h)$	N	8.8398	8.8395	8.8395	8.8395	8.8395	8.8408
	$\Delta_{k(sim)}$	0.0006	0.0007	0.0007	0.0007	0.0007	–
	$\Delta_{(k,k-1)}$	–	$2.47\cdot10^{-4}$	$4.54\cdot10^{-6}$	$2.02\cdot10^{-7}$	$2.06\cdot10^{-8}$	–
	$\Delta_{(6,k)}$	$2.46\cdot10^{-4}$	$4.49\cdot10^{-6}$	$2.05\cdot10^{-7}$	$2.06\cdot10^{-8}$	–	–