sciendo Studia Geotechnica et Mechanica, 2023; 45(s1); 333–349

Special Issue 19th KKMGiIG Open Access

Artur Góral*, Marek Lefik , Marek Wojciechowski

Reduction of Numerical Model in Some Geotechnical Problems

https://doi.org/10.2478/sgem-2023-0016
received March 1, 2023; accepted August 18, 2023.

Abstract: The concept of equivalence of the realistic, initial reference model and the simplified, reduced model is proposed. In reduced models, the action of the soil on the structure is replaced by the action of a layer with prescribed properties, defined by a set of parameters. The main difficulty here is to find the parameter values required by the simplified theory. The subject of this work is to find the dependence of the parameters of the reduced model on the parameters of the full model, including realistic soil behavior, in order to ensure the equivalence of both models. We show the potential of the method by presenting two examples: Winkler and Pasternak's model of a plate on the ground. We assume that both models are equivalent if they give identical results (displacements) at a finite number of observation points. An artificial neural network (ANN) is built in order to approximate and record the dependence of the parameters of the reduced model (at the network output) from the parameters of the full model (given at the network input). The complex network acts as a formula that assigns the parameters of the reduced model to a realistic description of the soil structure that is used for finite element method (FEM) modeling. The formalism we propose is quite general and can be applied to many engineering problems. The presented procedure is entirely numerical; it allows to calculate the parameters of the reduced model without resorting to symbolic calculations or additional theoretical considerations.

Keywords: ANN in geotechnics; Winkler model; Pasternak model; reduced models.

***Corresponding author: Artur Góral,** Division of Geotechnics and Engineering Structures, Department of Concrete Structures Lodz University of Technology, Al. Politechniki 6, 93-590 Łódź, E-mail: artur.goral@dokt.p.lodz.pl
Marek Lefik, Marek Wojciechowski, Division of Geotechnics and Engineering Structures, Department of Concrete Structures Lodz University of Technology, Al. Politechniki 6, 93-590 Łódź

1 Introduction

Despite the enormous progress in modeling of soil–structure interaction by discretization of the structure and the real domain of soil using finite element method (FEM), discrete element method (DEM), or finite difference method (FDM), the reduced models are still very popular in engineering practice and scientific literature (e.g. [1, 2, 6, 11, 12, 25, 27, 35] and many other articles). There are various sources of complexity in numerical models. For example, for composites, the model that takes into account the microstructure at the whole macroscale by a simple fitting the FE mesh would lead to huge numerical task, impossible in practical applications. In this case the homogenous-like, theoretically homogenized medium plays a role of the reduced model. In the field of civil engineering, there is an obvious need of analysis of the performance of the entire structure before it is built, with FE mesh defined for all elements of the structure. In geotechnics, the entire structure includes the structure itself and a part of soil's domain supposed to be in interaction with the structure. Traditional reduced models in geotechnics operate as the action of some boundary constraints on the structure's boundary, replacing the soil's domain. These constraints must properly reflect the interaction between the structure and the soil that depends on both constitutive properties of the structure and those of the soil. For example, the action of the soil on the structure is replaced by the action of a linear or nonlinear elastic layer (e.g. Winkler's model, [34, 35]; Pasternak's two-parameters model [22]) or a layer with other properties, described by a number of parameters [11, 12]. In the oldest model, that of Winkler, the unique parameter, stiffness of the Winkler's spring, was several times generalized [1, 18, 20, 23, 24, 32, 33]. In fact, this one-parameter model may require a few additional characteristics if the spring rate is to be different for compression and extension, or for varying spring's stiffnesses for different stress levels. In the dynamic version of this model, it is necessary to enter the mass of the spring, parameters of viscous and dry friction damping. Finally, as many as six parameters can be needed. The formulae for the simplest set of parameters are basing on arbitrary assumptions, like for example the depth of the soil's active layer, and thus are always disputable or uncertain. The same can be stated for the two-parameters model of Pasternak [22, 29, 30]. In this model also the second parameter can be interpreted in two different manners: as a stiffness of a membrane-like interface or Kirchhoff stiffness of the subgrade. The situation becomes more difficult if the strength of the springs must be accounted for, like for piles or retaining walls. In this last case, we need the surrogate of two Rankine's coefficients and the Jaki's type-parameter [35]. Each of these parameters results from individual analysis requiring various "a priori" assumptions. In the complex soil condition this reasoning requires a lot of engineering intuition. A significant number of works are devoted to this issue. Recommendations regarding the values of these parameters can be found in each of the articles cited so far, some of which are devoted exclusively to this topic [14, 20, 21, 22, 23, 24, 25]. Winkler stiffness calculators have been developed in programs used for engineering calculations. The various hypotheses adopted in the cited analyses regarding the physical nature of the soil response, the depth to which the calculations are carried out, qualitative assumptions about the interaction of the soil with the structure lead to different estimates of the physical parameters of traditional models (which can be easily seen by analyzing any engineering problem).

The subject of this article is to find the dependence of the parameters of the reduced model on the parameters of the full model in an automatic manner, via a numerical procedure. First, we formalize the equivalence problem of two different models, the initial (model of reference) and the reduced one. The algorithm of the procedure leading to the definition of the equivalent model will be described in turn. To obtain the equivalent properties of the reduced model, the numerical tool involving a superposition of two artificial neural networks (ANN) is developed. The first neural network, called ANN_1, approximates a direct solution of the model of reference. Here, the input will be soil and structure characteristics and mechanistic properties of the materials. The output will contain some selected elements of solution of the reference problems (e.g. beam deflection or pile settlement). The second neural network in this superposition, namely ANN_2^{-1}, approximates a solution of inverse problem related to the reduced model. This network is inverse with respect to the ANN_2 - the network approximating the direct solution of the reduced problem. Please note, that the symbol denoting the inverse function has been adopted here: if the network ANN approximates the function f, then the network that approximates the function inverse to f is indicated as ANN^{-1}. The inverse network ANN_2^{-1} is trained by presenting the sets of values of the displacement in the observation points obtained as a solution of the reduced theory at the input of the ANN_2^{-1} and the trial parameters of the reduced model are elements of its output. This will be explained in detail in Section 3.3.

ANNs have been used in geotechnics for a long time. Most often, they are a tool for modeling constitutive relationships. The pioneering article here is the work [19], in which a classical neural network simulates the constitutive properties of a stratified soil medium based on tests with the falling weight deflectometer (FWD) dynamic test. A review of constitutive relationship modeling strategies is a very interesting issue, but it is not related to the topic of this article. It seems that the formulation of the reduced model was not the subject of ANN application. If it is assumed that the homogenized homogeneous medium is a model reduced in relation to the nonhomogeneous model, then the work [37] should be mentioned, but there the classical neural network serves only as a surrogate for the calculation of the effective modules of the homogenized medium performed classically with the use of FEM. This network is trained in such a way that the periodicity cell described with the necessary number of parameters is assigned effective modulus of elasticity. It seems that the concept of using a complex neural network to evaluate the parameters of a reduced model is a novelty.

Even though the work deals with issues related to soil–structure interaction, we are convinced that the general scheme of conduct may be useful in constructing simplified models in a more general context. This procedure seems to be novel and applicable in many domains of engineering.

2 Equivalence of the complex model (model of reference) and the reduced one

The main difficulty in the definition of the reduced model is finding values of the parameters required by the reduced theory. The values of these parameters are obviously functions of the engineering properties of the soil and structure, and are given by algebraic formulas only for elementary cases.

Under the term "initial model" or "reference model" we will understand a complex, realistic description of soil and the structure interaction, formulated originally in the form of a boundary value problem (BVP). Engineering approximations of the solutions of this BVP (with rare exceptions) can only be efficiently obtained by discretizing it, using the finite elements, the finite difference, or the discrete element method. We limit ourselves to the case, when the reference model is always FEM discretization built for the analyzed engineering problem. Therefore, the reduced model parameters will be selected to imitate the FEM results. In this article, a Coulomb–Mohr model implemented by FEM is assumed to serve as a constitutive model of soils. However, other constitutive models are also possible to be applicable within the proposed framework. The typical set of parameters (p_{CM1},..., p_{CMn}) appearing in the Coulomb–Mohr reference model of a layered soil would consist of 5 elements by layer:

$$\left(E_{eff}\right), \text{thickness of the layer}$$
$$(h))_1, \dots ,((c), (\varphi), (\psi), \left(E_{eff}\right), \{(h))_n\},$$

where n is the number of layers. This set of known parameters can be extended as far as the FE code permits and the BVP requires.

The “reduced model” is formalized to be a pair of two elements: differential equation that it defines and an ordered set of parameters (p_1,..., p_n) appearing in this equation. The physical range of the reduced model’s parameter values should always be known. From this range, the test values of the parameters used to train the ANN are selected. Only within this range the equivalence of the models will be valid. We assume that the physical sense of the equation and the parameters is clear. The assumption also is that this equation is solvable, with the solution being an ordered set of fields of displacement. This solution will be called “direct solution of the reduced problem”.

It is important to formulate the criterion of equivalence of the two models. We assume that the two models are equivalent if they give identical (up to given tolerance) results in finite number of observation points. We assume that the reference model and the reduced one are formulated in displacements; thus, these results are simply the displacements in the observation points.

The choice of number and the placement of the observation points is not trivial in general. In all examples presented in this article (there are always beams or plates on the elastic support) this choice is inspired by the well-known FWD test. In FWD test, the given mass is dropped down on the surface of the pavement and the deflections due to this impact are measured in maximum nine points in some prescribed distances. It is assumed that the set of this deflections fully characterize the pavement and the (possibly layered) subgrade, so the mechanical properties of them can be computed by back analysis. More about this method can be found in [7, 21, 24, 28]. All sets of observation points in this article coincide with the ones from FWD test.

The equivalence criterion assures that a complex, realistic behavior of the soil is taken into account in the frame of the reduced model.

3 Scheme of the procedure

3.1 Approximator of direct problems

The crucial point of the proposed algorithm is the use of ANN, which is trained with results of FE computations of reference model for many sets of model’s parameters. ANNs are operators that process an input dataset into an output set, taking expected values. ANN acts here as a shortcut that, in the simple form and in a very low computational cost, replaces the solution of the complex model. If there exist a function of many variables, the arguments of which are the parameters of the reference model and the values are the displacements in the observation points, the ANN is able to approximate this function. It is guaranteed by the theorem that proves that the ANN is a universal approximator of any function, functional, or operator [4, 9].

Figure 1 depicts the schematic for the feed-forward ANN, which is a general approximator of multivalued functions of multiple variables [8, 10]. If the input and output of the network are interpreted as an argument and image of a certain operator or function that the ANN approximates, the coincidence of the output values with the expected values should be satisfied for an infinite number of all possible sets of arguments at input, with the error (in the sense of minimal squares) remaining below an acceptable threshold. To achieve this, the network parameters (weights) are selected through successive corrections in an iterative process called “learning or network training”. The training of the network minimizes the difference between the ANN output signal and the target signal expected at the ANN output. The most common procedure of weight’s tuning is a minimization of the sum of squared errors calculated for all output neurons and for all input patterns. The back propagation error (BPE) procedure for such a minimization was used in this article. The complexity of the relation between parameters and the resulting set of displacements is

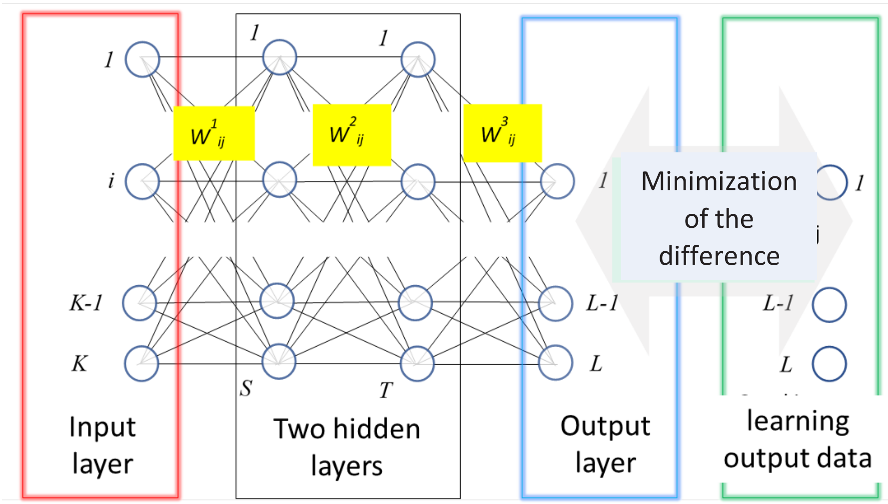


Figure 1: Scheme of the approximation of known learning output data (targets) by the transformation of the input signal on input layer by trained ANN. Segments between nodes are symbolic representations of weights (multipliers) that modify the nodal activities before attributing it to the next node.

adjusted by the number of layers and number of nodes in the layers that results with a set of approximation's parameters (weights and biases of the ANN), sufficiently rich. Detailed description of construction and operation of ANN have already been published in many textbooks or monographs [8, 9, 10, 13] and articles [14, 15, 16]. We refer to any of the above-listed manuals for details concerning the ANN.

In Figure 2, the approximation by ANN_1 of the FE solution for the initial model and the approximation by ANN_2 of the direct solution for the reduced model are shown in parallel.

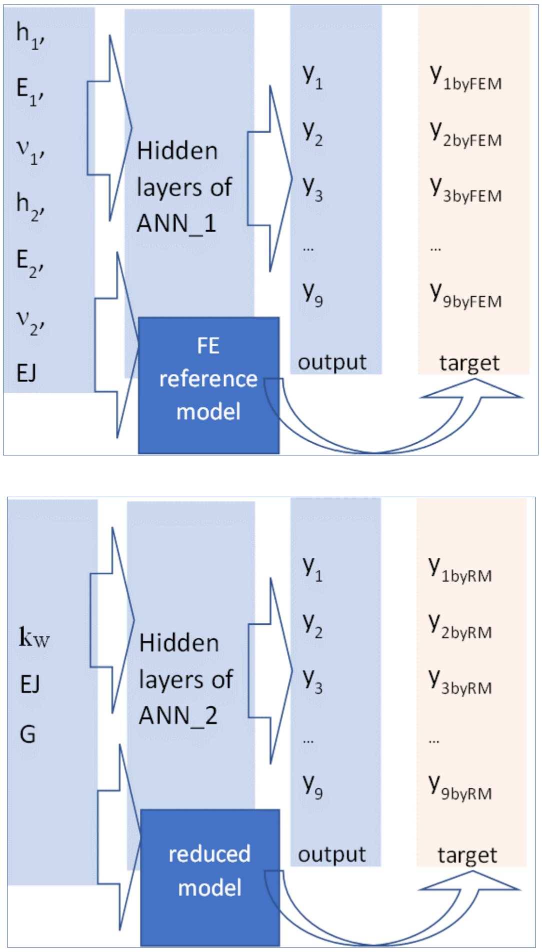
The training scheme of these networks is shown in the example: the reference model is a two-layer soil characterized by Young's modulus and Poisson's ratio with indices denoting the layer's number, and EJ – the stiffness of the beam resting on this substrate (Figure 2.a.). The trial values of these parameters are directed to the network input. The activation values of the network's output neurons are interpreted as the deflection values at the observation points. The network is trained using the deflection values at these points obtained by FEM calculations. The scheme of training a network ANN_2 that approximates the direct solution of the reduced model is presented in Figure 2.b. In this case, the three parameters of the model are the Winkler's ground stiffness, the beam stiffness, and the ground parameter related to the Pasternak model. Output neurons have the same interpretation as in Figure 2.a.

Both solutions are classical, direct solution of the BVP. In what follows, we will need a so-called inverse solution of the reduced model.

3.2 Approximator of inverse problem

The process of determining the parameters appearing in the "direct" problem through the interpretation of measurement data obtained as a result of the conducted physical experiment in the real world, consists of the formulation and solution of the inverse problem. Unknowns of the inverse problem are the parameters occurring in the direct problem and should be selected so that the solution obtained then is the closest one to the given, observed values. In our case we need the values of the parameters of the reduced model, such that for the given displacements in the observation points, the direct solution is the closest to them. The minimum of the distance is understood in terms of the sum of squared deviations from the given data in observation points. For any given direct problem (any typical BVP), there are various inverse problems, the mathematical nature of which is different than that of the direct one. In our previous articles [15, 16], we described an application of ANN for the solution of inverse problem.

It is to note, that in the algorithm we propose to construct the final, superposed neural network ANN_3 (defined later in Section 3.3) we execute at once two



a.

b.

Figure 2: Synthetic representation of the results of the model solution in observation points in the form of the ANN. Figure 2.a. represents the ANN approximation of the FE solution for the model of reference while in Figure 2.b. the same scheme applies for the reduced model.

trainings: the one illustrated in Figure 2.a. and the other described in Figure 3.

The most important advantage of this method is that it is not necessary even to formulate the inverse problem. It is enough to generate, using the well-known engineering tools, a set of direct solutions. These direct solutions are then used as an input of the ANN, the output of which are corresponding parameters known in training. The well-trained ANN in the "recall mode" attributes at output the correct parameters of the BVP for any direct solution presented at the input layer. In the case of inverse solution for reduced model, this procedure is illustrated in Figure 3. Unfortunately, the solution of inverse problem by well-trained ANN has also some disadvantages. The inverse solution is often nonunique and in this case the ANN requires many additional efforts to obtain all solutions.

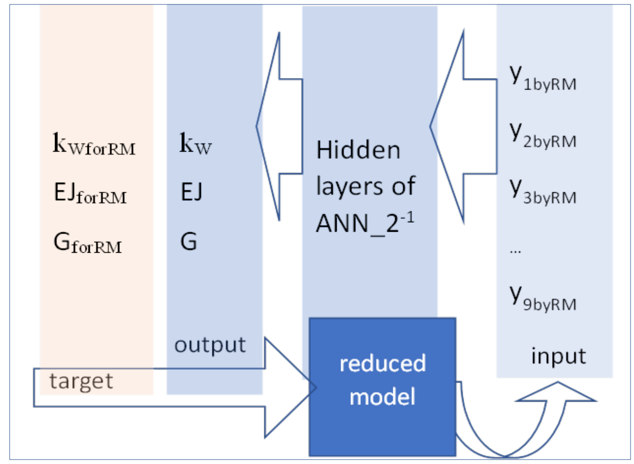


Figure 3: Scheme of numerical solution of inverse problem defined for the reduced model. In training, input to the "inverse ANN" are displacements in the observation points—the direct solutions of the reduced model while targets are the values of the parameters for which these displacements have been computed. In recall mode, the well-trained ANN responds with correct model parameters for the set of displacements obtained at the input.

3.3 Procedure description

Assuming, that for the reduced problem the inverse solution is unique, the parameters of the reduced model can be easily computed via composed ANN_3:

$$\text{ANN\_3} = \text{ANN\_1}@(\text{ANN\_2}^{-1}) \tag{1}$$

Symbol @ is used here to denote an action of an operator (ANN_1) on an object (ANN_2^{-1})).

The structure of the ANN_3 is illustrated in Figure 4. It is seen that the composed, complex ANN_3 that assigns the parameters of the reduced model to the parameters of the initial model is composed as follows:

- Input layer contains a number of neurons equal the number of parameters of the initial model.
- Block of hidden layers consists of:
  - all hidden layers from the direct ANN_1 (coding the direct solution of the initial problem);
  - additional layer of nine neurons (in general—as many as the number of control points is) with weights: at the layer input—taken from network ANN_1, at the layer output—taken from ANN_2^{-1};
  - all hidden layers from the inverse network ANN_2^{-1} (coding the inverse solution of the reduced problem).
- Output layer contains a number of neurons equal the number of parameters of the reduced model.

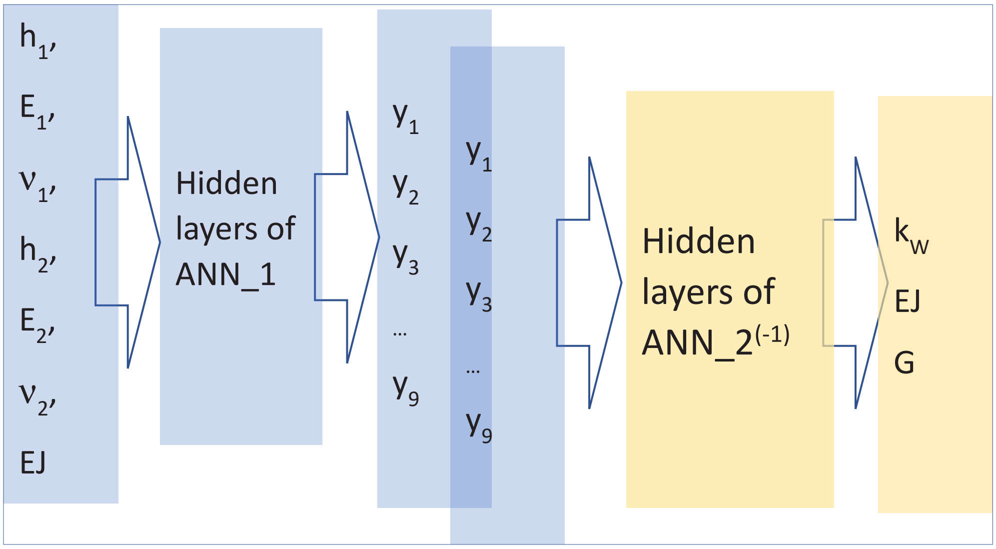


Figure 4: The complex ANN_1@(ANN_2^{-1}) acts as a formula that assigns the parameters of the reduced model to the realistic parametric descriptions of the problem that is used for its FE solution.

The following steps should be accomplished to define the complex network ANN_3:

– Execution of direct solutions of the initial problem, using FEM for representative set of input structural and soils parameters. In the results of this operation, an i^{ts} element of training patterns set for the network ANN_1 has the form:

$$\{\text{input, target}\}_i = \{\,(\text{combination of structural and soils parameters of BVP})_i,\ (y_1, y_2, .. y_i, .. y_9)_i\} \qquad (2)$$

– Execution of direct solutions of the reduced problem, for representative set of parameters of the reduced model. In the results of this operation an i^{ts} element of training patterns set for the network ANN_2^{-1} has the form:

$$\{\,(y_1, y_2, .. y_i, .. y_9)_i,\ (\text{corresponding combination of parameters of reduced model})_i\,\} \qquad (3)$$

– Training of the network ANN_1 with the set of training data (2);
– Training of the network ANN_2^{-1} with the set of training data (3);
– Creation of the resulting ANN_3 according to (1).

The application of complex network ANN_3 consists in introducing parameters of the initial, exact model of the problem at the network input. Then the parameters of the reduced problem will be calculated at the networks output.

4 Illustrative examples

In this section, two examples of application of the proposed procedure will be presented. The first one illustrates the application of the proposed procedure in the case of the Winkler model of the plate, in the second one we consider the beam on the two-parameter Pasternak soil model.

4.1 Stiffness coefficient for Winkler model of soil–structure interaction

The oldest reduced model is the model of a plate or beam resting on a Winkler elastic foundation. This model was formulated in 1876 by Winkler [34]. According to Winkler's formulation, the ground reaction vector at a certain point on the soil-building boundary is determined by the structure displacement vector at that point and is proportional to the beam deflection. The coefficient of proportionality—Winkler stiffness is denoted by k_w in the following. The assumptions of this model are well known and do not require discussion in this article.

4.1.1 Reference model

Figure 5 shows a diagram of the considered models. The concept of model corresponds to the finite concrete pavement on subgrade. Model A subgrade is represented by one layer of natural soil. Model B subgrade is represented by two layers of natural soil; each differ where thickness

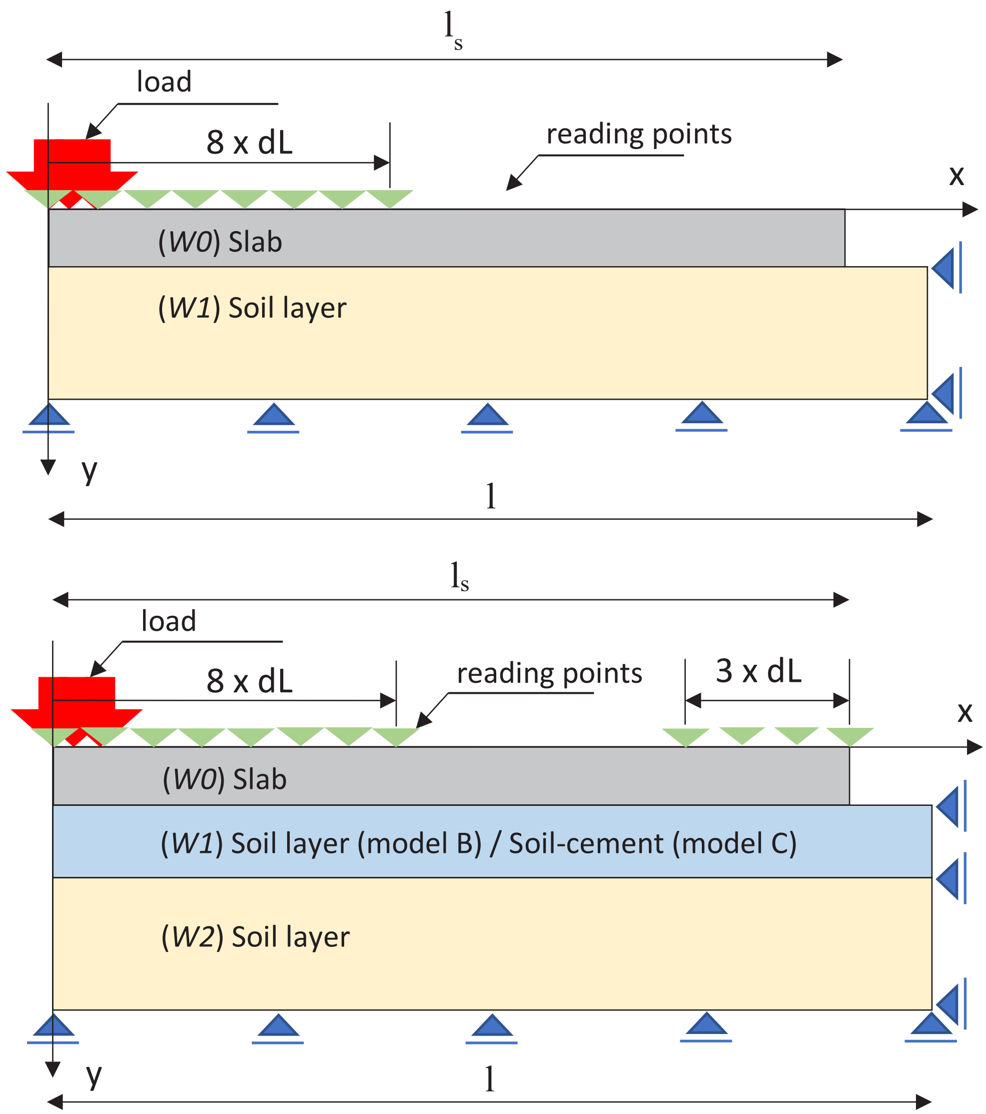


Figure 5: Diagram of the considered layered pavement models. From top: model A; model B, C.

of upper layer of natural soil change in a given range. Model C subgrade is represented from top: by a layer of stabilized soil (soil–cement mixture) and natural soil. Model A will be associated with the Winkler model of the plate. Model B and C will be associated with the Pasternak model of soil, respectively. The extract of data for the models are summarized in Table 1. Owing to the symmetry of the pavement model, only half of the geometry model was considered. Dimensions of the analyzed area of subgrade are l=8 m in horizontal direction and h depends on the model. The last layer of natural soil across models represents the last layer in elastic half-space theory and meets it requirements. This dimension ensures decay of general stresses according to Eurocode 7 guidelines, for example, recommends that integration be carried out to a depth where the effective stresses due to external loading are less than 20% of the effective primary stresses due to the soil's own weight σ_y. The horizontal dimension of top slab layer is l_s=4 m. This condition ensures proper decay of deflection along the horizontal direction of subgrade. As boundary condition horizontal displacements are blocked in the axis of symmetry and on the right edge of the model. At the bottom of the model, a full displacement lock is used. A static constant load is on the top surface with width of 15 cm and value of 700 kPa. The value and range of the load corresponds to the static vertical contact stresses apparent under the load plate of FWD vehicle. In engineering practice, the dynamic load from a falling mass is commonly replaced with an evenly distributed, equivalent static load [25]. Its realistic nonlinear course

Table 1: Summary of data for the applied model of the layered pavement.

	Model A			Model B			Model C		
Layer	h_j [cm]	v_j [-]	E_j [MPa]	h_j [cm]	v_j [-]	E_j [MPa]	h_j [cm]	v_j [-]	E_j [MPa]
W0	15÷25	0,2	32000	15÷25	0,2	32000	15÷25	0,2	32000
W1	300	0,25	50÷200	100÷200	0,25	50÷200	20	0,17	9100÷13850
W2	-	-	-	300	0,25	50÷200	380	0,25	50÷200

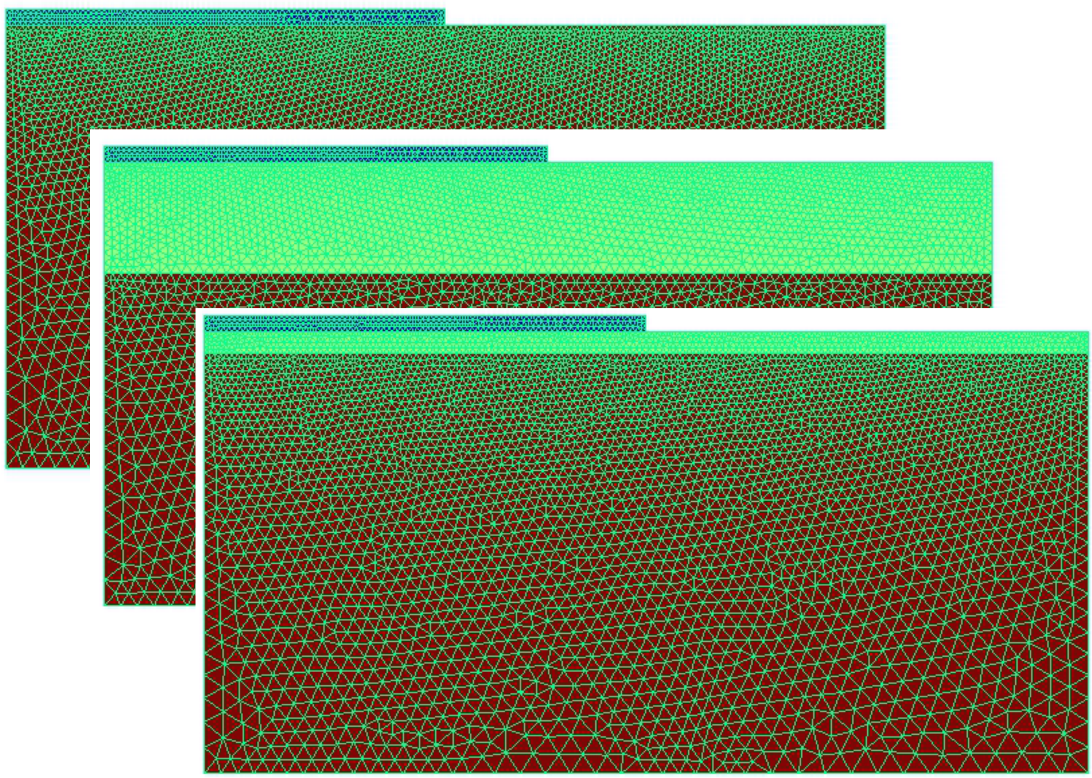


Figure 6: Examples of finite element meshes of the models.

leads to dynamic analysis, which is very interesting, but is beyond the scope of this article. Such an analysis was presented in a conference keynote presentation [17]. Standard surface quality testing using FWD test is only inspiration. FWD device is used to apply a dynamic load to the road, which simulates the pressure from the vehicle tire. Pavement deflection is measured using geophones. However, in this example, only the number and type of measurement data and their distance from the point where the load is applied will be taken from the real FWD test.

Figure 6 shows the finite element mesh adopted for the analyzed boundary problem. Six-node triangular elements were used. Edge size of equilateral triangle is 5 cm at the top surface of the model, evenly increased to 15 cm at the bottom surface. The FEM model and the calculations described below were performed using the *fempy* software [36].

For the models defined above, total 150 sets of data were randomly generated. Total thickness h_j for concrete slab or subgrade layer and deformation modules EJ for subgrade layers each time were random and it was assumed all other parameters of the task as invariant. The sampling was carried out using the Latin hypercube sampling (LHS) method. It is a statistical method of generating samples with a multivariate distribution [13]. It ensures that the random set of samples is representative for the given ranges of parameter variability (while the usual random sampling is just a collection of random numbers with no guarantees whatsoever). Then, calculations

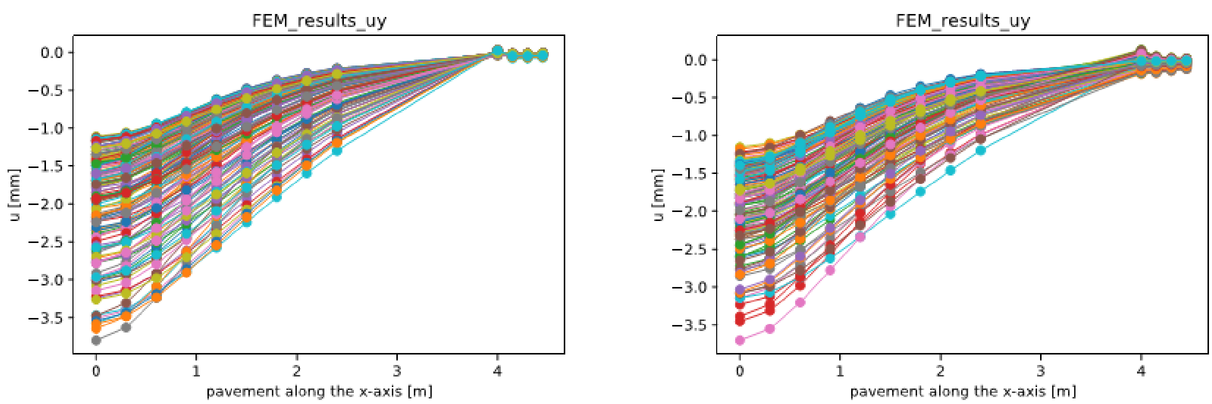


Figure 7: Deflections of the pavement for the selected sets of stiffnesses. From left: model A, model B. Deflections between (from 2.4 to 4.0) [m] are not recorded.

were performed for all generated samples, registering u_k deflections on the top surface of subgrade at 9 points 0, 30, 60, 90, 120, 150, 180, 210, and 240 cm from the plane of symmetry under concrete pavement. Next additional 4 deflections at 400, 415, 430, and 445 cm from the axis of symmetry on the top surface of subgrade. In these points, the influence of deflected plate on surrounding subgrade is registered for the Pasternak model. In this way, data was obtained in sequence ($E_j^{(p)}$, $u_k^{(p)}$ where $E_j^{(p)}$—thickness of slab layer or subgrade and elasticity modules of subgrade layers, $u_k^{(p)}$—deflections of top surface, $p = 1,\ldots, 150$, $j = 1, 2$ in model A, $j = 1, 2, 3$ in model B and C, $k = 1,\ldots, 13$.

4.1.2 Artificial Neural Networks that surrogate the results of the FE computations

Having the data from FE computations, the ANN that approximates the relation among the soil's parameters and the deflections of the plate in the nine observation points will be constructed. According to (1), this is the first component of the complex network, namely ANN_1. The structure of that network depends on the case of the reference model, A, B, or C, having from two to four, input nodes for Young's moduli of each of the layer and its widths (see Fig. 5). At the output we deal with nine or thirteen observable deflections in each case. In the case of 10 nodes in the hidden layer (model A) and 14 nodes (model B and C), the structure of the ANN_1 is similar for each of the analyzed examples. Network training was carried out on 125 sets of training data, the remaining 25 were used for network testing. The criterion for stopping the training is always the minimum of the root mean square error (RMSE) error for the testing set.

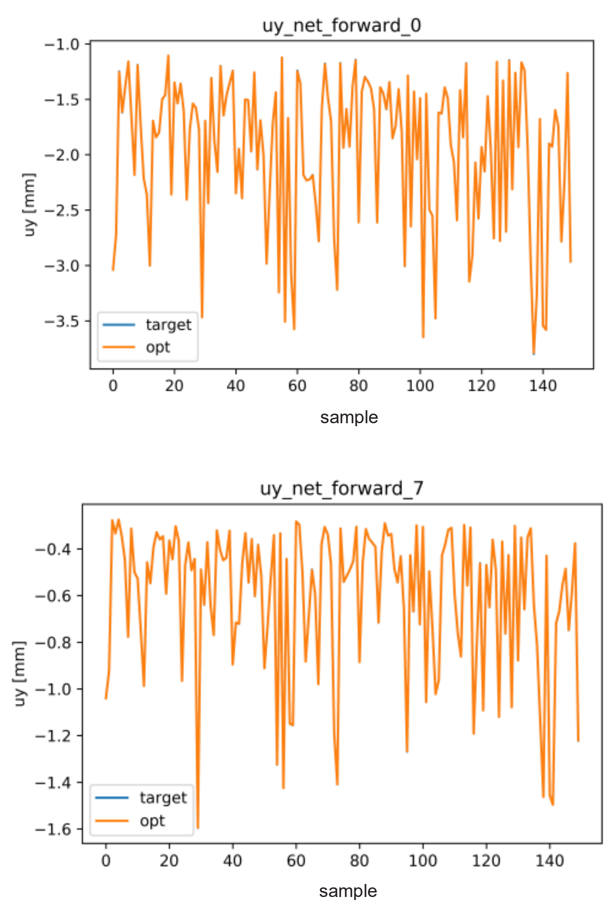


Figure 8: Direct N_k network learning results for two of all nine reading point of pavement deflection: 0 in title—1st reading point, 7 in title—8th reading point, target—reference deflection values, opt—deflection values identified by the trained ANN. On the horizontal axis—number of patterns. The target and output coincide (the blue line is not visible!)

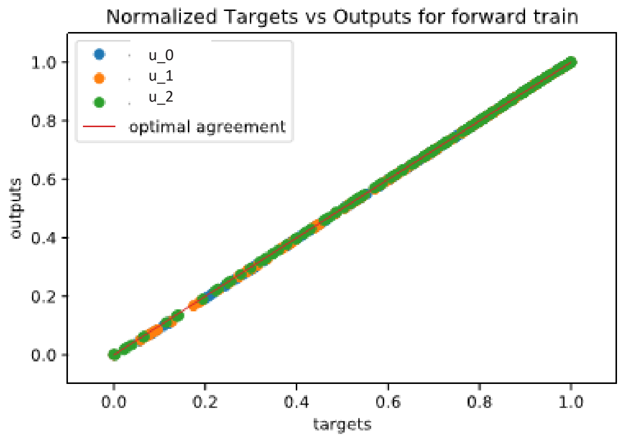


Figure 9: Normalized response of the network versus the pattern deflection values: u_0—results for 1st reading point, u_1—results for 2nd reading point, u_2—results for 3rd reading point. The points on the graph also contain test data.

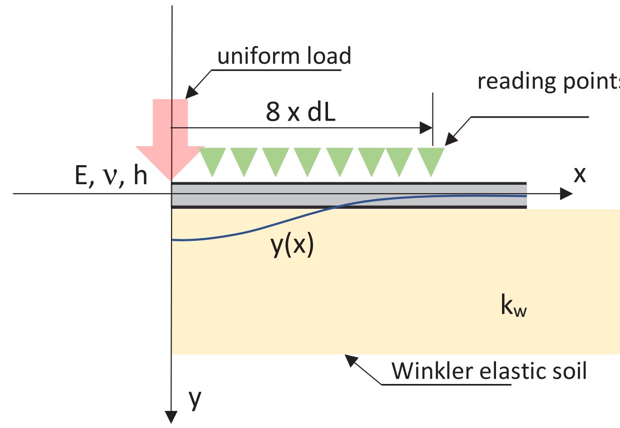


Figure 10: Scheme of the task of identifying the mechanical parameters of the surface and soil, description of symbols in the text.

The learning results are very good. A perfect match of the deflection values was achieved even though the ANNs have unfavorable architecture (more exits than inputs). It is worth noting that the quality of the fit on the test results does not differ from that on the training set, what indicates is that the ANN has been trained correctly.

4.1.3 Reduced model

The reduced model is here the classic Winkler model with bilateral soil reactions and the Euler–Bernoulli hypothesis for beam deflection was adopted. We consider a model of an elastic, homogeneous, finite beam band on a homogeneous Winkler elastic soil, loaded with a linear load on the plane of symmetry. In the inverse problem being solved, the thickness of the beam strip layer and the Winkler stiffness of the subsoil are identified.

In order to construct ANN_2[1], it is necessary to solve the inverse problem with respect to following boundary value problem (reduced model):

Find the deflection line y(x) for the finite plate strand:

$$0 \le x \le L,$$

where: L – half of the slab span [m].

The deflection line is a function of one variable that satisfies the equation:

$$\frac{d^4 y(x)}{dx^4} + 4\lambda^4 y(x) = \frac{p(x)}{EJ}, \quad \lambda = \sqrt[4]{\frac{k}{4EJ}} \tag{4.1}$$

The solution to the above equation is of the form:

$$w(x) = w_s(x) + e^{\delta x}(A \sin \delta x + B \cos \delta x) + e^{-\delta x}(C \sin \delta x + D \cos \delta x) \tag{4.2}$$

where: $w_s(x)$ *– is a particular integral of equation (4.1), the form of which depends on the form of the function p(x), while the constants A, B, C, and D are obtained from boundary conditions.*

We assume that *p(x)* is zero. In the reduced model, the load is concentrated at the symmetry plane.

Using suitable loops over the parameters, the Maple code computes these deflections for given x-coordinates. In order to solve the inverse problem, it is now necessary to construct a set of learning patterns (input patterns), allowing for the approximation of inverse relationship by ANN. At the input of this network, solutions calculated from the formula (4.2) should appear. The selected measurement points are, of course, geophones. The standard distance between geophones ΔL was assumed equal to 0.3 m. The output of the network should contain the parameters of the reduced problem for which the solution (4.2) was obtained. Natural questions arise here about how to choose the test parameters of the problem explicitly. There should be as few of them as possible so that the task of training the network is not time-consuming, and at the same time enough to make the approximation of the inverse relation accurate enough. In this example, there are two variables. The training set will contain pairs that match the notation:

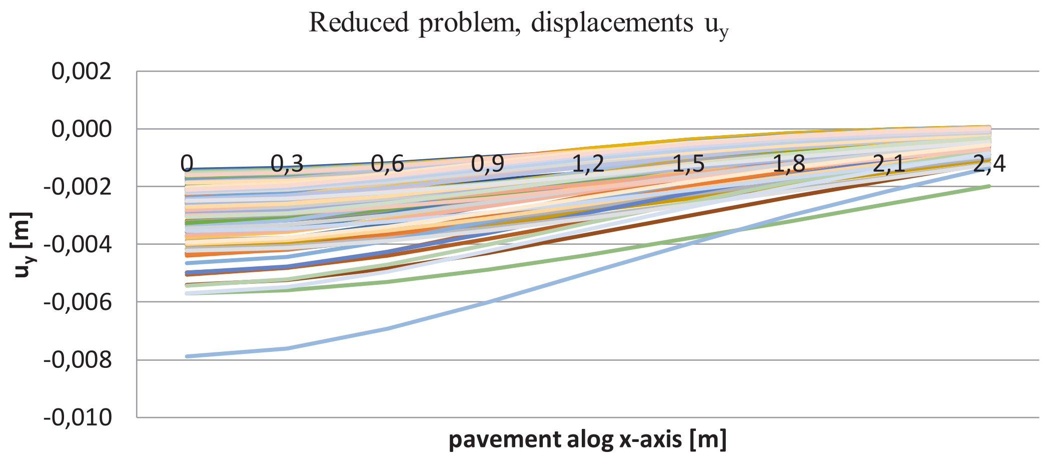


Figure 11: Trial deflections computed in the frame of the reduced model.

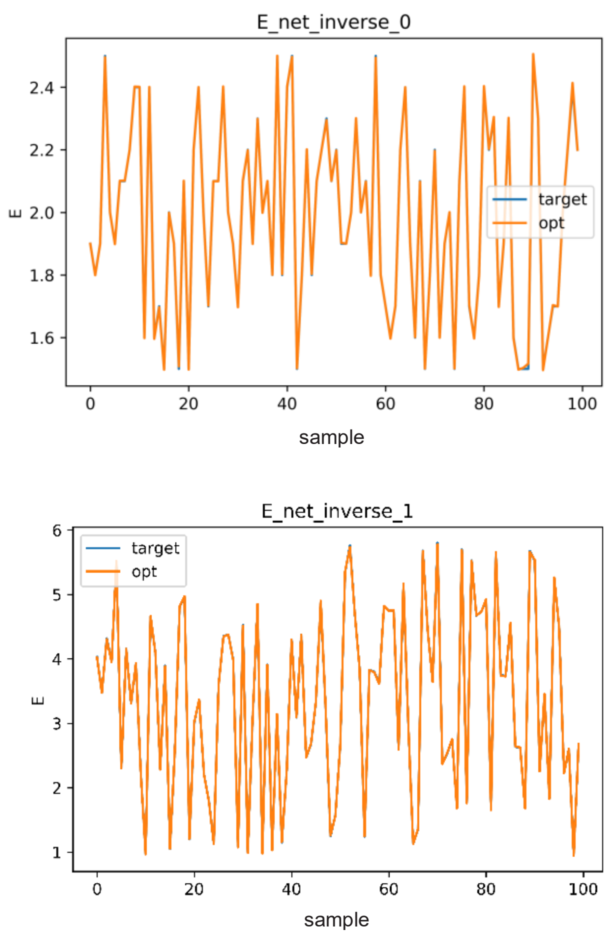


Figure 12: The results of learning the “inverse” network ANN_2^{-1} . Targets—are values of mechanical properties to learn, opt—values of mechanical properties identified by the trained net.

$$\{ (y_1, y_2, .. y_i, .. y_9)_i, \text{(corresponding combination of parameters of reduced model: } k, h)_i \} \qquad (5)$$

It should be noted that in any real technical problem related to the selected theoretical model, the range of identified quantities is usually well defined. For example, we can estimate with a high degree of certainty the range in which the Winkler stiffness or the value of the Pasternak constant, naturally related to the Kirchhoff constant of the soil medium, should lie. These assumptions are important because network training will only be performed for values in this range. It is generally believed that ANN extrapolates the results of the approximation very poorly beyond the interval for which it was trained. There are theoretical rules for sampling the parameter space, in the former example the LHS procedure has been used to this purpose. In this example, however, a simplified sampling method was used. All inputs were prepared using the Excell RAND() function, which returns evenly distributed random real numbers. In this simple example, we did not observe an unfavorable effect on the ANN’s ability to generalize. This means that the observed RMSE error on the test set decreased as fast as when using the LHS procedure in the previous example. In the solved example, the following limitations of the task parameters were adopted: the thickness of the beam is in the range [15 cm, 25 cm], and the stiffness of the Winkler subsoil is in the range [9 000 kPa, 60 000 kPa]. A simplified random method of sampling the space of these parameters, basing on Excel RAND() function was adopted. In Figure 11, a set of trial deflection of the beam-plate is shown.

4.1.4 Artificial Neural Networks that approximate the inverse relation for reduced model

According to (3), the network ANN_2[-1] has 9 input and 2 output neurons. The output parameters of pavement mechanical properties are *W0* for slab layer and *k* (Winkler modulus) for subgrade layer. The network architecture consists of five neurons in one hidden layer. Network training was carried out on 125 sets of training data, the remaining 25 were used for network testing.

Figures 12 and 13 show the results of the pattern fits and deflections determined by the trained ANN_2[-1] for all 150 samples.

As in the reference model (for direct dependence of the displacements on the soil's parameters), the quality of the approximation of the inverse problem for the reduced model is also excellent. Thus, the necessary condition for constructing the complex neural network ANN_3 (1) are fulfilled.

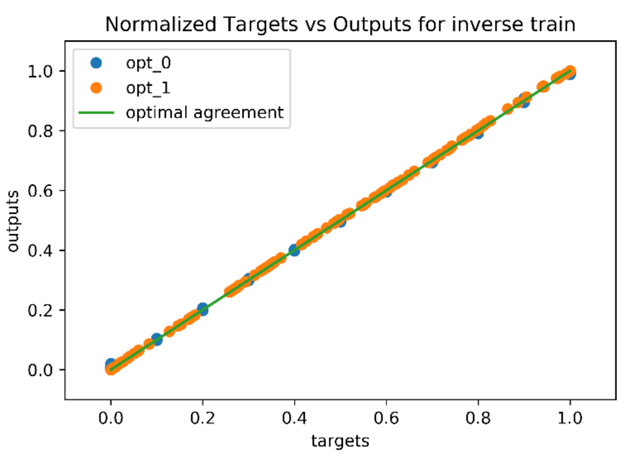


Figure 13: Normalized response of the "inverse" network versus expected output. The points on the graph also contain test data.

4.1.5 Quality of the solution

In this section, the ANN1 net was verified with three samples of input parameters. The output results will be used for ANN_2[-1] net verification. Figure 14 shows testing of the complex network.

It is seen from the Figure 14 that the deflections of the plate for reduced model and the reference model coincide.

4.2 Identification of two parameters of Pasternak model of soil–structure interaction

Pasternak's model (also called the two-parameter model), published in 1954 [16], should be mentioned as another, qualitatively different classical reduced model. Despite various physical interpretations, this model is based on introducing a second derivative of deflection into the beam equation. According to the well-known interpretation, the parameter *G* of the Pasternak's model is obtained by introducing a layer exhibiting an elastic shear response into the subsoil's idealization scheme. In a situation where an elastic layer of the Winkler type also appears in the soils scheme—such a model is associated with the name of Kerr. Another interpretation suggests that the coefficient accompanying the second derivative of the beam's deflection results from an action of the membrane underlying the beam or plate. While the first interpretation permits to guess the value of *G* as determined by Kirchhoff

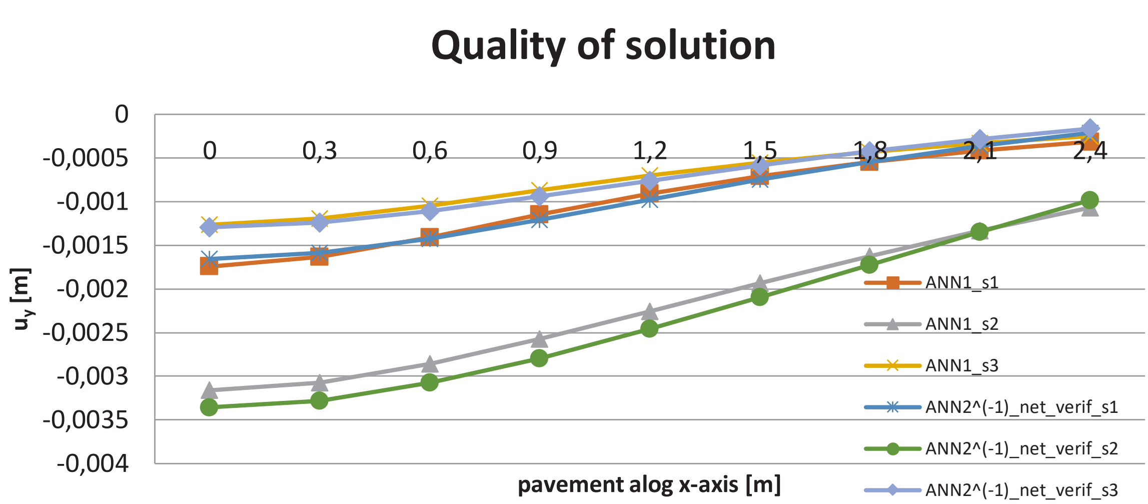


Figure 14: Reference (target) and ANN_3 identified (net_approx.) deflections of the pavement for three random test cases.

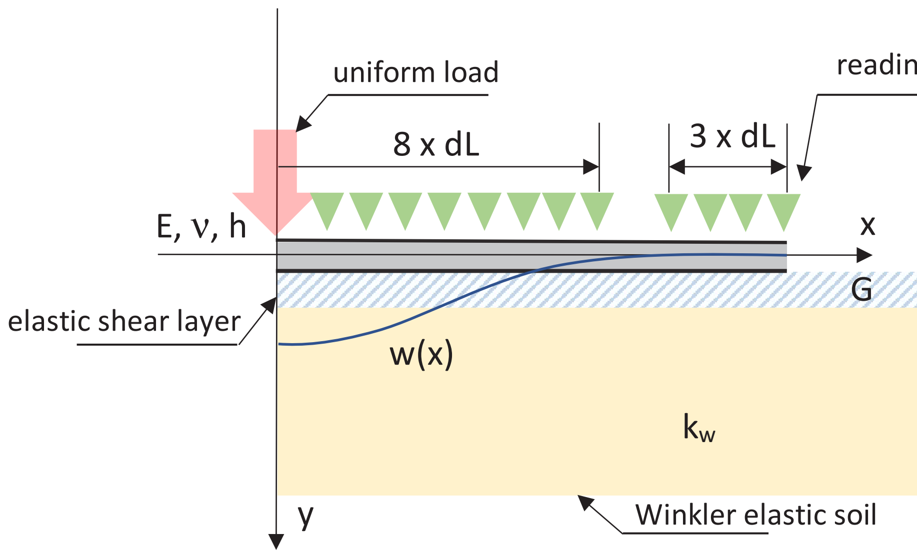


Figure 15: Idealization of the interaction of the pavement and soil, the positive direction of the axis and load as well as the assumed material parameters are marked. A detailed description of this scheme—in the text of the article.

modulus of the soil, the second one the value of *G* in a very speculative manner. This problem can be important in the modeling of soils reinforced with geogrids.

4.2.1 Reference model

The reference model is the same as for the Winkler model, described in Section 4.1.1. The only exception is that for the Pasternak model the last four points, external with respect to the plate strip, are taken into account. To keep the same number of displacements as in the previous example, we skip the four last observation points under the plate strip.

4.2.2 Reduced model

The scheme of the Pasternak's model is shown in Figure 15. We assume that the plate-strip deflection line *w(x)* is identical to the soil deflection line *u(x)* at all points under the beam. Apart from the beam, the soil deflection line satisfies other differential (6.2) and coincides with the plate-strip deflection.

Boundary problem written by equation (6), is associated with the scheme presented in Fig. 15 understood as the reduced model:

Find the deflection lines of the plate strip w(x) and the soil u(x) for x from the interval,

$$-L \le x \le L$$

The deflection line w(x) is a function of one variable that satisfies the equation:

$$\frac{d^4 w(x)}{dx^4} - \frac{G}{EJ}\frac{d^2 w(x)}{dx^2} + \frac{k_W}{EJ} w(x) = \frac{p(x)}{E\,J} \tag{6.1}$$

The soil deflection line u(x) is a function of one variable that satisfies the equation:

$$-G\frac{d^2 u(x)}{dx^2} + k_W u(x) = q(x) \tag{6.2}$$

These equations should be solved for a given load p(x) with boundary conditions assuming zeroing of u(x) at an infinite distance from the origin of the coordinate system.
The solution to the above equations is of the form:

$$w(x) = w_s(x) + e^{\delta x}(A \sin \delta x + B \cos \delta x) + e^{-\delta x}(C \sin \delta x + D \cos \delta x) \tag{6.3}$$

where: $w_s(x)$—is a particular integral of equation (6.1), the form of which depends on the form of the function p(x), while the constants A, B, C, and D are obtained from appropriate boundary conditions.

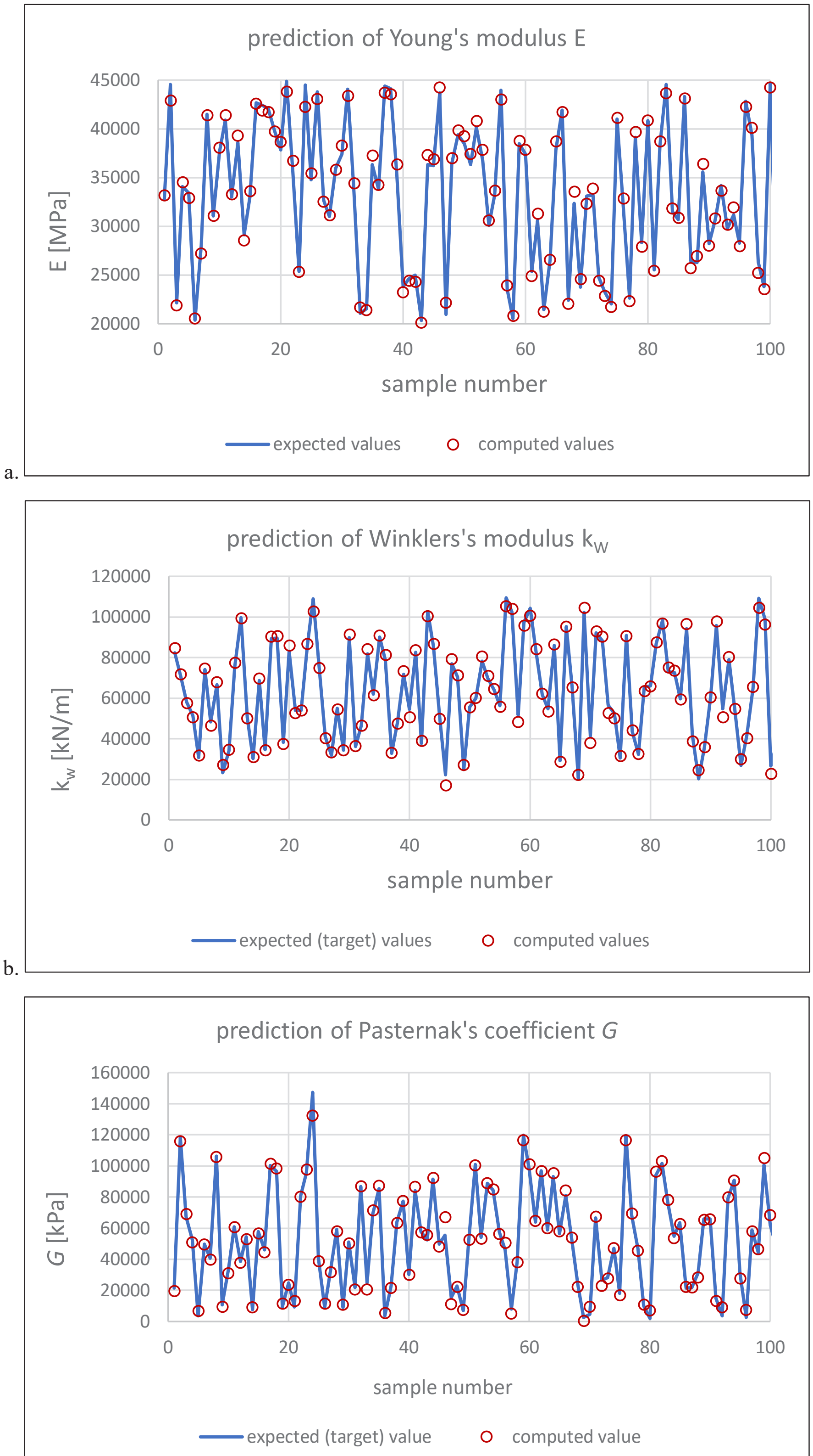


Figure 16: Comparison of the results of the network calculations with the known values of the material parameters of the Pasternak model: in Fig. 16.a, it is the Young's modulus of the stiffness of the E beam; in Figure 16.b it is the Winkler stiffness of the ground *kW*; in Figure 16.c it is Pasternak's constant *G*. These are the result of calculations in the reminder mode for a trained network with the structure of ANN_953, for 100 datasets, nine deflections each.

The adopted designations are defined as follows:

$$\alpha = \frac{1}{2}\sqrt{4\lambda^2 + G / EJ} \quad \beta = \frac{1}{2}\sqrt{4\lambda^2 - G / EJ} \quad \lambda = \sqrt[4]{\frac{k_W}{4EJ}} \qquad (7)$$

Two sets of 150 pairs (3) were generated for random values of all three parameters of the Pasternak model. The sampling ranges of the parameter values are as follows: E in the range from 20 000 to 45000 MPa, k_W in the range from 20 000 to 110 000 kPa.

4.2.3 Artificial Neural Networks that approximate the inverse relation for reduced model (the Pasternak one)

In this section, we discuss the training results of the network ANN_2[1]—the second part of the complex network ANN_3 (3). For a three-layer network with nine input neurons (for nine values of measured deflections), five neurons in the hidden layer and three output neurons, very good training results were obtained for all three parameters of the reduced model. The value of the correlation coefficient of the training set with the network response was at the level of 0.997, while the mean square error of the calculated parameter values was of the order of 0.02. The network was not optimized; however, it seems that the adopted ANN variant is very simple. You can certainly limit the number of neurons in the input layer. Nothing is lost in the precision of the approximation with six input neurons with the values of the first six measured deflections. Discussion regarding the necessary minimum number of experimental deflection data is omitted here. Figures 16.a, 16.b, and 16.c show a comparison of the network calculation results with the known values of material parameters E, G, and k_W. These comparisons are done in the recall mode for new, 200 verification datasets. None of these sequences were used during training. Only 100 such comparisons are presented, for the sake of clarity.

4.2.4 Quality of the reduced model

It should be stated that the obtained ANNs: ANN_1 and ANN_2[1] are very accurate approximation of the direct and inverse relation, i.e. the dependence of the model parameters on the parameters of the reference model. The parameters of Pasternak's model can therefore be easily calculated via ANN_3, without resorting to speculative theoretical formulas.

4.3 Possible applications and limitations of the proposed method of formulation of reduced models

The aim of this work is to present the method of determining the parameters of the reduced model rather than to present spectacular engineering applications. Therefore, the presented examples have the value of illustrating the proposed procedure. However, they illustrate three important elements of this procedure. Its success depends on the ability of the ANN_1 network to synthetically replace the FEM solution at several representative points, on the possibility of well-trained ANN_2[1] inverse network. The third, most important advantage of this procedure is the fact that it takes into account a very wide context of real ground conditions to determine the parameters of the reduced model. The examples show that the ANN_1 network is a very good surrogate for the FEN solution in the scope we need. It was also shown that the ANN_2[1] inverse network can be trained very well. In order to create the ANN_3 network, which generates the constitutive parameters of the reduced model as an output, it should be trained for a situation in which the FEM model is described by a large number of parameters describing, for example, a system of layers not parallel to the surface. This was not done in this article, but we are sure that ANN_1, even for an input composed of a large number of neurons, is a good approximation of the "straight" FEM solution at selected points [37]. The biggest limitation of the method is the ability to precisely train the inverse network. The inverse relationship is often not a one-to-one function and problems may arise with effective training of the network. We are convinced, however, that in a situation where a properly constructed reduced model is described with few parameters and each of these parameters characterizes a different aspect of the soil–structure interaction—the inverse network will always be a good approximator of the inverse relationship.

In the light of what was written above, the excellent agreement of the FEM solution with the approximate solution obtained using the reduced model was obtained only in a simple case built to illustrate the presented method. We are well aware of the fact that in the case of the subgrade under the slab, a model with variable Winkler stiffness should be adopted [20, 35]. We are convinced that also much more difficult and important problems can be solved using the developed method. For example, pile-bearing capacity calculation methods that require the use of transfer functions describing pile–soil interactions are, in our understanding, reduced models. The parameters of the transfer functions depend on the properties and state of the soil in the vicinity of the pile lateral surface. These parameters can be obtained as the output values of the ANN_3 network, the input of which should be given the characteristics of the soils in the vicinity of the node where the transfer function is to be defined. Unfortunately, in this case, even the reference model is not easy to construct.

Such research, as well as the examples in this article, is the content of the doctoral dissertation being prepared by the first author of this article. The concept of the method was proposed by the second author, the third author is the creator of the original FEM code, which, thanks to the attached scripts, can be used as a training data generator [36].

5 Conclusions

In this article, the new method of identification of the parameters of reduced model has been established and exemplified for two classical reduced models of soil–structure interactions in soil mechanics: model of Winkler and model of Pasternak.

The method assures that the parameters of the reduced model permit to generate the displacement (solutions of the reduced model) are very close to the solution of the reference model in selected points.

The method is fully automatic, the correspondence between two models (initial and reduced) is assured without resorting to speculative theoretical consideration related to the formulation of the reduced model.

The resulting artificial neural network ANN_3 (see (3)) can be treated as a symbolic formula. Since its explicit form is too complex to be printed, the Excel formula that simulates the action of ANN_3 in recall mode will be soon available (please mail to the first author).

The authors continue their work in order to apply the procedure presented in this article to simplified nonlinear models that allow modeling materials behavior in limit states.

Ancknowlegment

This paper has been completed while the first author was the Doctoral Candidate in the Interdisciplinary Doctoral School at the Lodz University of Technology

References

[1] Ataman, M., Zbiciak A., Szcześniak, W.: Drgania własne płyty cienkiej na podłożu Kerra – Jemielity, https://www.researchgate.net/publication/263414439, 2009
[2] Ataman, M.: Inercyjne, Sprężyste modele podłoża odkształcalnego w zadaniach z mechaniki konstrukcji, *Logistyka-nauka*, 6, s. 1515, 2014.
[3] Boso, D.P., Lefik, M. and Schrefler B. A.: Generalised self consistent homogenisation as an inverse problem. *Z. Angew. Math. Mech*., 90(10-11), pp. 847 – 860, 2010.
[4] Chen T., Chen H.: Universal Approximation to Non-linear Operators by Neural Networks with Arbitrary Activation Functions and Its Application to Dynamical Systems, *IEEE Trans. on Neural Networks*, Vol.6, No 4, 911-917, 1995.
[5] Favre, H.: Sur une loi régissant la déformation d'un sol horizontal sous l'action d'une charge répartie à la surface. *Comptes Rendus, Acad. Sci.,* 252, 1961, 2988–2990.
[6] Ghavanloo, E., Daneshmand, F. and Rafiei, M.: Vibration and instability analysis of carbon nanotubes conveying fluid and resting on a linear viscoelastic Winkler foundation, *Physica E,* vol. 42, no. 9, pp. 2218–2224, 2010.
[7] Gradkowski K.: Nośność podłoży nawierzchni dróg samochodowych wzmacnianych geosyntetykami. Badania doświadczalne, Oficyna Wydawnicza Politechniki Warszawskiej, Warszawa, 2008
[8] Hertz, J., A. Krogh., A., Palmer G.R.: Introduction to The Theory of Neural Computation, Lecture Notes Volume I, Santa Fe Institute Studies in the sciences of Complexity, Addison-Wesley, 1991.
[9] Hornik K., Stinchcombe M., White H.: Universal approximation of an unknown mapping and its derivatives using multilayered feedforward networks, Neural Networks, vol. 3, 551-560, 1990.
[10] Hou Z.Q. and Lucifredi, A.: On the Use of Neural Networks for Identification of Linear and Nonlinear Systems, Meccanica 30, 377-388, 1995.
[11] Kerr, A.D.: Elastic and viscoelastic foundation models, *J. Appl. Mech.,* 31, 1964, pp.491-498.
[12] Kerr, A.D.: On the formal development of elastic foundation models, *Ing. Arch,* 54, 1984, pp. 455–464. https://doi.org/10.1007/BF00537376
[13] Lefik M.: Przykłady rozwiązań zagadnienia odwrotnego metodą aproksymacji relacji odwrotnej, Wydawnictwo Politechniki Łódzkiej, 2016.
[14] Lefik M.: Rozwiązanie zagadnienia odwrotnego przy pomocy sztucznych sieci neuronowych. *Zastosowanie metod numerycznych w analizie wybranych zagadnień geotechniki i geotechniki środowiska.* Wydawnictwo Politechniki Łódzkiej, 2021.
[15] Lefik M. and Boso D. P.: Identification of parameters of adsorption by approximation of inverse relation and using artificial neural networks, Proceedings Solmech2016 Conference http://solmech2016.ippt.pan.pl /Abstracts/0177.pdf
[16] Lefik M. and Boso D. P.: Inverse problem - soft solution, in Bytes and Science, edited by G. Zavarise and D. P. Boso (© CIMNE, Barcelona, Spain 2012), pp. 10-34.
[17] M. Lefik, M. Wojciechowski, M.: Examples of inverse problem in geotechnics: comparative analysis of various strategies of

solutions involving artificial neural networks, International Symposium on Geomechanics and Applications for Sustainable Development, SIPS2019, Cyprus. October 2019
[18] Levinson, M.: On the linear, isotropic, elastic foundation of grade 4. *Mech. Res. Com.* 6, 1979, pp. 369–378.
[19] Meier R. W.: Backcalculation of Flexible Pavement Moduli from Falling Weight Deflectometer Data Using Artificial Neural Networks, Report prepared for U.S. Army Corps of Engineers, technical report GL-95-3, Washington DC, 1995.
[20] Meyer, Z.: Propozycja określania współczynnika podatności podłoża przy projektowaniu płyt fundamentowych, https://docplayer.pl/52076096-Propozycja-okreslania-wspolczynnika-podatnosci-podloza-przy-projektowaniu-plyt-fundamentowych.html
[21] Nita P.: Budowa i utrzymanie nawierzchni lotniskowych. WKŁ. Warszawa 2008
[22] Pasternak, P.L.: On a new method of analysis of an elastic foundation by means of two foundation constants, Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstuve i Arkhitekture, Moscow, 1954, (in Russian).
[23] Reissner, E.: A note on deflections of plates on a viscoelastic foundation, *J. Appl. Mech., ASME,* 25, 1958, pp.144-145.
[24] Ruta P., Krawczyk B., Szydło A.: Identification of pavement elastic moduli by means of impact test, *Engineering Structures*, 100, 2015, ss. 201-211.
[25] Sadrekarimi J., Akbarzad M.: Comparative study of methods of determination of coefficient of subgrade reaction. *The Electronic Journal of Geotechnical Engineering*, tom 14/2009, zeszyt E, 2009.
[26] Suchart Limkatanyu, Woraphot Prachasaree, Nattapong Damrongwiriyan, Minho Kwon, andWooyoung Jung.: Exact Stiffness for Beams on Kerr-Type Foundation: The Virtual Force Approach, *Journal of Applied Mathematics,* Article ID 626287, p. 13, 2013, http://dx.doi.org/10.1155/2013/626287.
[27] Szeptyński, P.: Praktyczne zagadnienia interakcji budowli z podłożem, *Inżynieria i Budownictwo* 72, nr 7 ss. 384-389, 2016.
[28] Szydło A.: Nawierzchnie drogowe z betonu cementowego. Wydawnictwo Polski Cement, Kraków 2004.
[29] Świniarski J.: Ocena podatności podłoża przy wymiarowaniu płyt fundamentowych. *Inżynieria Morska i Geotechnika*, nr 5/2003.
[30] Tanahashi, H.: Two-dimensional analysis of ground surface displacements using Pasternak Model, *J. Struct. Constr. Engrg., AIJ*, 530, 2000, pp.85-91
[31] Terzaghi K.: Evaluation of coefficients of subrade reaction, Inst. of Civil. Eng., London 1956;
[32] Vallabhan, C.V.G., and Das, Y.C.: Parametric study of beams on elastic foundations, *J. Engrg. Mech., ASCE,* 114(12), pp.2072-2082, 1988
[33] Vlasov, V.Z. and Leont'ev, N.N.: Beams, plates and shells on elastic foundations, Israel Program for Scientific Translations, Jerusalem (Translated from Russian; original Russian version published in 1960). https://archive.org/details/nasa_techdoc_19670004909/page/n43/mode/2up.
[34] Winkler, E.: Die Lehre von der Elastizität und Festigkeit, H. Dominicus, Prague, 1867 (in German).
[35] Wojciechowski, M.: Posadowienie na płycie fundamentowej – ustalanie współczynniów podatności podłoża, Wydawnictwo Politechniki Łódzkiej, 2016.
[36] Wojciechowski M.: Fempy - finite element method in python, https://github.com/mrkwjc/fempy, http://fempy.org (access: 15, 03, 2021).
[37] Wojciechowski, M., Lefik, M., D. Boso.: Inverse problems in the light of homogenisation methods: identification