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RANDOM WALK – FUZZY ASPECTS

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Abstract. Some beautiful and powerful mathematical ideas are hard to present to students because of the involved abstract language (notation, definitions, theorems, proofs, formulas) and lack of time. Animation and "mathematical experiments" provide a remedy. In the field of stochastics, the Galton board experiment presents several fundamental stochastic notions: a random event, independent random events, the binomial distribution, limit distribution, normal distribution, interpretation of probability, and leads to their better understanding. Random walk is a natural generalization of the Galton board. We use random walks as a motivation and presentation of basic principles of fuzzy random events and fuzzy probability. Fuzzy mathematics and fuzzy logic generalize classical (Boolean) mathematics and logic, reflect everyday experience and decision making and have broader applications. Experimenting with random walks also sheds light on the transition from classical to fuzzy probability.

1. Introduction

Probability and statistics are considered to be important and useful components of the general mathematical education. Unfortunately, due to the abstract language of mathematics (notation, definitions, theorems, proofs, formulas) and lack of time it is usually hard to present students with some beautiful and powerful stochastic ideas. A possible remedy is to work with animation and "stochastic experiments".

Internet sources provide numerous and detailed information about experiments with the Galton board, see for example:

http://en.wikipedia.org/wiki/Bean_machine,

http://animation.yihui.name/prob:bean_machi,

http://www.jcu.edu/math/isep/Quincunx/Quincunx.html,

http://mathworld.wolfram.com/GaltonBoard.html.

Experimenting with the Galton board enables us to present several fundamental stochastic notions and laws, for example, a random event, independent random events, the binomial, normal, and limit distributions, interpretation of probability and so on, in a natural way, and leads to their better understanding. Our goal is to study the Galton board and some of its generalizations from the viewpoint of random walks and fuzzy probability. We believe that our approach provides a vehicle to convey to students basic ideas of both classical and nonclassical fields of stochastics and sheds some light on the transition from classical to fuzzy probability (cf. [1], [4]). The latter one reflects everyday experience and decision making and has broader applications.

In this paper we concentrate on finite random walks, but the infinite ones constitute another important topic to be included into "stochastic experiments", see for example

http://en.wikipedia.org/wiki/Random_walk.

Here we would like to point out a surprising fact that even a very small change of the probability $p(l) = p(r) = 1/2$ (going left or right) in the symmetric one-dimensional random walk to $p(l) = 1/2 + 0.01$, $p(r) = 1/2 - 0.01$ leads to a very nonsymmetric behaviour, see

http://artax.karlin.mff.cuni.cz/ macim1am/pub/antoch/pdf.

The next steps in animation should be relationships between the Galton board and the Moivre-Laplace limit theorems leading to the normal distribution, various laws of large numbers, and limit theorems. "But that's another story", as Rudyard Kipling would say.

2. The Galton board

According to http://mathworld.wolfram.com/GaltonBoard.html *the Galton board, also known as a quincunx or bean machine, is a device for statistical experiments named after English scientist Sir Francis Galton. It consists of an upright board with evenly spaced nails (or pegs) driven into its upper half, where the nails are arranged in staggered order, and a lower half divided into a number of evenly-spaced rectangular bins. The front of the device is covered with a glass cover to allow viewing of both nails and slots. In the middle of the upper edge, there is a funnel into which balls can be poured, where the diameter of the balls must be much smaller than the distance between the nails.*

The funnel is located precisely above the central nail of the second row so that each ball, if perfectly centered, would fall vertically and directly onto the uppermost point of this nail's surface. Each ball follows a path starting at the center nail, then bounces either right or left and so on, and ultimately lands in one of the bins.

Schematically, it can be visualized via a graph starting with one vertex v_{00} on the level zero, continuing with two vertices v_{10}, v_{11} on the level one and so on, ending with $N+1$ vertices (bins) $v_{N0}, v_{N1}, \ldots, v_{NN}$ on the level N, see Figure 1.

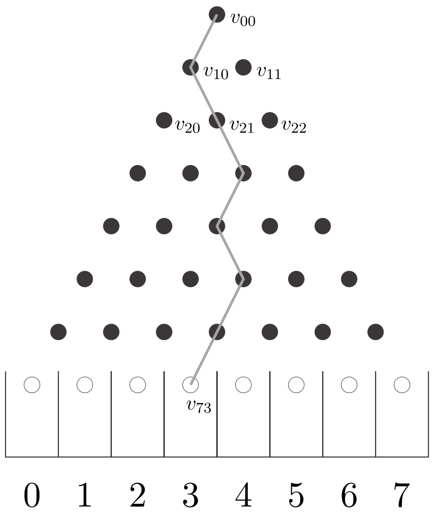

Figure 1: Visualization of of a ball path via a graph

The Galton board is connected to the binomial distribution in the following way. Each time a ball hits one of the nails, it can bounce left (or right) with some probability $p(l)$ (right with the probability $p(r) = 1 - p(l)$). For symmetrically placed nails, balls will bounce left or right with equal probability, so $p(l) = p(r) = 1/2$. The probability that a ball (after hitting $N-1$ nails) ends in the nth bin, $n = 1, 2, \ldots, N$, is

$$P(n) = \binom{N}{n} p(l)^n p(r)^{N-n}.$$

Even a novice in probability should be able to appreciate how experimenting with the Galton board is related to random walks. Indeed, the path of a ball can be viewed as a random walk on the graph of Galton board. The paths constitute a discrete probability space and we offer an alternative way how to calculate *the probability of a path.* To this end, we recall the notion of a conditional probability.

Let us repeat some random experiment N times independently (the outcomes do not depend on the previous experiments). We consider two events A and B, n_B is the number of occurrence of B (we assume $n_B > 0$), and $n_{A\cap B}$ is the number of their joint occurrences (we count only occurrences of A when B has occurred). Then

$$\frac{n_{A\cap B}}{n_B} = \frac{n_{A\cap B}/N}{n_B/N}$$

means that the *conditional probability* of A given B should be defined as

$$P(A|B) = \frac{P(A\cap B)}{P(B)}.$$

Of course, we assume that $P(B) > 0$ and the definition is based on the *interpretation of probability via relative frequency.*
QUESTION: What is the probability of the path $(v_{00}, v_{10}, \ldots, v_{N0})$?

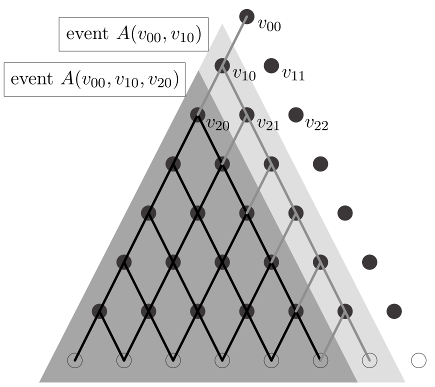

Figure 2: Events as sets of paths

Denote by $A(v_{00}, v_{10})$ the set of all paths going trough v_{10}, $A(v_{00}, v_{11})$ the set of all paths going trough v_{11}, and $A(v_{00}, v_{10}, v_{20})$ the set of all paths going trough v_{10} and then through v_{20}.

It is easy to see (Figure 2) that $P\big(A(v_{00}, v_{10})\big) = P\big(A(v_{00}, v_{11})\big) = 1/2$ and $A(v_{00}, v_{10}, v_{20}) \subset A(v_{00}, v_{10})$. Further,

$$P\big(A(v_{00}, v_{10}, v_{20})|A(v_{00}, v_{10})\big) = P\big(A(v_{00}, v_{10}, v_{21})|A(v_{00}, v_{10})\big) = 1/2.$$

Thus

$$P\big(A(v_{00}, v_{10}, v_{20})|A(v_{00}, v_{10})\big) = \frac{P\big(A(v_{00}, v_{10}, v_{20}) \cap A(v_{00}, v_{10})\big)}{P\big(A(v_{00}, v_{10})\big)}$$

implies that $P\big(A(v_{00}, v_{10}, v_{20})\big) = (1/2)^2$.

Repeating the reasoning, we arrive to the conclusion that the probability of the path $(v_{00}, v_{10}, \ldots, v_{N0})$ equals $(1/2)^N$. Analogously, we can calculate the probability of any other path: it is equal to $(1/2)^N$. So, we ended up with a classical (discrete) probability space the elementary events of which are exactly the paths of balls in the Galton experiment.

3. Walking on the Galton board

The random walk on *the classical Galton board* is rather simple. Each vertex v_{Nk}, $k = 0, 1, 2, \ldots, N$, is absorbing, other vertices are not. From any other vertex a ball can proceed to two adjacent vertices on the next level with equal probability 1/2. We can study "two step walks" or "k step walks" and ask about the corresponding conditional probabilities. In such cases *combinatorics suffices.* On a more complicated board, a ball at each vertex can proceed to more than two points on the next level and the conditional probabilities can vary from one level to the next level, and then *combinatorial methods do not suffice.* We believe (see the next section) that fuzzy probability offers a natural approach to such random experiments.

The original Galton board can be used for less traditional experiments. We mention two of them. First, let us imagine that inside the board there is another funnel pointing to some fixed vertex v_{ij} which forces all the balls to through it. We can study the relationships between the (discrete) probability spaces describing the modified experiment and the original one. Second, let us imagine that behind the board a magnet is placed to influence the fall of balls. This time it is impossible to calculate the probabilities of individual paths but, using statistical tests, we can carry out a large number of experiments and test whether the magnet has an impact on the experiment. Similar "statistical activities" can be carried out in the case of the first modified experiment.

A less traditional approach to walking on the Galton board is to study the transition of balls from a given level to the next one. Each level can be viewed as a discrete probability space, the transition can be studied as a transformation of one probability space into another, and the consecutive transitions can be chained as the compositions of transformations.

4. Transformations

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ be a finite set, let p be a probability function on Ω, i.e. $0 \le p(\omega_i) \le 1$ and $\sum_{i=1}^{m} p(\omega_i) = 1$. Then (Ω, p) is said to be a *discrete probability space.* Note that to each probability function p on Ω there corresponds a probability measure P defined on subsets of Ω and, for discrete probability spaces, there is a natural one-to-one correspondence between probability functions and probability measures. *In what follows, all the probability spaces will be discrete.*

Definition 1. *Let* (Ω, p) *and* (Ξ, q) *be probability spaces. Let* T *be a map of* $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ *into* $\Xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$ *such that*

$$q(\xi_j) = \sum_{\omega_i \in T^{\leftarrow}(\xi_j)} p(\omega_i) \quad \textit{for all} \quad j \in \{1, 2, \ldots, n\} \quad \textit{such that} \quad q(\xi_j) > 0.$$

Then T *is said to be a* transformation *of* (Ω, p) *to* (Ξ, q)*. If* Ξ *is a set of real numbers, then* T *is said to be a* random variable.

Each transformation T can be visualized as a system of pipelines going from Ω to Ξ, through which p flows and results in q, see Figure 3 (cf. [1], [2]).

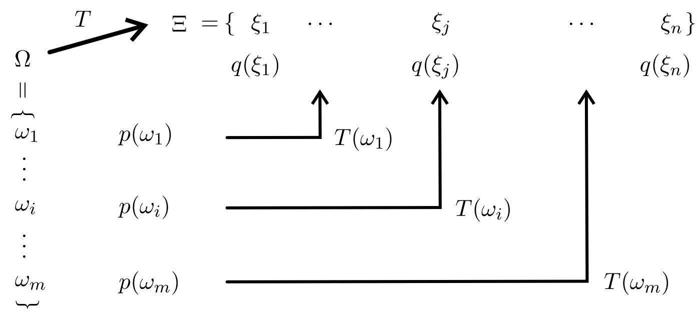

Figure 3: Transformation of a probability space

Let (Ω, p) and (Ξ, q) be probability spaces. It is natural to ask the following question: Does there always exist a transformation of (Ω, p) to (Ξ, q)? The answer is NO.

Indeed, for probability spaces (Ω, p) and (Ξ, q), if Ξ has more points than Ω, $p(\omega_i) = 1/m$ for all $i = 1, 2, \ldots, m$, and $q(\xi_j) = 1/n$ for all $j = 1, 2, \ldots, n$, then there is *no transformation* of (Ω, p) to (Ξ, q).

As shown in [1] and [2], if we replace the classical pipeline (sending the whole amount of each $p(\omega_i)$ to exactly one ξ_j) by a more complex pipeline (sending each $p(\omega_i)$ proportionally to several/all points of Ξ), then the answer is YES. The solution, called a “fuzzy transformation”, is based on Figure 4.

Observe that our complex pipeline is determined by a special matrix $\mathbf{A} = (a_{ij})_{m \times n}$, and the corresponding “fuzzy transformation” of p on Ω into q on Ξ can be described as follows: q (as a vector) is the (matrix) product of p (as a vector) and $\mathbf{A}$. The ith row of $\mathbf{A}$

$$q_i = (a_{i1}, a_{i2}, \ldots, a_{in})$$

is a probability function on Ξ, and a_{ij} can be interpreted as the probability of “transition” from ω_i to $\xi_j \in \Xi$, see [3].

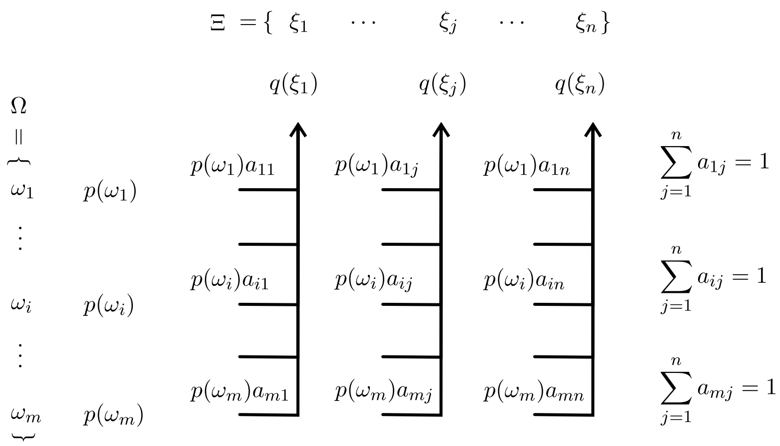

Figure 4: "Fuzzy transformation" of a probability space

Definition 2. *Let* $\mathbf{A} = (a_{ij})_{m\times n}$ *be an* m*-by-*n *matrix such that all* a_{ij} *are non-negative and* $\sum_{j=1}^{n} a_{ij} = 1$ *for all* i*,* $1 \le i \le m$*. Then* $\mathbf{A}$ *is said to be a* generalized stochastic matrix*. Further, if* $a_{i,j} \in \{0, 1\}$ *for all indexes, then* $\mathbf{A}$ *is said to be a* crisp generalized stochastic matrix*. If* $m = 1$*, then* $\mathbf{A}$ *is condensed to* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and is called a* stochastic vector*. If* $m = n$*, then* $\mathbf{A}$ *is called a* stochastic matrix*.*

Note that the product of two generalized stochastic matrices $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{n\times l}$ is a generalized stochastic m-by-l matrix (in particular, the product of a stochastic vector and a generalized stochastic matrix is a stochastic vector).

Definition 3. *Let* (Ω, p)*,* $\Omega = \{\omega_1, \ldots, \omega_m\}$*, and* (Ξ, q)*,* $\Xi = \{\xi_1, \ldots, \xi_n\}$*, be probability spaces. Let* $\mathbf{A} = (a_{ij})_{m\times n}$ *be a generalized stochastic matrix. Let* $T_{\mathbf{A}}$ *be a map of* Ω *into the set of all probability functions on* Ξ *defined by*

$$T_{\mathbf{A}}(\omega_i) = (a_{i1}, a_{i2}, \ldots, a_{in}), \quad i = 1, 2, \ldots, m.$$

If $q = p\mathbf{A}$*, then* $T_{\mathbf{A}}$ *is said to be a* fuzzy transformation *of* (Ω, p) *to* (Ξ, q)*.*

Let (Ω, p) and (Ξ, q) be probability spaces. Define $\mathbf{A} = (a_{ij})_{m\times n}$ as follows:

$$q = (a_{i1}, a_{i2}, \ldots, a_{in}), \quad i = 1, 2, \ldots, m.$$

Let $T_{\mathbf{A}}$ be the corresponding map of Ω sending each ω_i into q.

Lemma 1. $T_{\mathbf{A}}$ *is a fuzzy transformation of* (Ω, p) *to* (Ξ, q)*.*

Note that there are other (non trivial) fuzzy transformations of (Ω, p) to (Ξ, q), and fuzzy transformations are related to probability functions on the product $\Omega \times \Xi$ such that p and q are marginal probabilities, see [3].

Let (Ω, p), $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$, (Ξ, q), $\Xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$, and (Λ, r), $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ be discrete probability spaces. Let $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{n\times l}$ be generalized stochastic matrices such that $T_{\mathbf{A}}$ is a fuzzy transformation of (Ω, p) to (Ξ, q) and $T_{\mathbf{B}}$ is a fuzzy transformation of (Ξ, q) to (Λ, r). Let $\mathbf{C} = (c_{ij})_{m\times l} = \mathbf{A} \times \mathbf{B}$ and let $T_{\mathbf{C}}$ be the corresponding map of Ω into probability functions on Λ.

Lemma 2. *$T_{\mathbf{C}}$ is a fuzzy transformation of (Ω, p) to (Λ, r).*

5. Generalized random walk

A *generalized random walk* can be viewed as a finite series of successive fuzzy transformations "governed" via the product of constituent matrices. Indeed, for $l = 1, 2, \ldots, N$, let (Ω_l, p_l), $\Omega_l = \{\omega_{l1}, \omega_{l2}, \ldots, \omega_{lm_l}\}$, be discrete probability spaces and, for $l = 1, 2, \ldots, N-1$, let $\mathbf{A}_l$ be generalized stochastic matrices such that $T_{\mathbf{A}_l}$ is the corresponding fuzzy transformation of (Ω_l, p_l) to (Ω_{l+1}, p_{l+1}).

The starting point (top vertex) can be viewed as a trivial probability space (Ω_0, p_o), where Ω_0 consists of just one point $\{\omega_{00}\}$ and $p_0(\omega_{00}) = 1$. Formally, p_1 can be viewed as the "fuzzy image" of p_0 and (Ω_1, p_1) can be viewed as the fuzzy transformation of (Ω_0, p_o) (via T_{p_1}). Consequently,

$$p_N = p_1 \mathbf{A}_1 \mathbf{A}_2 \ldots \mathbf{A}_{N-1}$$

enables us to calculate the probability of "a generalized random walk starting at ω_{00} ends up at ω_{Nk}, $k = 1, 2, \ldots, m_N$".

The fact that we send a point (elementary event) to a probability measure has definitely a quantum nature and characterizes the transition from classical to fuzzy transformations [1].

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