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Analytical Models for Distribution of the Envelope and Phase of Linearly Modulated Signals in AWGN Channel

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***Abstract*—In this paper, analytical expressions for the distribution of the envelope and phase of linearly modulated signals such as BPSK, M-PSK, and M-QAM in AWGN are presented. We perform numerical simulations for different orders of signal constellations. The results show that the proposed theoretical models are in excellent agreement with the estimated distributions from various numerical experiments.**

***Keywords*—digital modulations, linear modulation, probability density function, phase, AWGN**

I. Introduction

MEASUREMENT of the probability density function (PDF) plays a very important role in digital communication applications, such as signal detection and modulation classification [1], [2]. The amplitude probability distribution function has been found to be useful in characterizing signals and evaluating effects of interference on victim receivers [3]. The theoretical derivation of the PDF of the envelope of baseband digital signals in narrowband (color) Gaussian noise has been presented in the literature [2]–[6]. In this paper, we consider additive white noise channel and drive the PDF of the envelopes of the linearly modulated signals (BPSK, M-PSK, and M-QAM) as well as the PDF of the phases of M-PSK signals.

II. Signal Model

Assume that the received signal at the output of the matched filter is modeled as,

$$x_n = Sa_n + w_n, \quad n = 1, 2, \ldots, N \tag{1}$$

$$a_n = a_{In} + ja_{Qn} \tag{2}$$

$$w_n = w_{In} + jw_{Qn} \tag{3}$$

where S is a scalar, a_n is the transmitted symbol, and w_n is the noise, the subscripts I and Q are the in-phase and quadrature components, respectively, and N is the number of samples during the observation interval. The noise component is modeled by a zero-mean Gaussian random variable with independent real and imaginary parts, each of which has a variance of σ^2. We assume that the transmitted symbols are independent and identically distributed and drawn from either an M-array PSK or a square QAM constellation [7].

Without loss of generality, we assume that the constellation has unit energy, i.e., $\mathrm{E}\{|a_n|^2\} = 1$.

For the N received samples, the SNR of interest is defined as [8],

$$\rho = \frac{S^2}{2\sigma^2} \tag{4}$$

Note that in the case of BPSK constellation, the imaginary parts defined in (2) and (3) are zero, a_n's are the symbols taking the values of ± 1 with equal probabilities to be 1 or -1; in this case, the SNR is defined by,

$$\rho = \frac{S^2}{\sigma^2} \tag{5}$$

To express the PDF of interest in terms of SNR, for simplicity, we assume that the AWGN has unit variance. In this case, the SNR (ρ) will be equal to S^2.

III. Distribution of the Signal Envelope

A. BPSK

As the received signal x_n in (1) is corrupted with an additive white Gaussian noise, one may find that the envelope of x_n's can be modeled as an absolute value of a normal variable with a mean of $\mu = S$. That is,

$$p_{|x|}(x) = \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{(-x-\sqrt{\rho})^2}{2}} + e^{-\frac{(x-\sqrt{\rho})^2}{2}}\right) \tag{6}$$

B. M-PSK

Assume that the a_n's in (1) takes complex values independently with equal probabilities, and w_n is modeled by a zero-mean Gaussian random variable with independent real and imaginary parts, w_{In} and w_{Qn}, each of which has unit variance. In this case, the SNR (ρ) will be equal to $S^2/2$. By calculating the square of x_n,

$$|x_n|^2 = (Sa_{In} + w_{In})^2 + (Sa_{Qk} + w_{Qn})^2 \tag{7}$$

Therefore, the square envelope $|x_k|^2$ can be modeled by a non-central chi square variable with two degrees of freedom and the non-centrality parameter of λ, where $\lambda = S^2$. That is,

$$p_{|x|^2}(x) = \frac{1}{2}e^{-(x+\lambda)/2}\sum_{i=0}^{\infty}\frac{(\lambda x)^i}{2^{2i}(i!)^2} \tag{8}$$

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TABLE I
THE WEIGHTS AND COEFFICIENTS USED IN EQ. (12).

M	W_M^j	K_M^j
8	1/2	0.3820
	1/2	1.9098
16	1/4	0.2229
	1/2	1.1146
	1/4	2.0062
32	1/5	1.9000
	1/5	1.4529
	1/10	1.0059
	1/5	0.5588
	1/10	0.1118
64	1/17	2.6453
	2/17	1.9975
	2/17	1.5656
	3/17	1.3497
	2/17	0.9178
	2/17	0.7018
	1/17	0.4859
	3/17	0.2699
	1/17	0.0540

As $\lambda = S^2 = 2\rho$, one can get,

$$p_{|x|^2}(x) = \frac{1}{2} e^{-(x+2\rho)/2} \sum_{i=0}^{\infty} \frac{(\rho x)^i}{2^i (i!)^2} \tag{9}$$

By using,

$$f_{Y(y)} = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|} \tag{10}$$

where $y = g(x) = \sqrt{x}$,
we arrive at the PDF of envelope of x_k's given as,

$$p_{|x|}(y) = y e^{-(y^2+2\rho)/2} \sum_{i=0}^{\infty} \frac{(\rho y^2)^i}{2^i (i!)^2} \tag{11}$$

C. M-QAM

In QAM signals, since there are different amplitudes in the constellation, the PDF of $|x_k|^2$ becomes mixture of non-central chi square.
That is,

$$p_{|x|^2}(x) = \frac{1}{2} \sum_j W_j^M \Big(e^{-(x+2K_j^M \rho)/2} \sum_{i=0}^{\infty} \frac{(2K_j^M \rho x)^i}{2^{2i}(i!)^2} \Big) \tag{12}$$

The weights W_j^M and coefficients K_j^M are related to the probability of symbols and values of amplitudes, and Tab. I tabulates the weights and the coefficients for a number of constellations.

With the help of (10) and (12), the PDF of envelope of x_k's will be,

$$p_{|x|}(y) = \sum_j W_j^M \Big(y e^{-(y^2+2K_j^M \rho)/2} \sum_{i=0}^{\infty} \frac{(2K_j^M \rho y^2)^i}{2^{2i}(i!)^2} \Big) \tag{13}$$

IV. DISTRIBUTION OF THE SIGNAL PHASE OF M-PSK SIGNALS

The phase angle for a given $I-Q$ pair is calculated by the arctangent function of the ratio of quadrature to the in-phase component of signal as,

$$\varphi = \tan^{-1}(Q/I) \tag{14}$$

where Q and I are given in (1).

From (1), it is seen that both the real (I) and imaginary (Q) parts of the M-PSK signal x_n can be modeled by Gaussian random variables with unit variance and mean of $S\cos(\pi/M)$ and $S\sin(\pi/M)$, respectively. Therefore, the ratio Q/I can be modeled by a mixture of ratio Gaussian distribution [9]. That is,

$$p_{Q/I}(x) = \frac{1}{M} \sum_{m=1}^{M} \frac{b(x)\cdot c(x)}{a^3(x)} \frac{1}{\sqrt{2\pi}} \Big[2\Phi\Big(\frac{b(x)}{a(x)}\Big) - 1 \Big] +$$

$$+ \frac{1}{\pi a(x)} e^{-\rho} \tag{15}$$

where

$$a(x) = \sqrt{x^2+1},$$

$$b(x) = \sqrt{2\rho\cos(\pi/m)}x + \sqrt{2\rho\sin(\pi/m)},$$

$$c(x) = e^{\frac{1}{2}\frac{b^2(x)}{a^2(x)} - \rho},$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du,$$

By putting $y = g(x) = \tan^{-1}(Q/I)$ into (10), one may get the PDF of signal phase φ as,

$$p_\phi(\varphi) = \frac{1}{M}(1+\tan^2(\varphi)) \sum_{m=1}^{M} \frac{b(\varphi)\cdot c(\varphi)}{a^3(\varphi)} \frac{1}{\sqrt{2\pi}} \cdot$$

$$\cdot \Big[2\Phi\Big(\frac{b(\varphi)}{a(\varphi)}\Big) - 1 \Big] + \frac{1}{\pi a(\varphi)} e^{-\rho} \tag{16}$$

where

$$a(\varphi) = \sqrt{1+\tan^2(\varphi)},$$

$$b(\varphi) = \sqrt{2\rho\cos(\pi/m)}\tan(\varphi) + \sqrt{2\rho\sin(\pi/m)},$$

$$c(\varphi) = e^{\frac{1}{2}\frac{b^2(\varphi)}{a^2(\varphi)} - \rho},$$

$$\Phi(\varphi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tan(\varphi)} e^{-\tan^2(\varphi)/2} d\varphi.$$

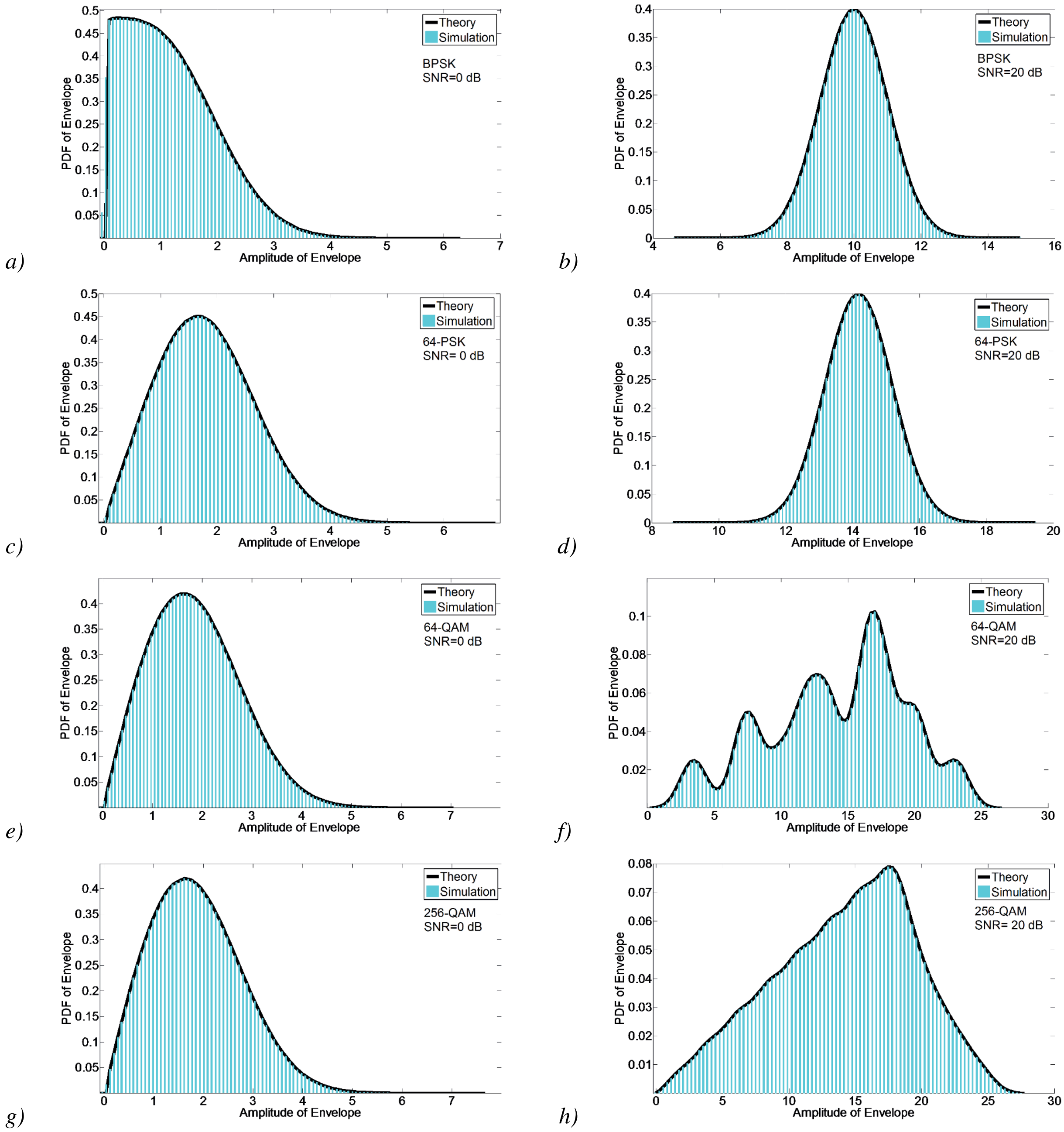

Fig. 1. The PDF of envelope of noisy linearly modulated signals, a: BPSK ($SNR = 0$dB), b: BPSK ($SNR = 20$dB), c: 64-PSK ($SNR = 0$dB), d: 64-PSK ($SNR = 20$dB), e: 64-QAM ($SNR = 0$dB), f: 64-QAM ($SNR = 20$dB), g: 256-QAM ($SNR = 0$dB), and h: 256-QAM ($SNR = 20$dB).

V. SIMULATION RESULTS

To verify the derived expressions, we perform computer simulations by generating BPSK, M-PSK, and M-QAM signals corrupted by AWGN with different SNR values, and comparing the derived theatrical model with the simulation results. In order to assure accurate estimations, close to the true PDF, we set the sample size to be 100,000,000. For the sake of space, we only include the results of BPSK, 64-PSK, 64-QAM and 256-QAM with SNR values of 0 and 20 dB for the PDF of envelopes, and QPSK, 16-PSK and 64-PSK with SNR values of 0 and 20 dB for the PDF of phases. The results are shown in Figs. 1 and 2.

As we see from all figures, the results from the proposed analytical models match well with the true PDF obtained experimentally. Also, from Fig. 2 it is seen that by increasing the order of modulation in M-PSK signals, the PDF of phase approaches to a uniform distribution even for high SNR values.

VI. CONCLUSION

In this paper, we showed that the square envelopes of M-PSK and M-QAM signals in AWGN channel follow non-central chi square, and mixture of non-central chi square distributions, respectively. We derived the PDFs of phase M-PSK signals in AWGN channel and found the proposed analytical models match well with the true PDFs.

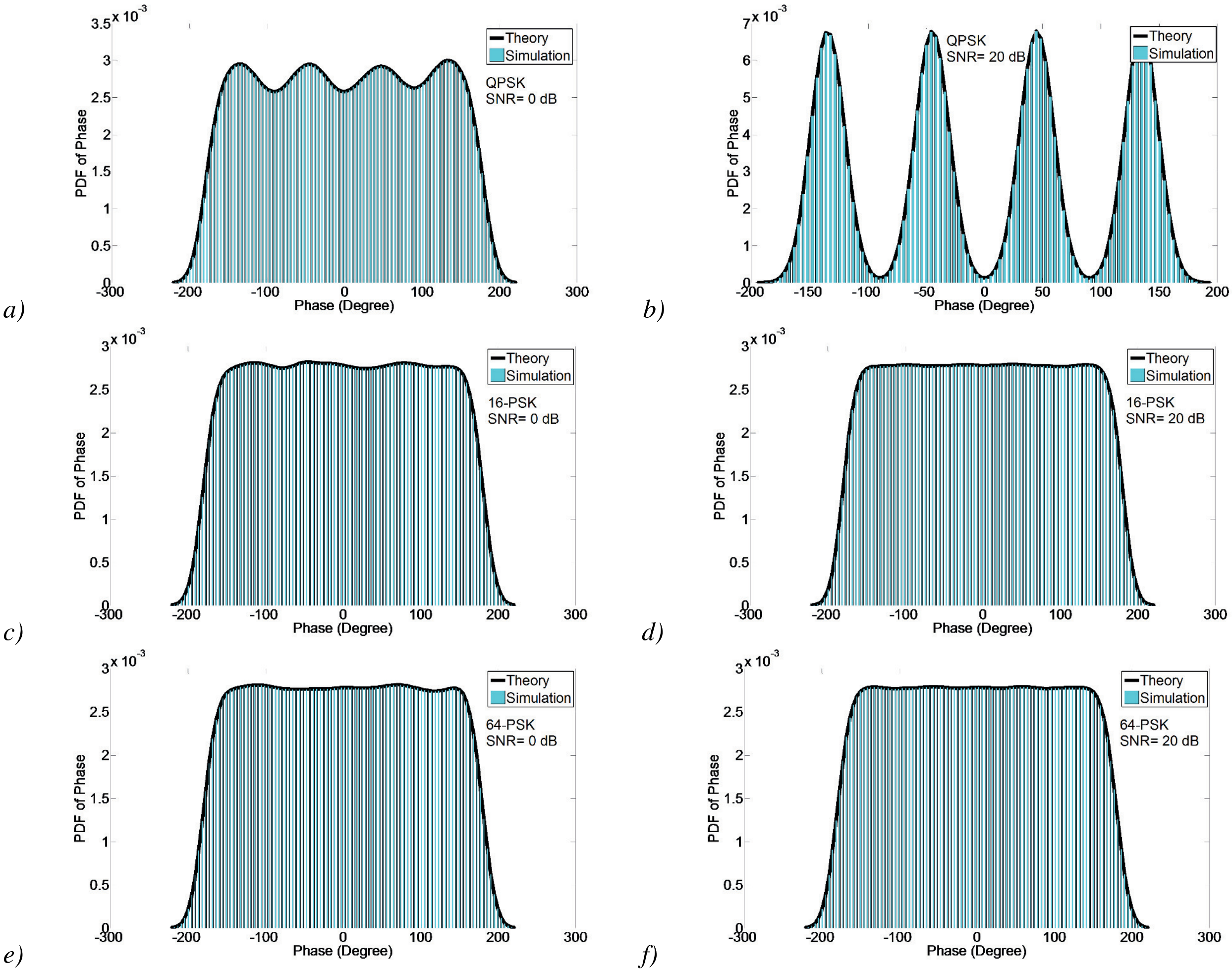

Fig. 2. The PDF of phase M-PSK signals, a:QPSK ($SNR = 0$dB), b: QPSK ($SNR = 20$dB), c: 16-PSK ($SNR = 0$dB), d: 16-PSK ($SNR = 20$dB), e: 64-PSK ($SNR = 0$dB), f: 64-PSK ($SNR = 20$dB).

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