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Dedicated to Professor Jan Stochel
on the occasion of his 70th birthday

FINITELY ADDITIVE FUNCTIONS IN MEASURE THEORY AND APPLICATIONS

Daniel Alpay and Palle Jorgensen

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Abstract. In this paper, we consider, and make precise, a certain extension of the Radon–Nikodym derivative operator, to functions which are additive, but not necessarily sigma-additive, on a subset of a given sigma-algebra. We give applications to probability theory; in particular, to the study of μ-Brownian motion, to stochastic calculus via generalized Itô-integrals, and their adjoints (in the form of generalized stochastic derivatives), to systems of transition probability operators indexed by families of measures μ, and to adjoints of composition operators.

Keywords: Hilbert space, reproducing kernels, probability space, Gaussian fields, transforms, covariance, Itô integration, Itô calculus, generalized Brownian motion.

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1. INTRODUCTION

Motivated by a diverse set of applications, in the context of probability theory, we present here a general result (Theorem 3.2) on finitely additive functions. We demonstrate its implications for the study of a stochastic calculus based on generalized Itô-integrals, and generalized derivatives, for a prescribed systems of sigma-finite positive measures, see especially Theorems 5.4, 5.6 and 5.7 below.

To provide motivation, consider the following example. Let $f \in \mathbf{L}^2(\mathbb{R}, \mathcal{B}, dx) \setminus \mathbf{L}^1(\mathbb{R}, \mathcal{B}, dx)$ (the classical Lebesgue spaces of the real line). The function

$$f(A) = \int_A f(x)dx \tag{1.1}$$

is additive on the algebra of finite length (measurable) sets, but will not be sigma-additive since f is not summable. The question we address more generally is the following:

Question 1.1. Given a measure space $(X, \mathcal{F}, \mu)$, where μ is sigma-finite, define $\mathcal{F}^{\text{fin}}$ to be the family of sets of finite measure for μ. The question is to give an intrinsic characterization of the functions of the form

$$M(A) = \int_A f(x)\mu(dx), \quad A \in \mathcal{F}^{\text{fin}}, \tag{1.2}$$

where $f \in \mathbf{L}^2(X, \mathcal{F}, \mu)$.

We note that $\mathcal{F}^{\text{fin}}$ generates $\mathcal{F}$ (in the sense that $\mathcal{F}$ is the smallest sigma-algebra containing $\mathcal{F}^{\text{fin}}$). This problem was first suggested, and discussed briefly, by S.D. Chatterji in [9], in dealing with cases when the derivative need not be assumed summable. First results of this kind seem to originate with the work of F. Riesz for the real line and the Lebesgue measure; see [20, §, 5 p. 462]. The motivation in that paper was the theory of convergence of martingales. Our motivation comes from the theory of composition operators. Consider a measure space $(X, \mathcal{F})$, an endomorphism σ of X, and a sigma-finite measure μ which is σ-invariant:

$$\mu \circ \sigma^{-1} = \mu, \tag{1.3}$$

meaning that

$$\mu(A) = \mu(\sigma^{-1}(A)), \quad A \in \mathcal{F}.$$

It follows that the composition map S:

$$f \mapsto f \circ \sigma \tag{1.4}$$

is an isometry from $\mathbf{L}^p(X, \mathcal{F}, \mu)$ into itself for $p \in [1, \infty)$. As we will illustrate in Section 6 in the case $p = 2$, the computation of the adjoint S^* involves the extension of the Radon–Nikodym theorem considered here.

The problem addressed in the present work is further motivated by a key idea from Itô calculus; in particular, on the fact that the Itô-integral is based on L^2 theory. Following for example [13, 14, 18] one notes that the Itô-integral takes the form of an isometry between the respective L^2-spaces. This is true also for the extension of Itô's theory which is based on a version of Brownian motion, or the Wiener process, $W^{(\mu)}$ governed by an arbitrary sigma-finite measure μ, as opposed to the more familiar case of Lebesgue measure; see Section 4 below. Denoting by V_μ the Itô-isometry calculated from $W^{(\mu)}$, it is then natural to view the adjoint operator V_μ^* (now a co-isometry) as a generalized derivative operator. But this entails a separate L^2 approach for such a generalized derivative; so one not relying on more familiar notions of Radon–Nikodym derivatives for μ. Here we present such a theory, accompanied with applications which in turn entail a new stochastic analysis based on families of sigma-finite measures, and their associated Itô-calculus.

The paper consists of five sections besides the introduction. In Section 2 we review some properties of the reproducing kernel Hilbert space with reproducing kernel $\mu(A \cap B)$, where μ is a sigma-finite measure and A, B run in $\mathcal{F}^{\text{fin}}$. The main result of the paper is proved in Section 3. In the last three sections we consider applications, to the μ-Brownian motion, transition probability systems and adjoint of composition operators respectively.

2. THE REPRODUCING KERNEL HILBERT SPACE $\mathfrak{H}(\mu)$

In preparation to Theorem 3.2, we recall the definition of the reproducing kernel Hilbert space associated to a sigma-finite measure and some of its properties. We refer to [4,5] for further details. With the notation of the introduction, we have the following result, see [5].

Theorem 2.1. *The function $K^{(\mu)}(A,B) = \mu(A \cap B)$ is positive definite on $\mathcal{F}^{\rm fin}$ and the associated reproducing kernel Hilbert space consists of the functions of the form (1.2), where $f \in \mathbf{L}^2(X,\mathcal{F},\mu)$, with norm $\|M\| = \|f\|_2$ (where $\|f\|_2$ denotes the norm of f in $\mathbf{L}^2(X,\mathcal{F},\mu)$).*

Definition 2.2. We denote by $\mathfrak{H}(\mu)$ the reproducing kernel Hilbert space with reproducing kernel $\mu(A \cap B)$, $A, B \in \mathcal{F}^{\rm fin}$.

It follows from (1.2) that $M(A) = 0$ when $\mu(A) = 0$. When $\mu(X) < \infty$, M is a signed measure and the function f is equal to the Radon–Nikodym derivative $\frac{\mathrm{d}M}{\mathrm{d}\mu}$. This latter interpretation fails when μ is not finite, and we will see in Theorem 3.2 another characterization of M.

The following lemma will be used in the proof of Theorem 3.2.

Lemma 2.3. *Let $B_1, \ldots, B_J \in \mathcal{F}^{\rm fin}$, and let $c_1, \ldots, c_J \in \mathbb{C}$. The sum $\sum_{j=1}^J b_j 1_{B_j}$ can be rewritten in the form $\sum_{n=1}^N a_n 1_{A_n}$ where the sets A_n are pairwise disjoint.*

Proof. The proof is a repeated use of the formula $A = (A \setminus B) \cup (A \cap B)$. We use induction. For $N = 2$, one writes

$$\begin{aligned} b_1 1_{B_1} + b_2 1_{B_2} &= c_1\left(1_{B_1 \setminus B_2} + 1_{B_1 \cap B_2}\right) + c_2\left(1_{B_2 \setminus B_1} + 1_{B_2 \cap B_1}\right) \\ &= c_1 1_{B_1 \setminus B_2} + c_2 1_{B_2 \cap B_1} + (c_1 + c_2) 1_{B_1 \cap B_2}. \end{aligned}$$

Assuming the result true at rank J we have

$$\begin{aligned} \sum_{j=1}^{J+1} b_j 1_{B_j} &= \sum_{m=1}^{J} b_j 1_{B_j} + b_{J+1} 1_{B_{J+1}} = \sum_{n=1}^{N} a_n 1_{A_n} + b_{J+1} 1_{B_{J+1}} \\ &= \sum_{n=1}^{N} a_n 1_{A_n \setminus B_{J+1}} + \sum_{n=1}^{N} a_n 1_{A_n \cap B_{J+1}} + b_{J+1} 1_{B_{J+1} \setminus \bigcup_{n=1}^N A_n} \\ &\quad + b_{J+1} 1_{B_{J+1} \cap \left(\bigcup_{n=1}^N A_n\right)} \\ &= \sum_{n=1}^{N} a_n 1_{A_n \setminus B_{J+1}} + \sum_{n=1}^{N} a_n 1_{A_n \cap B_{J+1}} + b_{J+1} 1_{B_{J+1} \setminus \bigcup_{n=1}^N A_n} \\ &\quad + \sum_{n=1}^{N} b_{J+1} 1_{B_{J+1} \cap A_n} \\ &= \sum_{n=1}^{N} a_n 1_{A_n \setminus B_{J+1}} + \sum_{n=1}^{N} (a_n + b_{J+1}) 1_{A_n \cap B_{J+1}} + b_{J+1} 1_{B_{J+1} \setminus \bigcup_{n=1}^N A_n}. \qquad \square \end{aligned}$$

The following result is a special case of the characterization of the elements of a reproducing kernel Hilbert space; see e.g. [7].

Theorem 2.4. *A function M defined on $\mathcal{F}^{\text{fin}}$ belongs to $\mathfrak{H}(\mu)$ with norm less or equal to $\sqrt{C}$ if and only if the kernel*

$$\mu(A\cap B)-\frac{1}{C}M(A)\overline{M(B)}$$

is positive definite on $\mathcal{F}^{\text{fin}}$.

3. THE MAIN RESULT

Let as above $\mathcal{F}$ be a sigma-algebra on a set X, let μ be a sigma-finite measure, and let $\mathcal{F}^{\text{fin}}$ denote the sets of finite measure for μ. In [9] the following problem was considered:

Problem 3.1. Given is a complex-valued additive function M on $\mathcal{F}^{\text{fin}}$ which is absolutely continuous with respect to μ in the sense that:

$$\forall A\in\mathcal{B}^{\text{fin}}:\ \mu(A)=0\implies M(A)=0.$$

The problem was to characterize M in alternative ways.

Theorem 3.2. *The following three conditions are equivalent:*

(1) *There exists $h\in\mathbf{L}^2(X,\mathcal{F},\mu)$ such that*

$$M(A)=\int_A h(x)\mu(dx),\quad A\in\mathcal{F}^{\text{fin}}. \tag{3.1}$$

(2) *There exists a constant $C<\infty$ such that, for every $N\in\mathbb{N}$ and every family $A_1,\ldots,A_N$ of pairwise disjoint elements in $\mathcal{F}^{\text{fin}}$, it holds that*

$$\sum_{n=1}^N\frac{|M(A_n)|^2}{\mu(A_n)}\le C. \tag{3.2}$$

(3) *$M\in\mathfrak{H}(\mu)$.*

The proof of the equivalence between (1) and (2) is outlined in [9, Theorem 6, p. 17] for $p\in(1,\infty)$. Here, for $p=2$, we prove this equivalence and add the equivalence with (3). The difference with the arguments in [9, pp. 17–18] is that we do not prove directly that (2) implies (1), but that (2) implies (3), and then prove that (3) implies (1). The case of general $p\in(1,\infty)$ is recalled below (see Theorem 3.5).

Proof of Theorem 3.2. We first show that (1) implies (2). Assume (1) holds. Let $A \in \mathcal{F}^{\mathrm{fin}}$. By Cauchy–Schwarz inequality we have:

$$\begin{aligned}|M(A)|^2 &= \left| \int_X 1_A(x)(1_A(x)h(x))\mu(dx) \right|^2 \\ &\le \left(\int_X 1_A^2(x)\mu(dx) \right) \left(\int_X 1_A^2(x)|h(x)|^2\mu(dx) \right) = \mu(A) \int_A |h(x)|^2\mu(dx).\end{aligned}$$

Thus, for $A_1, \ldots, A_N$ pairwise disjoint elements of $\mathcal{F}^{\mathrm{fin}}$, we can write:

$$\sum_{n=1}^N \frac{|M(A_n)|^2}{\mu(A_n)} \le \sum_{n=1}^N \int_{A_n} |h(x)|^2 dx \le \int_X |h(x)|^2 dx,$$

so that (2) holds. Assuming (2), in order to prove (3) we will show that the kernel

$$\mu(A \cap B) - \frac{M(A)\overline{M(B)}}{C}$$

is positive definite on $\mathcal{F}^{\mathrm{fin}}$ for $C = \|h\|_2^2$. Let $b_1, \ldots, b_J$ be complex numbers and $B_1, \ldots, B_J$ be in $\mathcal{F}^{\mathrm{fin}}$. Using Lemma 2.3 we rewrite

$$\sum_{j=1}^J b_j 1_{B_j} = \sum_{n=1}^N a_n A_n$$

where $a_1, \ldots, a_N \in \mathbb{C}$ and now the A_n are pairwise disjoint. By Cauchy–Schwarz inequality,

$$\begin{aligned}\frac{1}{C}\left| \sum_{n=1}^N a_n M(A_n) \right|^2 &= \frac{1}{C}\left| \sum_{n=1}^N \frac{M(A_n)}{\sqrt{\mu(A_n)}} a_n \sqrt{\mu(A_n)} \right|^2 \\ &\le \frac{1}{C}\left(\sum_{n=1}^N \frac{M(A_n)|^2}{\mu(A_n)} \right) \left(\sum_{n=1}^N |a_n|^2 \mu(A_n) \right) \\ &\le \sum_{\ell,n=1}^N a_\ell \overline{a_n} \mu(A_\ell \cap A_n)\end{aligned}$$

since the A_n are pairwise disjoint. By Theorem 2.4, the function $A \mapsto M(A)$ belongs to $\mathfrak{H}(\mu)$ with norm at least C. The fact that (3) implies (1) forms the content of Theorem 2.1. $\square$

Definition 3.3. We will use the notation ∇_μ for the map which to $M \in \mathfrak{H}(\mu)$ associates h as in (3.1), and call it the Krein–Feller derivative. We can therefore rewrite (3.1) as

$$M(A) = \int_A (\nabla_\mu M)(x)\mu(dx), \quad A \in \mathcal{F}^{\mathrm{fin}}. \tag{3.3}$$

Our motivation for this terminology comes from analysis on fractals, where various variants of the operator of differentiation by μ appear in the theory of the Krein–Feller diffusion. The generator of the Krein–Feller diffusion is a variant of $L \coloneqq \frac{d^2}{dxd\mu}$. See for instance [1, 11, 12, 15, 16]. See also [10].

As a consequence of Theorem 3.2 we have:

Corollary 3.4. *The map ∇_μ is unitary from $\mathfrak{H}(\mu)$ onto $\mathbf{L}^2(X, \mathcal{F}, \mu)$, with adjoint given by*

$$(\nabla_\mu^* g)(A) = \int_A g(x)\mu(dx), \quad A \in \mathcal{F}^{\mathrm{fin}},\ g \in \mathbf{L}^2(X, \mathcal{F}, \mu). \tag{3.4}$$

Proof. By Theorem 2.1 the map (3.4) is unitary from $\mathbf{L}^2(X, \mathcal{F}, \mu)$ onto $\mathfrak{H}(\mu)$. Denoting temporarily this map by I_μ, we take $M \in \mathfrak{H}(\mu)$ of the form $M(A) = \int_A h(x)\mu(dx)$ (with $h \in \mathbf{L}^2(X, \mathcal{F}, \mu)$). The fact that $I_\mu = \nabla_\mu^*$ follows from

$$\langle I_\mu g, M\rangle_{\mathfrak{H}(\mu)} = \langle g, h\rangle_\mu = \langle g, \nabla_\mu M\rangle_\mu, \quad g \in \mathbf{L}^2(X, \mathcal{F}, \mu). \qquad \square$$

In the general case where $p \in (1, \infty)$ the two first items in Theorem 3.2 are still equivalent, as we now prove; we follow, with a bit more details, the arguments in [9, pp. 17–18]. One could replace the third condition in Theorem 3.2 by introducing pairs of spaces in duality (see [2, 3, 8] for the latter), but this will not be done here.

Theorem 3.5 (the case $p \in (1, \infty)$)**.** *The following are equivalent:*

(1) *There exists $h \in \mathbf{L}^p(X, \mathcal{F}, \mu)$ such that*

$$M(A) = \int_A h(x)\mu(dx), \quad A \in \mathcal{F}^{\mathrm{fin}}. \tag{3.5}$$

(2) *There exists a constant $C < \infty$ such that, for every $N \in \mathbb{N}$ and every family $A_1, \ldots, A_N$ of pairwise disjoint elements in $\mathcal{F}^{\mathrm{fin}}$, it holds that*

$$\sum_{n=1}^{N} \frac{|M(A_n)|^p}{(\mu(A_n))^{p-1}} \leq C. \tag{3.6}$$

Proof. The proof follows the proof of Theorem 3.2, but now uses Hölder inequality. Assume first that (1) is in force. We have for $A \in \mathcal{F}^{\mathrm{fin}}$:

$$\begin{aligned}
|M(A)| &\leq \int_X 1_A(x)1_A(x)|h(x)|\mu(dx) \\
&\leq \left(\int_X (1_A(x))^q \mu(dx)\right)^{1/q} \left(\int_X (1_A(x))^p |h(x)|^p \mu(dx)\right)^{1/p} \\
&\leq (\mu(A))^{1/q} \left(\int_A |h(x)|^p \mu(dx)\right)^{1/p}.
\end{aligned}$$

Since $p/q = p-1$ we have

$$|M(A)|^p \le (\mu(A))^{p-1} \int_A |h(x)|^p \mu(dx).$$

Hence, for $A_1, \ldots, A_N$ pairwise disjoint elements of $\mathcal{F}^{\text{fin}}$ we have:

$$\sum_{n=1}^{N} \frac{|M(A_n)|^p}{(\mu(A_n))^{p-1}} = \int_{\bigcup_{n=1}^{N} A_n} |h(x)|^p \mu(dx) \le \int_X |h(x)|^p \mu(dx).$$

Assume now that (3.6) is in force and define a map on the linear span of the functions 1_A, $A \in \mathcal{F}^{\text{fin}}$, by

$$\varphi(f) = \sum_{n=1}^{N} c_n M(A_n),$$

where $f = \sum_{n=1}^{N} c_n 1_{A_n}$, the sets $A_1, \ldots, A_N$ being moreover pairwise disjoint. Then, by Hölder's inequality (and with $1/p + 1/q = 1$)

$$\begin{aligned}
|\varphi(f)| &\le \sum_{n=1}^{N} |c_n| \mu(A_n)^{1/q} \frac{|M(A_n)|}{(\mu(A_n))^{1/q}} \\
&= \left(\sum_{n=1}^{N} |c_n|^q \mu(A_n) \right)^{1/q} \left(\sum_{n=1}^{N} \frac{|M(A_n)|^p}{(\mu(A_n))^{p-1}} \right)^{1/p} \\
&\le C \left(\sum_{n=1}^{N} |c_n|^q \mu(A_n) \right)^{1/q} \\
&= C \left(\int_X |f(x)|^q \mu(dx) \right)^{1/q}
\end{aligned}$$

since $p/q = p-1$ Hence φ extends to a continuous functional on $\mathbf{L}^q(X, \mathcal{F}, \mu)$. The claim follows then from Riesz theorem. □

As a corollary we have:

Theorem 3.6. *Let $p \in (1, \infty)$. A function f defined on the real line is of the form $f(x) = \int_0^x g(u)du$ where $g \in \mathbf{L}^p(\mathbb{R}, \mathcal{B}, du)$ (the Borel sets and the Lebesgue measure) if and only if there exists $C > 0$*

$$\sum_{n=1}^{N} \frac{|f(x_{n+1}) - f(x_n)|^p}{|x_{n+1} - x_n|^{p-1}} \le C$$

for all $N \in \mathbb{N}$ and any ordered set of real points $x_1 < x_1 < \ldots < x_N$.

4. APPLICATION TO THE μ-BROWNIAN MOTION

Given a measure space $(X, \mathcal{F})$ and a sigma-finite measure μ on X, one introduces in a natural way three Hilbert spaces:

(1) the Hilbert space $\mathbf{L}^2(X, \mathcal{F}, \mu)$,
(2) the reproducing kernel Hilbert space $\mathfrak{H}(\mu)$ of functions defined on $\mathcal{F}^{\mathrm{fin}}$ with reproducing kernel $K_\mu(A, B) = \mu(A \cap B)$,
(3) a probability space $\mathbf{L}^2(\Omega, \mathcal{C}, \mathbb{P})$ in which is constructed the μ-Brownian motion $W^{(\mu)}$ with covariance function $K^{(\mu)}(A, B)$,

$$\mathbb{E}(W_A^{(\mu)} W_B^{(\mu)}) = \mu(A \cap B), \quad A, B \in \mathcal{F}^{\mathrm{fin}}, \tag{4.1}$$

and associated Itô-type stochastic integrals

$$V_\mu(f) = \int\limits_X h(x) dW_x^{(\mu)}, \quad h \in \mathbf{L}^2(X, \mathcal{F}, \mu). \tag{4.2}$$

Remark 4.1. In fact, V_μ is isometric into any L^2 probability space for which (4.1) is satisfied.

We recall the following (see [5]):

Proposition 4.2. *The map V_μ is an isometry from $\mathbf{L}^2(X, \mathcal{F}, \mu)$ into $\mathbf{L}^2(\Omega, \mathcal{C}, \mathbb{P})$.*

In the commutative diagram in Figure 1 the maps V_μ and ∇_μ were defined below and the map T_μ is defined by

$$V_\mu \nabla_\mu T_\mu = I_{\mathbf{L}^2(\Omega, \mathcal{C}, \mathbb{P})},$$

so that

$$T_\mu = \nabla_\mu^* V_\mu^* \tag{4.3}$$

since V_μ is an isometry and ∇_μ is unitary.

$$\begin{tikzcd}
\mathbf{L}^2(X, \mathcal{F}, \mu) \arrow[rr, "V_\mu"] & & \mathbf{L}^2(\Omega, \mathcal{C}, \mathbb{P}) \arrow[dl, "T_\mu"] \\
& \mathfrak{H}(\mu) \arrow[ul, "\nabla_\mu"] &
\end{tikzcd}$$

Fig. 1. Maps V_μ, ∇_μ and T_μ

Proposition 4.3. *The map T_μ and its adjoint are given by*

$$(T_\mu\psi)(A) = \mathbb{E}\left(\psi W_A^{(\mu)}\right), \tag{4.4}$$

$$T_\mu^* M = \int_X h(x) dW_x^{(\mu)} \tag{4.5}$$

where $h \in \mathbf{L}^2(X, \mathcal{F}, \mu)$ and $M \in \mathfrak{H}(\mu)$ is defined by $M(A) = \int_A h(x)\mu(dx)$.

Proof. Using (4.3) we have for $\psi \in \mathbf{L}^2(\Omega, \mathcal{C}, \mathbb{P})$ and $A \in \mathcal{F}^{\text{fin}}$:

$$(T_\mu\psi)(A) = \int_A (V_\mu^*\psi)(x)\mu(dx) = \langle V_\mu^*\psi, 1_A\rangle_\mu = \langle \psi, W_A^{(\mu)}\rangle_{\mathbb{P}} = \mathbb{E}(\psi W_A^{(\mu)}).$$

Furthermore,

$$T_\mu^* M = V_\mu \nabla_\mu M = V_\mu h = \int_X h(x) dW_x^{(\mu)}. \tag{4.6}$$

□

Corollary 4.4. *It holds that*

$$T_\mu^*(K^{(\mu)}(\cdot, A)) = W_A^{(\mu)}, \quad A \in \mathcal{F}^{\text{fin}}. \tag{4.7}$$

Proof. This is a special case of (4.6), with $M(A) = K^{(\mu)}(\cdot, A)$, corresponding to $h(x) = 1_A(x)$. □

In Theorem 3.2 we have defined a new kind of derivative, that allows us to give a precise characterization of V_μ^*, which has the flavor of a derivative operator, and is presented in the following corollary:

Corollary 4.5. *The adjoint of the map V_μ is given by*

$$V_\mu^*\psi = \nabla_\mu \mathbb{E}(\psi W_\cdot^{(\mu)}), \tag{4.8}$$

which we will also write as

$$(V_\mu^*\psi)(x) = \frac{d\mathbb{E}(\psi W_\cdot^{(\mu)})}{d\mu}(x), \tag{4.9}$$

the precise meaning of this expression being given in terms of the operator ∇_μ.

Remark 4.6. It follows from the above and from Proposition 4.2 that

$$\int_X (\nabla_\mu^*\psi)(x) dW_x^{(\mu)} = \mathbb{E}(\psi|\mathcal{C}_\mu) \tag{4.10}$$

holds for all $\psi \in \mathbf{L}^2(\Omega, \mathcal{C}, \mathbb{P})$. Hence, (4.10) justifies calling $\psi \mapsto V_\mu^*\psi$ an Itô derivative.

We now interpret some of the previous results in terms of a conditional expectation in the underlying probability space $\mathbf{L}^2(\Omega, \mathcal{C}, \mathbb{P})$.

Proposition 4.7. *Let $\mathcal{C}_\mu$ denote the sigma-algebra generated by the random variables $W_A^{(\mu)}$, $A \in \mathcal{F}^{\text{fin}}$. Then,*

$$T_\mu^* T_\mu = \mathbb{E}(\cdot|\mathcal{C}_\mu). \tag{4.11}$$

Proof. In view of (4.3), and since ∇_μ is unitary, it is enough to show that

$$V_\mu V_\mu^* \psi = \mathbb{E}(\psi|\mathcal{C}_\mu), \quad \psi \in \mathbf{L}^2(\Omega, \mathcal{C}, \mathbb{P}), \tag{4.12}$$

i.e. the $\mathcal{C}_\mu$ conditional expectation. This in turn is equivalent to verify that

$$\langle V_\mu V_\mu^* \psi, \int_X f(x) dW_x^{(\mu)} \rangle_{\mathbb{P}} = \mathbb{E}\Big(\psi \Big(\overline{\int_X f(x) dW_x^{(\mu)}}\Big)\Big), \quad \forall f \in \mathbf{L}^2(X, \mathcal{F}, \mathbb{P}), \tag{4.13}$$

i.e.

$$\langle V_\mu V_\mu^* \psi, V_\mu f \rangle_{\mathbb{P}} = \mathbb{E}_P\Big(\psi\Big(\overline{\int_X f(x) dW_x^{(\mu)}}\Big)\Big). \tag{4.14}$$

We can restrict f to be of the form 1_A with $A \in \mathcal{F}^{\text{fin}}$. Since V_μ is an isometry (4.14) becomes equivalent to

$$\int_A (V_\mu^* \psi)(x) \mu(dx) = \mathbb{E}(\psi W_A).$$

Applying ∇_μ on both sides we get

$$V_\mu^* \psi = \nabla_\mu (\mathbb{E}(\psi W_A)),$$

which is nothing but $V_\mu^* = \nabla_\mu T_\mu$, which holds in view of (4.3). □

Remark 4.8. We set $Q_\mu = V_\mu V_\mu^*$. Note that

$$Q_\mu V_\mu = V_\mu. \tag{4.15}$$

5. TRANSITION-PROBABILITY SYSTEMS

There are diverse approaches to the following general question: Given some stochastic data, then find an appropriate probability space $(\Omega, \mathcal{C}, \mathbb{P})$ that realizes what is needed for the particular data at hand. Below we make this precise and we offer a brief outline with citations, especially [17, 21]. By probability space $(\Omega, \mathcal{C}, \mathbb{P})$ we mean a triple consisting of a sample set Ω, a sigma-algebra of events, and a probability measure $\mathbb{P}$ defined on $\mathcal{C}$. Of the following four approaches to the problem, for our present purpose, number (ii) is best suited. The list of four is: (i) via Kolmogorov consistency, (ii) via Gaussian Hilbert space, (iii) via transition kernels, and with the use of (iv) generalized Gelfand triples. While for many purposes, the Kolmogorov consistency construction (i) is more constructive; here (ii) is better, i.e., via (ii) we obtain a probability space $(\Omega, \mathcal{C}, \mathbb{P})$ from the following Gaussian Hilbert space construction: Starting with a Hilbert space $\mathcal{H}$, we select a realization of the vectors $h \in \mathcal{H}$ as a canonical Gaussian process W_h. Hence, the realization of a Gaussian Hilbert space in some $(\Omega, \mathcal{C}, \mathbb{P})$ has its associated covariance kernel equal to the inner product from $\mathcal{H}$. Here we may use construction (ii) on the canonical and universal Hilbert space in the sense of Nelson and Schwartz. We recall that this universal Hilbert space is a Hilbert space of specific equivalence classes of pairs.

Discussion of (ii): For our present applications, we begin with a given generalized measure space $(X, \mathcal{F})$, where $\mathcal{F}$ is a prescribed sigma-algebra. Consider systems of positive measures (μ) . We note that the positive measures μ will be based on the same $\mathcal{F}$, but of course the ring $\mathcal{F}^{\text{fin}}(\mu)$ will depend on μ. Of the books covering Gaussian Hilbert space and their applications, we stress [17]. Summary of details for the construction leading from $\mathcal{H}$ to $(\Omega, \mathcal{C}, \mathbb{P})$ has: (a) We let $\mathcal{H} = \mathfrak{H}_{(X,\mathcal{F})}$ to be the universal Hilbert space in the sense of Nelson and Schwartz, and then: (b), via an associated system of Itô-isometries, we pass to a choice of a "universal" $(\Omega, \mathcal{C}, \mathbb{P})$ probability space. Specializing to two μ-Brownian motions, say $W^{(\mu_i)}$, $i = 1, 2$, for $(\Omega, \mathcal{C}, \mathbb{P})$ they will be independent if and only if the two measures μ_i are mutually singular.

In brief summary, the remaining two approaches are as follows: (iii) Fix a system (μ), create an associated system of transition kernels, and then construct $(\Omega, \mathcal{F}, \mathbb{P})$ from the combined Markov kernels. For other purposes, of course, we have (iv) Gelfand triple constructions, see e.g. [6]. For completeness we recall the construction of the universal Hilbert space; see [17].

Definition 5.1. Given a fixed measure space $(X, \mathcal{F})$, the associated universal Hilbert space $\mathfrak{H}_{(X,\mathcal{F})}$ consists of equivalence classes of pairs (f, μ), where μ is a positive measure on $(X, \mathcal{F})$ and $f \in \mathbf{L}^2(X, \mathcal{F})$. One says that $(f_1, \mu_1) \sim (f_2, \mu_2)$ if there exists a positive measure ν on $(X, \mathcal{F})$ such that $\mu_1 \ll \nu$ and $\mu_2 \ll \nu$ and

$$f_1 \sqrt{\frac{\mathrm{d}\mu_1}{\mathrm{d}\nu}} = f_2 \sqrt{\frac{\mathrm{d}\mu_2}{\mathrm{d}\nu}}, \quad \nu \ a.e. \tag{5.1}$$

It is known (see [17]) that (5.1) is indeed an equivalence relation, and an equivalence class for this relation will be denoted by $f\sqrt{\mu}$. The set of equivalence classes endowed with the norm $\|f\sqrt{\mu}\|^2_{\mathfrak{H}_{(X,\mathcal{F})}} = \int_X |f(x)|^2 \mu(dx)$ where $(f, \mu) \in f\sqrt{\mu}$ is a Hilbert space. We denote by $\mathbf{L}^2(\Omega, \mathcal{C}, \mathbb{P})$ the associated universal probability space, constructed as follows: One considers an orthonormal basis $(e_a)_{a \in A}$ of $\mathfrak{H}_{(X,\mathcal{F})}$ and build

$$\Omega = \prod_{a \in A} \left(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right),$$

endowed with the cylinder algebra; see [19, pp. 38–39].

Theorem 5.2. *The Itô integrals pass through the equivalence relation, meaning that the map*

$$f\sqrt{\mu} \quad \mapsto \quad V_\mu f \in \mathbf{L}^2(\Omega, \mathcal{C}, \mathbb{P})$$

is a well defined isometry from the universal Hilbert space into the associated universal probability space.

Proof. It holds that

$$\begin{aligned}\int_X f_1(x) dW_x^{(\mu_1)} &= \int_X f_1(x)\sqrt{\frac{\mathrm{d}\mu_1}{\mathrm{d}\nu}}(x) dW_x^{(\nu)}\\ &= \int_X f_2(x)\sqrt{\frac{\mathrm{d}\mu_2}{\mathrm{d}\nu}}(x) dW_x^{(\nu)} = \int_X f_2(x) dW_x^{(\mu_2)}.\end{aligned}$$

□

As a corollary we have:

Corollary 5.3. *Given two sigma-finite measures μ_1 and μ_2 on X. The following are equivalent:*

(1) *μ_1 and μ_2 are mutually singular,*
(2) *the corresponding μ-Brownian motions $W^{(\mu_1)}$ and $W^{(\mu_2)}$ are independent.*

Theorem 5.4 (transition probability systems)**.** *Using notation* (4.9) *we have:*

$$V_{\mu_1}^*(W_B^{(\mu_2)})(x) = \frac{\mathrm{d}\mathbb{E}(W_\cdot^{(\mu_1)} W_B^{(\mu_2)})}{\mathrm{d}\mu_1}(x), \tag{5.2}$$

$$V_{\mu_2}^*(W_A^{(\mu_1)})(y) = \frac{\mathrm{d}\mathbb{E}(W_\cdot^{(\mu_2)} W_A^{(\mu_1)})}{\mathrm{d}\mu_2}(y). \tag{5.3}$$

Proof. The first formula is a special case of (4.8) with $\mu = \mu_1$ and $\psi = W_B^{(\mu_2)}$. The second formula interchanges the indices 1 and 2. □

Notation 5.5. We set

$$\frac{\mathrm{d}\mathbb{E}(W_\cdot^{(\mu_1)} W_B^{(\mu_2)})}{\mathrm{d}\mu_1}(x) = P(x, B), \tag{5.4}$$

$$\frac{\mathrm{d}\mathbb{E}(W_\cdot^{(\mu_2)} W_A^{(\mu_1)})}{\mathrm{d}\mu_2}(y) = Q(y, A). \tag{5.5}$$

Theorem 5.6. *It holds that*

$$\left(V_{\mu_1}^* V_{\mu_2} f_2\right)(x) = \int_X P(x, dy) f_2(y), \quad f_2 \in \mathbf{L}^2(X, \mathcal{F}, \mu_2), \tag{5.6}$$

$$\left(V_{\mu_2}^* V_{\mu_1} f_1\right)(y) = \int_X Q(y, dx) f_1(x), \quad f_1 \in \mathbf{L}^2(X, \mathcal{F}, \mu_1). \tag{5.7}$$

Proof. See Figure 2. It is enough to prove these formulas with $f_2 = 1_B$ in the first case and $f_1 = 1_A$ in the second case. (5.6) reduces then to (5.4). Formula (5.7) follows in a similar way from (5.5). $\square$

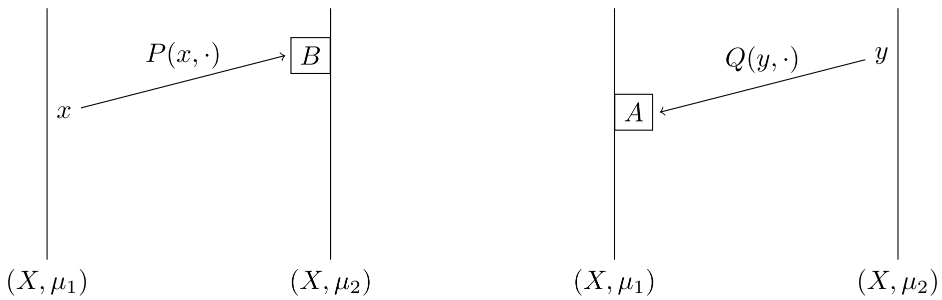

Fig. 2. The interpretation of action of $P(x,\cdot)$ and $Q(y,\cdot)$

Theorem 5.7 (reversibility)**.** *In the above notations, the following holds:*

$$\int_A P(x,B)\mu_1(dx) = \int_B Q(y,A)\mu_2(dy) = \mathbb{E}(W_A^{(\mu_1)}W_B^{(\mu_2)}), \quad A,B \in \mathcal{F}^{\text{fin}}. \tag{5.8}$$

Proof. This is just an application of (3.3) to the functions M and N defined by $M(A) = N(B) = \mathbb{E}(W_A^{(\mu_1)}W_B^{(\mu_2)})$. $\square$

We now study a related Markov property, and begin with a definition. In the statement, and as in Proposition 4.7, we denote by $\mathcal{C}_\mu$ the sigma-algebra generated by the μ-Brownian motion.

Definition 5.8. Let μ_2 and μ_3 be two sigma-finite positive measures on $(X,\mathcal{F})$. We say that the transition $\mu_2 \longrightarrow \mu_3$ is *anticipating* if

$$\mathcal{C}_{\mu_3} \subset \mathcal{C}_{\mu_2}. \tag{5.9}$$

With Q_μ as in Remark 4.8 we can rewrite (5.9) in terms of orthogonal projections as

$$Q_{\mu_3} = Q_{\mu_2}Q_{\mu_3}. \tag{5.10}$$

We set for $f_2 \in \mathbf{L}^2(X,\mathcal{F},\mu_2)$

$$(V_1^*V_2f_2)(x) = \int_X P_{1\mapsto 2}(x,dy)f_2(y), \quad \mu_1\, a.e.$$

see (5.6), and similarly for other indices.

Theorem 5.9. *Given three positive sigma-finite measures μ_1, μ_2, μ_3 on X the following are equivalent:*

(1) *the transition equation*

$$P_{1\mapsto 3}(x,B) = \int_X P_{1\mapsto 2}(x,dy)P_{2\mapsto 3}(y,B), \quad B \in \mathcal{F}^{\mathrm{fin}}, \tag{5.11}$$

holds,

(2) *μ_3 is anticipating μ_2.*

Proof. The Markov property (5.11) follows from the operator identity

$$(V_1^* V_2)(V_2^* V_3) = V_1^* V_3, \tag{5.12}$$

which we rewrite as

$$V_1^* Q_2 Q_3 V_3 = V_1^* Q_3 V_3.$$

It is immediate that the converse implication holds as well. □

6. ADJOINT OF THE COMPOSITION MAP

We now go back to the example presented in the introduction. The setting consists of a measure space $(X, \mathcal{F})$, an endomorphism σ of X, and a sigma-finite measure μ which is σ-invariant (see (1.3)). We compute the adjoint of the composition map (1.4) using the Krein–Feller derivative.

Theorem 6.1. *The adjoint of the operator S is given by the Krein–Feller derivative of the map*

$$A \mapsto M_g(A) = \int_X 1_A(\sigma(x))g(x)\mu(dx). \tag{6.1}$$

Proof. Let $f \in \mathbf{L}^2(X, \mathcal{F}, \mu)$. The composition map S is isometric and therefore there exists $h \in \mathbf{L}^2(X, \mathcal{F}, \mu)$ such that

$$\int_X f(\sigma(x))\overline{g(x)}\mu(dx) = \int_X f(x)\overline{h(x)}\mu(dx). \tag{6.2}$$

Taking conjugate and setting $f = 1_A$ with $A \in \mathcal{F}^{\mathrm{fin}}$ we have:

$$\int_X 1_A(\sigma(x))g(x)\mu(dx) = \int_A h(x)\mu(dx). \tag{6.3}$$

If follows from Theorem 3.2 that the function (6.1) belongs to $\mathfrak{H}(\mu)$ and has Krein–Feller derivative h. □

Remark 6.2. In an informal way one sometimes uses the notation $(g\mu)\circ\sigma^{-1}$ for the map M_g, and the adjoint is given by the formula

$$S^*g = \nabla_\mu \left((g\mu)\circ\sigma^{-1}\right). \tag{6.4}$$

Remark 6.3. We now check directly that the map (6.1) satisfies (3.2). To that purpose, let $N\in\mathbb{N}$ and let $A_1,\ldots,A_N$ be non-intersecting elements of $\mathcal{F}^{\text{fin}}$. We note that

$$1_A(\sigma(x)) = 1_{\sigma^{-1}(A)}(x)$$

and rewrite M_g as $M_g(A) = \int_X 1_{\sigma^{-1}(A)}(1_{\sigma^{-1}(A)}g(x))\mu(dx)$. We then obtain from Cauchy–Schwarz inequality

$$\begin{aligned}|M_g(A_n)|^2 &\le \mu(\sigma^{-1}(A_n)) \int\limits_{\sigma^{-1}(A_n)} |g(x)|^2\mu(dx)\\ &= \mu(A_n) \int\limits_{\sigma^{-1}(A_n)} |g(x)|^2\mu(dx)\end{aligned}$$

since μ is σ-invariant. Furthermore, the sets $\sigma^{-1}(A_n)$ are disjoints since the A_n are pairwise disjoints. Hence,

$$\begin{aligned}\sum_{n=1}^N \frac{|M_g(A_n)|^2}{\mu(A_n)} &\le \int\limits_{\sigma^{-1}(A_n)} \sum_{n=1}^N \frac{\mu(A_n)}{\mu(A_n)}|g(x)|^2\mu(dx)\\ &\le \int\limits_{\bigcup_{n=1}^N \sigma^{-1}(A_n)} |g(x)|^2\mu(dx) \le \|g\|_2.\end{aligned}$$

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REFERENCES

[1] S. Albeverio, L. Nizhnik, *A Schrödinger operator with a δ'-interaction on a Cantor set and Krein–Feller operators*, Math. Nachr. **279** (2006), no. 5–6, 467–476.

[2] D. Alpay, *On linear combination of positive functions, associated reproducing kernel spaces and a non hermitian Schur algorithm*, Arch. Math. (Basel) **58** (1992), 174–182.

[3] D. Alpay, *A theorem on reproducing kernel Hilbert spaces of pairs*, Rocky Mountain J. Math. **22** (1992), 1243–1258.

[4] D. Alpay, P. Jorgensen, *New characterizations of reproducing kernel Hilbert spaces and applications to metric geometry*, Opuscula Math. **41** (2021), no. 3, 283–300.

[5] D. Alpay, P. Jorgensen, *mu-Brownian motion, dualities, diffusions, transforms, and reproducing kernel spaces*, J. Theoret. Probab. **35** (2022), no. 4, 2757–2783.

[6] D. Alpay, P. Jorgensen, D. Levanony, *On the equivalence of probability spaces*, J. Theoret. Probab. **30** (2017), no. 3, 813–841.

[7] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[8] N. Aronszajn, *Quadratic forms on vector spaces*, Proc. Internat. Sympos. Linear Spaces, Jerusalem, 1960, 29–87.

[9] S.D. Chatterji, *Les martingales et leurs applications analytiques*, [in:] *École d'Été de Probabilités: Processus Stochastiques (Saint Flour, 1971)*, Lecture Notes in Math., vol. 307, Springer Verlag, 1973, 27–164.

[10] H. Dym, H.P. McKean, *Gaussian Processes, Function Theory and the Inverse Spectral Problem*, Probability and Mathematical Statistics, vol. 31, Academic Press, New York–London, 1976.

[11] W. Feller, *On boundaries defined by stochastic matrices*, Proc. Sympos. Appl. Math. **7** (1957), 35–40.

[12] U. Freiberg, *Refinement of the spectral asymptotics of generalized Krein Feller operators*, Forum Math. **23** (2011), no. 2, 427–445.

[13] T. Hida, *White noise analysis and applications in random fields*, [in:] *Dirichlet forms and stochastic processes (Beijing, 1993)*, de Gruyter, Berlin, 1995, 185–189.

[14] K. Itô, *Stochastic Processes*, Springer-Verlag, Berlin, 2004.

[15] P. Jorgensen, F. Tian, *Reproducing kernels and choices of associated feature spaces, in the form of L^2-spaces*, J. Math. Anal. Appl. **505** (2022), no. 2, Article no. 125535.

[16] L.A. Minorics, *Spectral asymptotics for Krein–Feller operators with respect to V-variable Cantor measures*, Forum Math. **32** (2020), no. 1, 121–138.

[17] E. Nelson, *Topics in Dynamics I: Flows*, Mathematical Notes, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1969.

[18] E. Nelson, *Stochastic mechanics of particles and fields*, [in:] *Quantum Interaction*, Lecture Notes in Comput. Sci., vol. 8369, Springer, Heidelberg, 2014, 1–5.

[19] J. Neveu, *Processus aléatoires gaussiens*, Number 34 in Séminaires de Mathématiques Supérieures, Les Presses de l'Université de Montréal, 1968.

[20] F. Riesz, *Untersuchungen über Systeme integrierbarer Funkionen*, Math. Ann. **69** (1910), 449–497.

[21] L. Schwartz, *Probabilités cylindriques et fonctions aléatoires*, [in:] *Séminaire Laurent Schwartz 1969–1970: Applications radonifiantes, Exp. No. 6*, École Polytechnique, 1970, 1–8.

Daniel Alpay (corresponding author)
alpay@chapman.edu
https://orcid.org/0000-0002-7612-3598

Schmid College of Science and Technology
Chapman University
One University Drive Orange, California 92866, USA

Palle Jorgensen
palle-jorgensen@uiowa.edu
https://orcid.org/0000-0003-2681-5753

The University of Iowa
Department of Mathematics
14C McLean Hall, Iowa City, IA 52246, USA

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