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## ANALYSIS OF THE PARALLEL RESONANCE CIRCUIT WITH SUPERCAPACITOR

The article presents an analysis of phase resonance conditions for the parallel  $RLC_\alpha$  circuit with supercapacitor. Supercapacitors behaviour somehow differs from classic dielectric capacitors, therefore it is necessary to develop their new mathematical models, using fractional – order differential equations. A simple fractional – order supercapacitor model has been assumed, taking into account its internal equivalent series resistance ESR too. A lossless inductance has also been assumed and relations for equivalent admittance and resonance occurrence conditions have been derived in the considered parallel  $RLC_\alpha$  circuit with supercapacitor. The analysis was conducted for different fractional-order parameter  $\alpha$  values. Derived relations have been illustrated by simulation examples. The existence of resonance frequency depends largely on the supercapacitor series resistance value.

KEYWORDS: phase and magnitude resonance, parallel  $RLC_\alpha$  circuit, supercapacitor.

### 1. INTRODUCTION

It is known from the classic circuit theory that the resonance frequency  $f_r$  of  $RLC$  circuits including real inductors and capacitors differs (compare Figs. 1 b, c, d) from the resonance frequency of an ideal resonance  $LC$  circuit. Depending on the resistance value in the considered circuit (see Fig. 1) resonance phenomenon may but does not have to occur. This issue becomes more complicated in systems containing supercapacitors [1 - 3] and fractional – order elements  $L_\beta, C_\alpha$  [4].

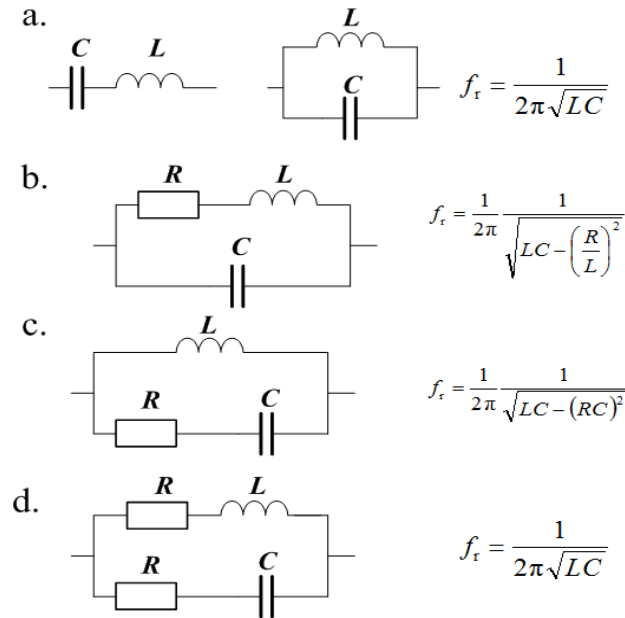
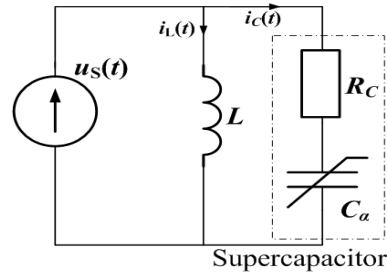
Analysis of properties of a series resonance  $RLC_\alpha$  circuit with supercapacitor was carried out in [5 - 6]. The article is a continuation of earlier works concerning studies on resonance phenomena in systems with fractional – order elements [7]. It concerns a parallel circuit with lossless inductance and a supercapacitor modelled as a fractional – order element.

### 2. MODEL OF THE SYSTEM

Model of the analyzed parallel  $RLC_\alpha$  circuit with supercapacitor in frequency domain is shown in Fig. 2.

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Fig. 1. Resonance frequencies  $f_r$  for selected simple parallel  $RLC$  circuitsFig. 2. Model of the parallel  $RLC_\alpha$  circuit with supercapacitor

In system from Fig. 2 a voltage source  $u_s(t)$  is included, which consists of both constant voltage source  $u_0$ , polarizing the supercapacitor and alternating voltage source  $u(t)$  of adjustable frequency. Voltage source supplying concerned circuit can be written in time domain as:

$$u_s(t) = u_0 + u(t) = u_0 + \sqrt{2}|U|\sin(\omega t + \phi), \quad (1)$$

where:  $|U|$  - RMS value of the alternating component,  $\phi$  - voltage phase.

The circuit admittance is given by a relation:

$$Y(j\omega) = \frac{1}{j\omega L} + \frac{1}{R_C + \frac{1}{(j\omega)^\alpha C}}. \quad (2)$$

Developing formula (2), the real and imaginary parts of the admittance can be written as:

$$\operatorname{Re}\{Y(j\omega)\} = \frac{\omega^{2\alpha} C^2 R_c + \omega^\alpha C \cos\left(\frac{\pi}{2}\alpha\right)}{\omega^{2\alpha} C^2 R_c^2 + 2\omega^\alpha C R_c \cos\left(\frac{\pi}{2}\alpha\right) + 1}, \quad (3)$$

and:

$$\operatorname{Im}\{Y(j\omega)\} = -\frac{\left(\omega^{2\alpha} C^2 R_c^2 + 2\omega^\alpha C R_c \cos\left(\frac{\pi}{2}\alpha\right) - \omega L \omega^\alpha C \sin\left(\frac{\pi}{2}\alpha\right) + 1\right)}{\omega L \left(\omega^{2\alpha} C^2 R_c^2 + 2\omega^\alpha C R_c \cos\left(\frac{\pi}{2}\alpha\right) + 1\right)}. \quad (4)$$

Subscribing the equivalent admittance  $Y(j\omega)$  of the system from Fig. 2 in exponential form, its module  $|Y(j\omega)|$  and phase  $\varphi(\omega)$  are:

$$|Y(j\omega)| = \frac{\sqrt{\left(\omega L\right)^2 \left(\omega^{2\alpha} C^2 R_c + \omega^\alpha C \cos\left(\frac{\pi}{2}\alpha\right)\right)^2 + \left(\omega^{2\alpha} C^2 R_c^2 + 2\omega^\alpha C R_c \cos\left(\frac{\pi}{2}\alpha\right) + 1\right)^2}}{\omega L \left(\omega^{2\alpha} C^2 R_c^2 + 2\omega^\alpha C R_c \cos\left(\frac{\pi}{2}\alpha\right) + 1\right)}, \quad (5)$$

$$\varphi(\omega) = \operatorname{arctg} \frac{\left(\omega^{2\alpha} C^2 R_c^2 + 2\omega^\alpha C R_c \cos\left(\frac{\pi}{2}\alpha\right) - \omega L \omega^\alpha C \sin\left(\frac{\pi}{2}\alpha\right) + 1\right)}{\omega L \left(\omega^{2\alpha} C^2 R_c^2 + \omega^\alpha C \cos\left(\frac{\pi}{2}\alpha\right)\right)}. \quad (6)$$

Derived relations for equivalent admittance of the parallel  $RLC_\alpha$  circuit with supercapacitor have been simulated for exemplary parameters and illustrated in Figs. 4 – 5. Illustrations are presented in the following sections.

### 3. ANALYSIS OF PHASE RESONANCE CONDITIONS

The phase resonance frequency  $f_r$  in a parallel  $RLC_\alpha$  circuit with supercapacitor can be determined from the general phase resonance condition  $\operatorname{Im}\{Y(j\omega)\} = 0$ . Then, the non-linear equation given by formula (7) must be solved:

$$\omega^{2\alpha} C^2 R_c^2 - \omega^{\alpha+1} L C \sin\left(\frac{\pi}{2}\alpha\right) + 2\omega^\alpha C R_c \cos\left(\frac{\pi}{2}\alpha\right) + 1 = 0. \quad (7)$$

Missing supercapacitor internal series resistance ESR (or introducing its large capacitance, of thousands F, then its ESR is small, of about 0,1  $\Omega$ ) the resonance frequency  $f_r$  can be written as ( $R_C = 0$ ):

$$f_r = \frac{1}{2\pi^{1+\alpha}} \sqrt{\frac{1}{LC} \frac{1}{\sin\left(\frac{\pi}{2}\alpha\right)}}. \quad (8)$$

For  $\alpha = 1$ , it means for lossless, ideal capacitor, formula (8) takes the form of:

$$f_r = \frac{1}{2\pi} \frac{1}{\sqrt{LC}}, \quad (9)$$

which describes the resonance frequency for a classic series and parallel  $RLC$  ( $LC$ ) circuit (see Fig. 1a). Graph from Fig. 3 presents relation (8), that is the resonance frequency as a function of fractional – order coefficient  $\alpha$  for selected values of  $1/LC$  factor.

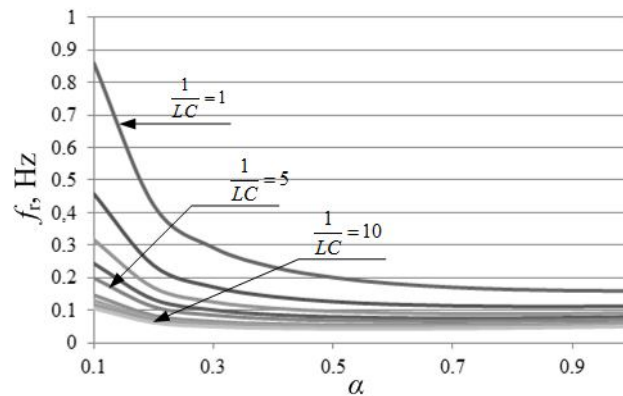


Fig. 3. Graph of the function  $f_r(\alpha)$  based on the formula (8) for  $\alpha \in \langle 0,1 \rangle$

In case of the series internal resistance ESR appearing in the parallel circuit, there can also occur a special case, when  $\alpha = 1$ . Then equation (7) can be converted to a closed form formula for the resonance frequency  $f_r$ :

$$f_r = \frac{1}{2\pi} \frac{1}{\sqrt{LC - (CR_C)^2}}, \quad (10)$$

which agrees with the formula of the classic system from Fig. 1c. In general case, when  $R_C \neq 0$  and  $\alpha \in \langle 0,1 \rangle$  equation (7) can be solved numerically for specified values of parallel  $RLC_\alpha$  circuit parameters, by transforming and looking for the intersection of two functions:

$$f_1(\omega) = \omega^{2\alpha} C^2 R_C^2 - \omega^{\alpha+1} LC \sin\left(\frac{\pi}{2}\alpha\right) + 2\omega^\alpha CR_C \cos\left(\frac{\pi}{2}\alpha\right), \quad (11)$$

and:

$$f_2(\omega) = -1. \quad (12)$$

As it turns out, the second phase resonance condition  $\text{Im}\{Z(j\omega)\} = 0$  leads to an identical form of non-linear equation, as equation (7). It means that for a given parallel  $RLC_\alpha$  circuit with supercapacitor, there is only one phase resonance frequency.

In the following section an example of parallel  $RLC_\alpha$  circuit with supercapacitor has been calculated and based on it, graphical method of determining the resonance frequency has been illustrated.

#### 4. EXAMPLE

Based on previous studies, simulations on exemplary parallel  $RLC_\alpha$  circuit with supercapacitor, modelled as a fractional-order parameter, have been performed. Parameters of this circuit are: the inductance  $L = 500$  mH and the supercapacitor with a nominal capacity  $C = 10$  F and resistance  $R_C = 0,18 \Omega$  [8]. The derived admittance relations (1 – 4) have been presented in graphs in Figs. 4 - 5.

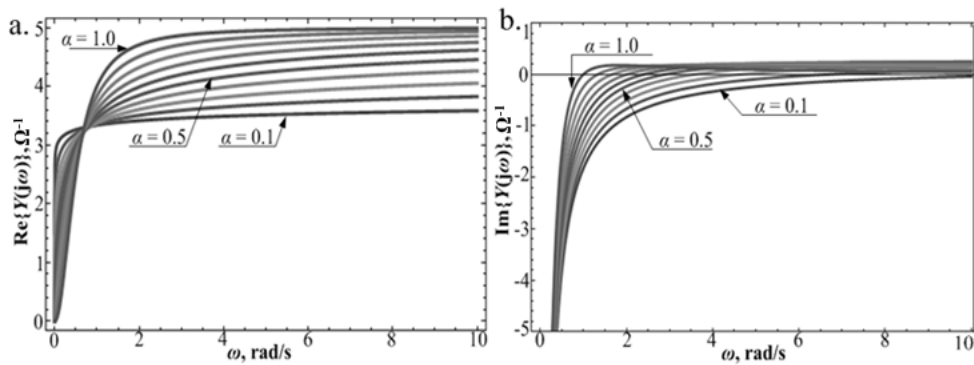


Fig. 4. Graphs of a function a.  $\text{Re}\{Y(j\omega)\}$  and b.  $\text{Im}\{Y(j\omega)\}$  based on formulas (3) and (4) for  $\alpha \in \langle 0,1 \rangle$

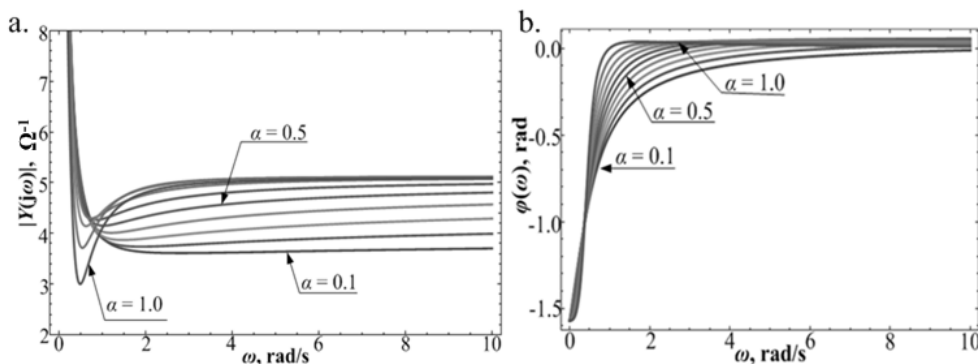


Fig. 5. Graphs of a function a.  $|Y(j\omega)|$  and b.  $\varphi(\omega)$  based on formulas (5) and (6) for  $\alpha \in \langle 0,1 \rangle$

It can be noticed from Fig. 4b, that the imaginary part of the admittance reaches zero value for a given phase resonance radial frequency. Designation of a specified value of resonance radial frequency is possible by solving equation (7) numerically. Fig. 6 presents a graphical way of finding the solution, described in previous section (see formulas (11) and (12)). Fig. 7 shows a resonance radial frequency graph of a system from Fig. 1 as a function of parameter  $\alpha$ . The graph indicates that depending on the supercapacitor resistance  $R_C$  value, resonance does not always occur.

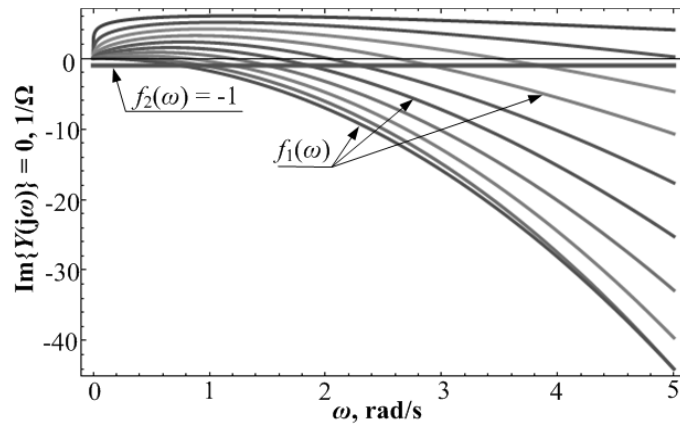


Fig. 6. Graphical way of finding the resonance radial frequency  $\omega_r$

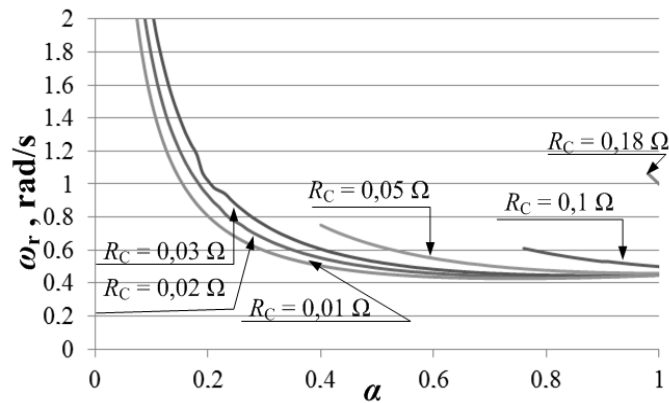


Fig. 7. Dependence of resonance radial frequency as a function of coefficient  $\alpha$  for selected values of  $R_C$

For maximum nominal value of  $R_C = 0,18 \Omega$ , phase resonance phenomenon can occur when the supercapacitor leakage losses are vanishingly small, that is, when  $\alpha \rightarrow 1$ . For a lower ESR resistance, resonance occurrence is also possible for smaller values of the coefficient  $\alpha$ .

## 5. CONCLUSION

The article analyzes phase resonance conditions in a parallel  $RLC_\alpha$  circuit with lossless inductor and a supercapacitor, modelled as a fractional – order electrical element. Relations for equivalent admittance have been derived and from the general resonance condition, relations describing resonance (radial) frequency have been derived too. In specific cases, formulas reduce to those describing classic parallel  $RLC$  circuit (of integer order). In general, values of supercapacitor nominal capacitance  $C$ , its internal series resistance  $R_C$ , fractional-order coefficient  $\alpha$ , and the inductance  $L$  have an impact on the possibility of resonance occurrence. Too high value of series resistance ESR may prevent the resonance occurrence in the parallel  $RLC_\alpha$  circuit with supercapacitor.

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