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ON 1-ROTATIONAL DECOMPOSITIONS OF COMPLETE GRAPHS INTO TRIPARTITE GRAPHS

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Abstract. Consider a tripartite graph to be any simple graph that admits a proper vertex coloring in at most 3 colors. Let G be a tripartite graph with n edges, one of which is a pendent edge. This paper introduces a labeling on such a graph G used to achieve 1-rotational G-decompositions of K_{2nt} for any positive integer t. It is also shown that if G with a pendent edge is the result of adding an edge to a path on n vertices, then G admits such a labeling.

Keywords: graph decomposition, 1-rotational, vertex labeling.

Mathematics Subject Classification: 05C78, 05C51.

1. INTRODUCTION

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \varnothing$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. Let K_n denote the complete graph on n vertices and let P_n denote a path on n vertices. We call a graph *tripartite* if its chromatic number is at most 3 (not just strictly equal to 3 as is also a common use of this term). If v is a vertex of a graph G, then we use $G \ominus v$ to denote the induced subgraph on $V(G) \setminus \{v\}$. For $A, B \subseteq \mathbb{N}$ with $a < b$ for all $a \in A$ and $b \in B$, we say $A < B$. We similarly define $A \leq B$ to mean $a \leq b$ for all $a \in A$ and $b \in B$. If $A \leq B$ with either $A = \{a\}$ or $B = \{b\}$, then we simply write $a \leq B$ or $A \leq b$, respectively.

1.1. LABELINGS FOR CYCLIC DECOMPOSITIONS

Let G and H be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Gamma = \{G_1, G_2, \ldots, G_t\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. If a G-decomposition of H exists, we also say that G *decomposes* H. Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. The *length* of an edge $\{i,j\} \in E(G)$ is defined as $\min\{|i-j|, k-|i-j|\}$. By *clicking* G, we mean applying the permutation $i \mapsto i+1$ to $V(G)$. A G-decomposition Γ of K_k is *cyclic* if clicking is an automorphism of Γ.

For any graph G, a one-to-one function $f\colon V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [7], Rosa introduced a hierarchy of labelings. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f}\colon E(G) \to \mathbb{N} \setminus \{0\}$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u,v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{f}(E(G)) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,
(ℓ2) $f(V(G)) \subseteq [0, n]$,
(ℓ3) $\bar{f}(E(G)) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1,n]$ either $x_i = i$ or $x_i = 2n+1-i$,
(ℓ4) $\bar{f}(E(G)) = [1, n]$.

If in addition G is bipartite with vertex bipartition $\{A, B\}$, consider also the following conditions established in [5]:

(ℓ5) for each $\{a,b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,
(ℓ6) there exists an integer λ such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1) and (ℓ3) is called a *ρ-labeling*;
(ℓ1) and (ℓ4) is called a *σ-labeling*;
(ℓ2) and (ℓ4) is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. Suppose G is bipartite. If a ρ-, σ-, or β-labeling of G satisfies condition (ℓ5), then the labeling is called *ordered* and is denoted by ρ^+, σ^+, or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is called *uniformly-ordered* and is denoted by ρ^{++}, σ^{++}, or β^{++}, respectively.

A β-labeling is better known as a *graceful labeling* and a uniformly-ordered β-labeling is an *α-labeling* as introduced in [7]. Labelings of the types above are called *Rosa-type labelings* because of Rosa's original article [7] on the topic (see [4] for a survey of Rosa-type labelings). A dynamic survey on general graph labelings is maintained by Gallian [6].

We present here some results on Rosa-type labelings of certain trees and forests that are useful for the application seen in this paper. First, we define a *caterpillar* to be any tree where the induced subgraph on the vertices not of degree 1 is a path. Second, a *linear forest* is a forest where all components are paths. The following two results are from [7] and [5], respectively.

Theorem 1.1. *Every caterpillar admits an α-labeling.*

Theorem 1.2. *If G is the vertex-disjoint union of graphs that separately admit α-labelings, then G admits a σ^+-labeling.*

Since all paths are caterpillars, the following result on linear forests naturally follows:

Corollary 1.3. *Every linear forest admits a σ^+-labeling.*

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [7] and [5], respectively.

Theorem 1.4. *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G admits a ρ-labeling.*

Theorem 1.5. *Let G be a bipartite graph with n edges. If G admits a ρ^+-labeling, then there exists a cyclic G-decomposition of K_{2nt+1} for any positive integer t.*

An obvious advantage of the more restrictive ρ^+-labeling is that it leads to infinitely many decompositions. A loosening of the bipartite restriction can be found in [1] where a definition for a γ-labeling, which yields results similar to those found in Theorem 1.5, was given. Both γ- and ρ^+-labelings were eventually subsumed by the following labeling introduced in [2]. Let G be a tripartite graph with n edges having the vertex tripartition $\{A, B, C\}$. A *ρ-tripartite labeling* of G is a one-to-one function $h\colon V(G) \to [0, 2n]$ that satisfies the following:

(r1) h is a ρ-labeling of G;
(r2) if $\{a, v\} \in E(G)$ with $a \in A$, then $h(a) < h(v)$;
(r3) if $e = \{b, c\} \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = \{b', c'\} \in E(G)$ with $b' \in B$ and $c' \in C$ such that

$$|h(c') - h(b')| + |h(c) - h(b)| = 2n;$$

(r4) if $b \in B$ and $c \in C$, then $|h(b) - h(c)| \neq 2n$.

The following result also appeared in [2].

Theorem 1.6. *Let G be a tripartite graph with n edges. If G admits a ρ-tripartite labeling, then there exists a cyclic G-decomposition of K_{2nt+1} for any positive integer t.*

1.2. LABELINGS FOR 1-ROTATIONAL DECOMPOSITIONS

The Rosa-type labelings described above all yield decompositions of complete graphs of orders that are 1 more than a multiple of twice the number of edges of a graph G. However, it is often desirable to find G-decompositions of complete graphs of other orders. Namely, complete graphs of orders that are a multiple of $2 \cdot \big|E(G)\big|$ often fall into the spectrum for a given graph G. One approach to finding such decompositions is through the use of a fixed point, commonly denoted with vertex label ∞, around which the rest of the vertices still act as in cyclic decompositions.

Let $V(K_k) = \mathbb{Z}_{k-1} \cup \{\infty\}$ and let G be a subgraph of K_k. The *length* of an edge $\{i, j\} \in E(G)$ where $\{i, j\} \not\ni \infty$ is (still) defined as $\min\{|i-j|, k-1-|i-j|\}$. Similarly, *clicking* G still implies applying the permutation $i \mapsto i+1$ to $V(G)$, but we now incorporate the convention that $\infty + 1 \mapsto \infty$. A G-decomposition Γ of K_k is then *1-rotational* if clicking is an automorphism of Γ.

Through the use of edge lengths and clicking, a 1-rotational decomposition can be viewed through the lens of Rosa-type labelings. Let G be a graph with n edges, no isolated vertices, and a vertex w of degree 1. A *1-rotational ρ-labeling* of G is a one-to-one function $f\colon V(G) \to [0, 2n-2] \cup \{\infty\}$ where $f(w) = \infty$ and such that f is a ρ-labeling of $G \ominus w$. The following is then a natural analogue of Theorem 1.4.

Theorem 1.7. *Let G be a graph with n edges. There exists a 1-rotational G-decomposition of K_{2n} if and only if G admits a 1-rotational ρ-labeling.*

Proof. Let $w \in V(G)$ have degree 1. Sufficiency follows directly from Theorem 1.4 and the fact that a 1-rotational ρ-labeling with $w \mapsto \infty$ induces a ρ-labeling on $G \ominus w$. To show necessity, consider a 1-rotational G-decomposition of K_{2n}, say Γ. We note that K_{2n} with vertex set $\mathbb{Z}_{2n-1} \cup \{\infty\}$ has exactly $2n-1$ edges of each length $\ell \in [1, n-1]$. Furthermore, edge length is preserved under clicking. Thus, any $G' \in \Gamma$ must consist of no more than one edge of each length $\ell \in [1, n-1]$. Since G' must have n edges to be isomorphic to G, it must have exactly one edge of each length in K_{2n} and one edge incident with ∞. Therefore, the vertices in G' induce a 1-rotational ρ-labeling of G. □

In this manuscript we further explore a 1-rotational counterpart of the ρ-tripartite labeling from [2] and give the corresponding result that yields infinitely many 1-rotational decompositions. We also show that any graph that results from adding an edge to a path admits such a 1-rotational labeling.

2. MAIN RESULTS

Let G be a tripartite graph with n edges, vertex tripartition $\{A, B, C\}$, and edge $\{u, w\}$ such that $\deg w = 1$. A *1-rotational ρ-tripartite labeling* of G is a one-to-one function $h\colon V(G) \to [0, 2n-2] \cup \{\infty\}$ that satisfies the following conditions:

(r′1) h is a 1-rotational ρ-labeling of G with $h(w) = \infty$;
(r′2) if $\{a, v\} \in E(G) \setminus \{u, w\}$ with $a \in A$, then $h(a) < h(v)$;
(r′3) if $e = \{b, c\} \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = \{b', c'\} \in E(G)$ with $b' \in B$ and $c' \in C$ such that

$$|h(c') - h(b')| + |h(c) - h(b)| = 2n.$$

Note that e and e' in (r′3) need not be distinct. Also, in order to satisfy (r′3), either u or w must be in set A of the vertex tripartition.

Theorem 2.1. *Let G be a tripartite graph with n edges and a vertex of degree 1. If G admits a 1-rotational ρ-tripartite labeling, then there exists a 1-rotational G-decomposition of K_{2nx} for any positive integer x.*

Proof. Let h be a 1-rotational ρ-tripartite labeling of a graph G with n edges, vertex tripartition $\{A, B, C\}$, and edge $\{u, w\}$ as in the definition. Without loss of generality, we may assume that $u, w \in A \cup B$. If $x = 1$, the result follows from Theorem 1.7, so we now assume $x \geq 2$.

Let $B_1, B_2, \dots, B_x$ and $C_1, C_2, \dots, C_x$ be x vertex-disjoint copies of B and C, respectively. For $i \in [1, x]$ the vertices in B_i and C_i corresponding to $b \in B$ and $c \in C$ are denoted b_i and c_i, respectively. If $w \in A$, let $\tilde{A} = A \cup \{w'\}$ where w' is unique from all previously described vertices; otherwise, if $w \notin A$, let $\tilde{A} = A$. Let $\tilde{B} = \bigcup_{i=1}^{x} B_i$ and $\tilde{C} = \bigcup_{i=1}^{x} C_i$. We define a new graph $\tilde{G}$ with vertex set $\tilde{A} \cup \tilde{B} \cup \tilde{C}$ and edges $\{a, v_i\}$, for $i \in [1, x]$ whenever $a \in A$ and $\{a, v\}$ is an edge of G, and edges $\{b_i, c_i\}$, for $i \in [1, x]$ whenever $b \in B$, $c \in C$, and $\{b, c\}$ is an edge of G. In the case where $w \in A$, we further modify $\tilde{G}$ as follows:

(i) contract the unique edge $\{w, u_1\}$ where $u_1 \in B_1$ corresponds to $u \in B$,
(ii) call the resulting vertex u',
(iii) replace the loop resulting from the contraction with edge $\{u', w'\}$.

Hence, vertex $u' \in V(\tilde{G})$ is adjacent to $u_i \in B_i$ for $i \in [2, x]$ as well as any neighbors of the previously defined u_1 and w, but $u_1, w \notin V(\tilde{G})$. (We note that u' is not being assigned here as belonging to either $\tilde{A}$ or $\tilde{B}$, but this does not change the proof in any way. Furthermore, $\tilde{G}$ is still tripartite with vertex tripartition $\{\tilde{A}, C_1 \cup \tilde{B} \setminus B_1, B_1 \cup \tilde{C} \setminus C_1\}$.) Alternatively, in the case where $w \notin A$, we make no modifications to $\tilde{G}$ but simply refer to the vertex in B_1 that corresponds to $w \in B$ as vertex w', while any other vertex $b_i \in B_i$, for $i \in [2, x]$, that corresponds to $w \in B$ is referred to as w_i.

We note that $\tilde{G}$ is composed of x copies of G where only one edge, namely $\{w, u_1\}$, from the first copy is replaced with another, i.e. $\{u', w'\}$, and only in the case where $w \in A$. In that case u' has the same neighbors as u_1 from the first copy of G and w' has only u' as a neighbor. Hence $\tilde{G}$ has nx edges, G decomposes $\tilde{G}$, and $\deg w' = 1$.

The plan of the proof is to show that $\tilde{G}$ admits a 1-rotational ρ-labeling, so that $\tilde{G}$ decomposes K_{2nx} via a 1-rotational decomposition, and thus so does G. We define a labeling $\tilde{h}$ on $\tilde{G}$ by

$$\tilde{h}(v) = \begin{cases} \infty, & v = w', \\ h(v), & v \in A \setminus \{w\}, \\ h(u), & v = u', \\ h(b) + (i-1)2n, & v = b_i \in B_i \setminus \{u', w', w_i\}, \\ h(u) + (i-1)2n, & v = w_i,\ i \in [2, x], \\ h(c) + (x-i)2n, & v = c_i \in C_i. \end{cases}$$

A demonstration of the labeling function $\tilde{h}$ on a graph $\tilde{G}$ where $w \in A$ and where $w \notin A$ can be seen in Figures 1 and 2, respectively, at the end of the proof.

In the case where $w \in A$, we note that

$$\begin{aligned} \tilde{h}(u') &= h(u) \in [0, 2n-2], \\ \tilde{h}(\tilde{A}) &= h(A) \subseteq [0, 2n-2] \cup \{\infty\}, \\ \tilde{h}(B_1 \cup C_x) &= h(B \cup C \setminus \{u\}) \subseteq [0, 2n-2] \setminus \{h(u)\}, \end{aligned}$$

and for $i \in [2, x]$

$$\tilde{h}(B_i \cup C_{x+1-i}) = h(B \cup C) + (i-1)2n \subseteq [2n(i-1), 2ni-2].$$

Similarly, in the case where $w \notin A$, we note that $\tilde{h}(\tilde{A}) = h(A) \subseteq [0, 2n-2]$, while $\tilde{h}(B_1 \cup C_x) = h(B \cup C) \subseteq [0, 2n-2] \cup \{\infty\}$ and for $i \in [2, x]$

$$\tilde{h}(B_i \cup C_{x+1-i}) = \big(\{h(u)\} \cup h(B \cup C) \setminus \{\infty\}\big) + (i-1)2n \subseteq [2n(i-1), 2ni-2].$$

In either case, these sets of vertex labels do not intersect because h is one-to-one. Thus $\tilde{h}$ is one-to-one from $V(\tilde{G})$ to $[0, 2nx-2] \cup \{\infty\}$.

Suppose G has an edge $\{s, t\}$ between B and C. For $j \in [1, x]$ let

$$f(j) = (j-1)2n - (x-j)2n = (2j - x - 1)2n,$$

and note that $f(x+1-j) = -f(j)$. Define $k = k(s, t, j)$ to be j or $x + 1 - j$ according as $s \in B$ or $s \in C$. Then in any case there exists some $k \in [1, x]$ such that

$$\tilde{h}(s_k) - \tilde{h}(t_k) = h(s) - h(t) + f(j). \tag{2.1}$$

Now let $1 \le i \le nx - 1$. We will show that $\tilde{G}$ has an edge with label either i or $2nx - 1 - i$. The proof will be by cases depending on q and r, where q and r are integers such that

$$i = qn + r, \quad 1 \le r \le n, \quad 0 \le q < x, \quad (q, r) \ne (x-1, n).$$

In the proof we will use vertices v_j, where $v \in B \cup C$. In all cases it can be checked that j is an integer and $1 \le j \le x$.

Case 1. q is even.

Note that if $1 \le r \le n-1$, then G has an edge e with label r or $2n-1-r$; otherwise, if $r = n$, then G has an edge e with label $n-1$ or $2n-1-(n-1) = r$.

Subcase 1a. $r = n$ and e has label $n - 1$.

If $e = \{a, v\}$, $a \in A$ and $v \in B \cup C$, then note that if $v \in B$, then

$$\tilde{h}(v_{x-q/2}) - \tilde{h}(a) = n - 1 + (x - q/2 - 1)2n = 2nx - 1 - nq - n = 2nx - 1 - i,$$

while if $v \in C$, then

$$\tilde{h}(v_{1+q/2}) - \tilde{h}(a) = n - 1 + (x - (1 + q/2))2n = 2nx - 1 - nq - n = 2nx - 1 - i.$$

There remains the case $e = \{s, t\}$ with $s, t \in B \cup C$ and $|h(s) - h(t)| = n - 1$. Let e' be as in the definition of a 1-rotational ρ-tripartite labeling. Then $e' = \{s', t'\} \subseteq B \cup C$ and $|h(s) - h(t)| + |h(s') - h(t')| = 2n$. Without loss of generality, we can assume that $h(s) - h(t) = n - 1$ and $h(s') - h(t') = 2n - (n-1) = n + 1$. If $q \equiv 0 \pmod 4$, then set $j = x - q/4$. Then, by Eq. (2.1), for some $k \in [1, x]$

$$\begin{aligned} |\tilde{h}(s_k) - \tilde{h}(t_k)| &= |h(s) - h(t) + f(j)| \\ &= |n - 1 + (2(x - q/4) - x - 1)2n| = |2nx - 1 - i| = 2nx - 1 - i. \end{aligned}$$

If $q \equiv 2 \pmod 4$, then set $j = (q+2)/4$. Then for some k

$$\begin{aligned} |\tilde{h}(s'_k) - \tilde{h}(t'_k)| &= |h(s') - h(t') + f(j)| \\ &= |n + 1 + (2(q+2)/4 - x - 1)2n| = |qn + n - 2nx + 1| \\ &= |i - 2nx + 1| = 2nx - 1 - i. \end{aligned}$$

Subcase 1b. e has label r.
If $e = \{a, v\}$, $a \in A$ and $v \in B \cup C$, then note that if $v \in B$, then

$$\tilde{h}(v_{1+q/2}) - \tilde{h}(a) = r + (q/2)2n = i,$$

while if $v \in C$, then

$$\tilde{h}(v_{x-q/2}) - \tilde{h}(a) = r + (x - (x - q/2))2n = i.$$

There remains the case $e = \{s, t\}$ with $s, t \in B \cup C$ and $h(s) - h(t) = r$. Let e' be as in the definition of a 1-rotational ρ-tripartite labeling. Then $e' = \{s', t'\} \subseteq B \cup C$ and $|h(s') - h(t')| = 2n - r$. First assume $q/2 \equiv x + 1 \pmod 2$. Set $j = (q/2 + x + 1)/2$. Then, by Eq. (2.1), for some $k \in [1, x]$

$$\begin{aligned}\tilde{h}(s_k) - \tilde{h}(t_k) &= h(s) - h(t) + f(j)\\ &= r + (2(q/2 + x + 1)/2 - x - 1)2n = nq + r = i.\end{aligned}$$

If $q/2 \equiv x \pmod 2$, then set $j = (x - q/2)/2$. Then for some k

$$\begin{aligned}|\tilde{h}(s'_k) - \tilde{h}(t'_k)| &= |h(s') - h(t') + f(j)|\\ &= |2n - r + (2(x - q/2)/2 - x - 1)2n| = |-qn - r| = |-i| = i.\end{aligned}$$

Subcase 1c. e has label $2n - 1 - r$.
First assume $e = \{a, v\}$ for $a \in A$. Take $j = x - q/2$. Then we compute that $\tilde{h}(v_j) - \tilde{h}(a) = 2nx - 1 - i$ for $v \in B$, while $\tilde{h}(v_{x+1-j}) - \tilde{h}(a) = 2nx - 1 - i$ for $v \in C$.

Now assume that G has edges $e = \{s, t\}$ and $e' = \{s', t'\}$ with $s, t, s', t' \in B \cup C$ such that $h(s) - h(t) = 2n - 1 - r$ and $h(s') - h(t') = 2n - (2n - 1 - r) = r + 1$. If $q \equiv 0 \pmod 4$, set $j = x - q/4$. Then, by Eq. (2.1), for some $k \in [1, x]$

$$\tilde{h}(s_k) - \tilde{h}(t_k) = 2n - 1 - r + f(j) = 2nx - 1 - i.$$

On the other hand, if $q \equiv 2 \pmod 4$, set $j = (q + 2)/4$. Then for some k

$$|\tilde{h}(s'_k) - \tilde{h}(t'_k)| = |r + 1 + f(j)| = |i - 2nx + 1| = 2nx - 1 - i.$$

Case 2. q is odd.
Thus $q \geq 1$. Note that if $1 \leq r \leq n - 2$, then $1 \leq n - 1 - r < n - 1$, and G has an edge e with label either $n - 1 - r$ or $2n - 1 - (n - 1 - r) = n + r$; otherwise, $r \in \{n - 1, n\}$.

Subcase 2a. $r \in \{n - 1, n\}$.
First, suppose $r = n - 1$ and let $j = x - (q - 1)/2$. Note that $j \neq 1$ since $x \geq 2$. If $w \in A$, we find that

$$\tilde{h}(u_j) - \tilde{h}(u') = (j - 1)2n = 2nx - qn - n = 2nx - 1 - i,$$

while if $w \in B$ we find

$$\tilde{h}(w_j) - \tilde{h}(u) = (j - 1)2n = 2nx - 1 - i.$$

Second, suppose $r = n$ and let $j = (q+3)/2$. Note that $j \neq 1$ since $q \geq 1$. If $w \in A$, we find that

$$\tilde{h}(u_j) - \tilde{h}(u') = (j-1)2n = qn + n = i,$$

while if $w \in B$ we find

$$\tilde{h}(w_j) - \tilde{h}(u) = (j-1)2n = i.$$

Subcase 2b. e has label $n-1-r$.
First suppose $e = \{a, v\}$ with $a \in A$. If $v \in B$ we take $j = x - (q-1)/2$ and find that

$$\tilde{h}(v_j) - \tilde{h}(a) = n - 1 - r + (j-1)2n = 2nx - 1 - i,$$

while if $v \in C$ we take $j = (q+1)/2$ and find

$$\tilde{h}(v_j) - \tilde{h}(a) = n - 1 - r + (x-j)2n = 2nx - 1 - i.$$

Otherwise we can assume G contains edges $e = \{s, t\}$ and $e' = \{s', t'\}$ between B and C such that $h(s) - h(t) = n - 1 - r$ and $h(s') - h(t') = 2n - (n-1-r) = n+r+1$. If $q \equiv 1 \pmod 4$ take $j = x - (q-1)/4$. Then, by Eq. (2.1), for some $k \in [1, x]$

$$\tilde{h}(s_k) - \tilde{h}(t_k) = n - 1 - r + f(j) = 2nx - 1 - i.$$

If $q \equiv 3 \pmod 4$ take $j = (q+1)/4$. Then for some k

$$|\tilde{h}(s'_k) - \tilde{h}(t'_k) = |n + r + 1 + f(j)| = |i - (2nx-1)| = 2nx - 1 - i.$$

Subcase 2c. e has label $n+r$.
First suppose $e = \{a, v\}$ with $a \in A$. If $v \in B$ we take $j = (q+1)/2$, and find that

$$\tilde{h}(v_j) - \tilde{h}(a) = n + r + (j-1)2n = i,$$

while if $v \in C$ we take $j = x - (q-1)/2$, making

$$\tilde{h}(v_j) - \tilde{h}(a) = n + r + (x-j)2n = i.$$

Otherwise we can assume G contains edges $e = \{s, t\}$ and $e' = \{s', t'\}$ between B and C such that $h(s) - h(t) = n + r$ and $h(s') - h(t') = 2n - (n+r) = n - r$. If $(q-1)/2 \equiv x \pmod 2$ take $j = (2x+1-q)/4$. Then, by Eq. (2.1), for some $k \in [1, x]$

$$|\tilde{h}(s'_k) - \tilde{h}(t'_k)| = |n - r + f(j)| = |-i| = i.$$

If $(q-1)/2 \equiv x + 1 \pmod 2$ take $j = (2x+1+q)/4$. Then for some k

$$\tilde{h}(s_k) - \tilde{h}(t_k) = n + r + f(j) = i.$$

This concludes the proof. □

Figures 1 and 2 both show 1-rotational ρ-tripartite labelings of a graph G with 4 edges along with the starters for a 1-rotational G-decomposition of K_{24} constructed as in the proof of Theorem 2.1. Note that the base graphs are identical from one figure to the next, but the choice in vertex tripartition changes: In Figure 1 the degree-1 vertex is in set A of the tripartition, and in Figure 2 the degree-1 vertex is in set B.

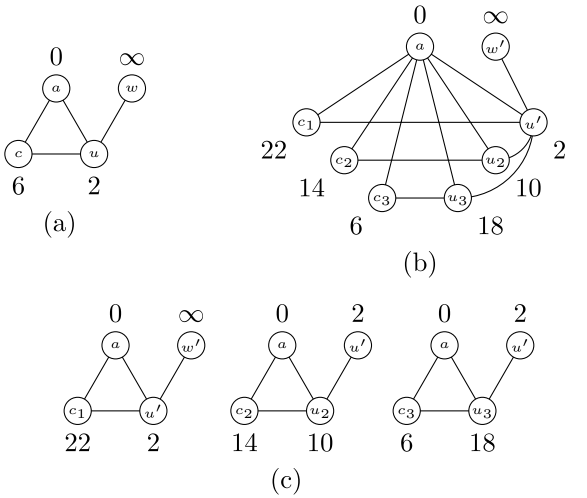

Fig. 1. Demonstrating a 1-rotational ρ-tripartite labeling of $K_3 + e$. (a) A 1-rotational ρ-tripartite labeling of a graph G. (b) The graph $\tilde{G}$ with a 1-rotational ρ-labeling as described in the proof for Theorem 2.1. (c) The three copies of G that decompose $\tilde{G}$, i.e., starter blocks for a 1-rotational decomposition of K_{24}.

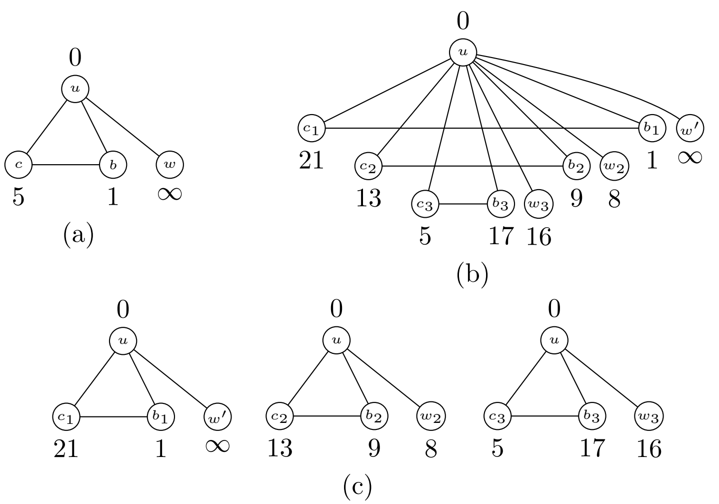

Fig. 2. Demonstrating a 1-rotational ρ-tripartite labeling of $K_3 + e$ choosing a different vertex tripartition than that found in Figure 1. (a) A 1-rotational ρ-tripartite labeling of a graph G. (b) The graph $\tilde{G}$ as described in the proof for Theorem 2.1. (c) The three copies of G that decompose $\tilde{G}$, i.e., starter blocks for a 1-rotational decomposition of K_{24}.

We note that condition (ℓ5) on a labeling of a bipartite graph satisfies condition (r$'$2) for a 1-rotational ρ-tripartite labeling. Hence, we arrive at the following corollary to Theorem 2.1.

Corollary 2.2. *Let G be a bipartite graph with n edges and with $w \in V(G)$ such that $\deg w = 1$. If $G \ominus w$ admits an ordered ρ-labeling, then there exists a 1-rotational G-decomposition of K_{2nx} for any positive integer x.*

3. APPLICATION

3.1. SOME NOTATION

Let G be a graph that admits a ρ-labeling. If m is the label of an edge, we define $m^* = \min\{m, 2|E(G)| + 1 - m\}$. Hence, m^* is the length of an edge with label m. If S is a set of labels of edges of G, let $S^* = \{m^* : m \in S\}$ denote the corresponding set of edge lengths. Thus if the set of vertex labels of G is a subset of $[0, 2n]$ and the set E of edge labels of G satisfies $E^* = [1, n]$, then conditions (ℓ1) and (ℓ3) are satisfied, and G has a ρ-labeling.

We denote the path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, for $i \in [0, k-1]$, by $(x_0, x_1, \ldots, x_k)$. The *first vertex* of this path is x_0, the *second vertex* is x_1, and the *last vertex* is x_k. If $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$ are paths with $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$.

Let $P(k)$ be the path with k edges and $k+1$ vertices $0, 1, \ldots, k$ given by $(0, k, 1, k-1, 2, k-2, \ldots, \lceil k/2 \rceil)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, \lfloor k/2 \rfloor]$, $B = [\lfloor k/2 \rfloor + 1, k]$, and every edge joins a vertex of A to one of B. Furthermore, the set of labels of the edges of $P(k)$ is $[1, k]$.

Now let a and b be nonnegative integers with $a \le b$ and let us add a to all the vertices of A and b to all the vertices of B. We denote the resulting graph by $P(a, b, k)$. Note that this graph has the following properties.

(P1) $P(a, b, k)$ is a path with first vertex a and second vertex $k + b$. Its last vertex is $a + k/2$ if k is even and $b + (k+1)/2$ if k is odd.
(P2) Each edge of $P(a, b, k)$ joins a vertex of $A' = [a, \lfloor k/2 \rfloor + a]$ to a larger vertex of $B' = [\lfloor k/2 \rfloor + 1 + b, k + b]$.
(P3) The set of edge labels of $P(a, b, k)$ is $[b - a + 1, b - a + k]$.

Now consider the path $Q(k)$ obtained from $P(k)$ by replacing each vertex i with $k - i$. The new graph is the path $(k, 0, k-1, 1, \ldots, \lfloor k/2 \rfloor)$. The set of vertices of $Q(k)$ is $A \cup B$, where $A = k - [\lfloor k/2 \rfloor + 1, k] = [0, \lceil k/2 \rceil - 1]$ and $B = k - [0, \lfloor k/2 \rfloor] = [\lceil k/2 \rceil, k]$, and every edge joins a vertex of A to one of B. The set of edge labels is still $[1, k]$.

Again, we add a to the vertices of A and b to vertices of B, where a and b are integers, $0 \le a \le b$. This graph is $(k + b, a, k + b - 1, a + 1, \ldots)$, which we denote by $Q(a, b, k)$. Note that this graph has the following properties.

(Q1) $Q(a,b,k)$ is a path with first vertex $k+b$. Its last vertex is $b+k/2$ if k is even and $a+(k-1)/2$ if k is odd.
(Q2) Each edge of $Q(a,b,k)$ joins a vertex of $A' = [a, \lceil k/2 \rceil - 1 + a]$ to a larger vertex of $B' = [\lceil k/2 \rceil + b, k+b]$.
(Q3) The set of edge labels of $Q(a,b,k)$ is $[b-a+1, b-a+k]$.

Some examples of the path notation are presented in Figure 3.

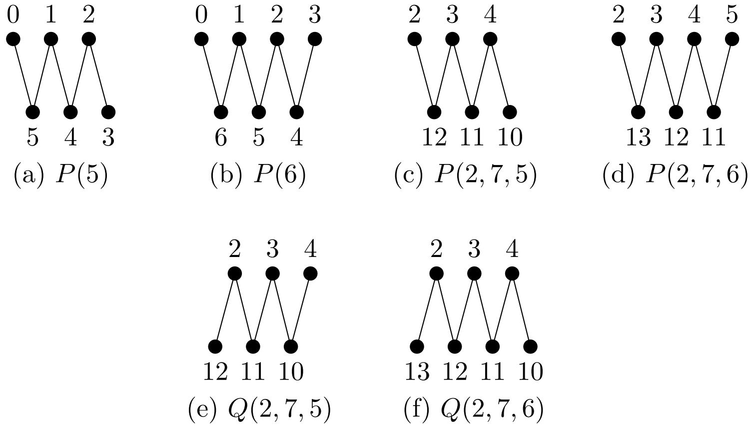

Fig. 3. Examples of the path notation

3.2. PATH PLUS AN EDGE

We now turn our attention to the class of graphs that results from adding an edge to a path. It was shown in [3] that any such graph that is not bipartite, besides K_3, admits a γ-labeling, which is necessarily a ρ-tripartite labeling. This ultimately led to the following result.

Theorem 3.1. *If G is the graph with n edges formed by adding the edge $\{v_x, v_{2y}\}$ to the path $(v_0, v_1, \ldots, v_{x+2y+z})$ where $x, y, z \in \mathbb{N}$ with $y \geq 1$, then there exists a cyclic G-decomposition of K_{2nt+1} for any positive integer t.*

Now, we similarly consider graphs G that result from adding an edge to a path, and we settle the analogous result for 1-rotational G-decompositions of $K_{2\cdot|E(G)|\cdot t}$. Recall that in order for such a 1-rotational G-decomposition to exist, G must have a degree-1 vertex, so the added edge cannot connect the endvertices of the original path. However, unlike the result found in [3], the following statement is not restricted to non-bipartite graphs.

Theorem 3.2. *Let G be a (simple) graph formed by adding an edge to P_n where $n \geq 2$. If G is not an n-cycle, then there exists a 1-rotational G-decomposition of K_{2nt} for any positive integer t.*

Proof. Let $\hat{e}$ denote the edge added to the path P_n to form G. Since G is not an n-cycle, $\hat{e}$ is not incident with both endvertices of the original P_n. Let w be an endvertex of P_n that is not incident with $\hat{e}$ in G. Hence, $\deg_G w = 1$.

First, we consider when $\hat{e}$ is not incident with any vertices of the original P_n (i.e., $\hat{e}$ and P_n are vertex disjoint). If $n = 2$, then $\hat{e} = \{0, 1\}$ and $P_n = (2, \infty)$ is a 1-rotational ρ-tripartite labeling of G. If $n > 2$, then $G \ominus w$ is a linear forest, which admits an ordered σ-labeling by Theorem 1.2. Similarly, when $\hat{e}$ is incident with exactly one vertex of the original P_n (i.e., $\hat{e}$ is a pendent edge to P_n), then $G \ominus w$ is a caterpillar, which admits an α-labeling by Theorem 1.1. In either case, G is bipartite, and the result follows from Corollary 2.2.

What remains is the case where $\hat{e}$ is incident with two vertices (not both of which being endvertices) in the original P_n. For this latter case, we now describe P_n as the edge-disjoint union of the four paths P_x, P_y, P_z, and (u, w) where

(i) P_x and P_z are vertex-disjoint, but each has a common endvertex with P_y,
(ii) $\hat{e}$ is incident with both endvertices of P_y,
(iii) vertex u is an endvertex of either P_x or P_z.

It follows that $x+y+z = n+1$ with $x, z \geq 1$ and $y \geq 3$. Note that in such a description of P_n with x, z both even (and necessarily at least 2), $\hat{e}$ is not incident with either endvertex of the original P_n. There then exists an alternative description of P_n with the other endvertex of the path P_n, say w', identified as the degree-1 vertex to be assigned vertex label ∞. Such an alternative description would consist of paths $P_{x\pm1}$, P_y, $P_{z\mp1}$, and (u', w') meeting the conditions listed above, i.e., an alternative description with odd orders for the first and third paths. Hence, we may assume at least one of x or z is odd without loss of generality.

Now let G' be the induced subgraph on $V(G) \setminus \{w\}$. That is, G' consists of just the paths P_x, P_y, and P_z along with the added edge $\hat{e}$. The remainder of the proof is broken into cases dependent on the value of y modulo 4. In each case, the idea of the proof is to assign vertex w the label ∞ and show that G', which has $n - 1 = x + y + z - 2$ edges, admits a ρ-labeling that satisfies the latter two conditions of the definition for a 1-rotational ρ-tripartite labeling. The result then follows from Theorem 2.1.

Case 1. $y \equiv 0 \pmod 4$.
Without loss of generality, we assume x is odd. Let $x = 2i + 1$ and let $y = 4j$ for some integers $i \geq 0$ and $j \geq 1$. Now let $P_x = G_1$, $P_y = G_2 + G_3$, $P_z = G_4$, and $\hat{e} = \{i + 4j + z - 1, i + 2j - 1\}$, where

$$\begin{aligned}
G_1 &= Q(0, 4j + z - 1, 2i),\\
G_2 &= Q(i, i + 2j + z, 2j - 1),\\
G_3 &= P(i + j - 1, i + j + z - 2, 2j),\\
G_4 &= P(i + 2j - 1, i + 2j - 1, z - 1).
\end{aligned}$$

An example of the induced vertex labeling on G can be seen in Figure 4.

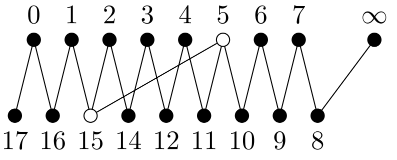

Fig. 4. A 1-rotational ρ-tripartite labeling of $P_5+P_8+P_6+(u,w)$ with the shared endvertices of $\hat{e}$ shown in white

Note that by (P1) and (Q1), the last vertex of G_1 is $i+4j+z-1$; the first vertex of G_2 is $i+4j+z-1$, and the last is $i+j-1$; the first vertex of G_3 is $i+j-1$, and the last is $i+2j-1$; and the first vertex of G_4 is $i+2j-1$. For $1 \le r \le 4$, let A_r and B_r denote the sets labeled A' and B' in (P2) and (Q2), corresponding to the path G_r. We then compute

$$\begin{aligned} A_1 &= [0, i-1], & B_1 &= [i+4j+z-1, 2i+4j+z-1], \\ A_2 &= [i, i+j-1], & B_2 &= [i+3j+z, i+4j+z-1], \\ A_3 &= [i+j-1, i+2j-1], & B_3 &= [i+2j+z-1, i+3j+z-2], \\ A_4 &= [i+2j-1, i+2j-1+\lfloor (z-1)/2 \rfloor], & B_4 &= [i+2j+\lfloor (z-1)/2 \rfloor, i+2j+z-2]. \end{aligned}$$

Thus,

$$\begin{aligned} 0 \le A_1 < A_2 \le A_3 \le i+2j-1 \le A_4 \\ < B_4 < B_3 < B_2 \le i+4j+z-1 \le B_1 \\ \le x+y+z-2 = n-1. \end{aligned} \tag{3.1}$$

Note that besides $V(G_2) \cap V(G_3) = \{i+j-1\}$, the only intersections of the vertex sets are at the endpoints of $\hat{e}$, which coincide with the endvertices of P_y and an endvertex of each of P_x and P_z.

Next, let E_r denote the set of edge labels in G_r for $1 \le r \le 4$. By (P3) and (Q3), we have edge labels

$$\begin{aligned} E_1 &= [4j+z, 2i+4j+z-1], \\ E_2 &= [2j+z+1, 4j+z-1], \\ E_3 &= [z, 2j+z-1], \\ E_4 &= [1, z-1]. \end{aligned}$$

Moreover, the edge $\hat{e}$ has label $2j+z$. Thus the subgraph G' has one edge of each label $\ell \in [1, 2i+4j+z-1] = [1, n-1]$, and the defined labeling is a β-labeling of G'.

Finally, let $A = \bigcup_{r=1}^{4} A_r$ and $B = \bigcup_{r=1}^{4} B_r$. Then, $\{A, B\}$ is a bipartition of $V(G')$. Condition (r′2) of a 1-rotational ρ-tripartite labeling is clear from inequality (3.1), and since G' is bipartite (i.e., $C = \varnothing$), condition (r′3) also holds. (In fact, we have an α-labeling of G'.)

Case 2. $y \equiv 1 \pmod 4$.
Let $y = 4j+1$ for some integer $j \ge 1$.

Subcase 2a. $j = 1$.
In the case where $x = z = 1$, it is easy to check that $P_y = (4, 7, 0, 1, 10)$ along with $\hat{e} = \{4, 10\}$ satisfy conditions (r′2) and (r′3) of a 1-rotational ρ-tripartite labeling if we use vertex tripartition $\big\{\{0\}, \{1, 4\}, \{7, 10\}\big\}$. See Figure 5 for the induced labeling on such a graph G.

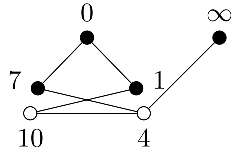

Fig. 5. A 1-rotational ρ-tripartite labeling of $P_1 + P_5 + P_1 + (u, w)$ with the shared endvertices of $\hat{e}$ shown in white

We now assume $x \geq 2$ and let $P_x = (2x + 2z + 6, x + z + 4) + G_1$, $P_y = (2x + 2z + 6, 0, 2x + 2z + 5, 1, x + z + 2)$, $P_z = G_2$, and $\hat{e} = \{2x + 2z + 6, x + z + 2\}$, where

$$G_1 = P(x + z + 4, x + z + 7, x - 2),$$
$$G_2 = Q(2, x + 3, z - 1).$$

An example of the induced vertex labeling on G can be seen in Figure 6.

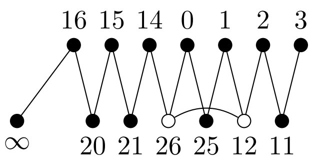

Fig. 6. A 1-rotational ρ-tripartite labeling of $(w, u) + P_6 + P_5 + P_4$ with the shared endvertices of $\hat{e}$ shown in white

Note that by (P1) and (Q1), the first vertex of G_1 is $x + z + 4$, and the first vertex of G_2 is $x + z + 2$. For $1 \leq r \leq 2$, let A_r and B_r denote the sets labeled A' and B' in (P2) and (Q2), corresponding to the path G_r. We then compute

$$A_1 = [x + z + 4, x + z + 3 + \lfloor x/2 \rfloor], \qquad B_1 = [x + z + 7 + \lfloor x/2 \rfloor, 2x + z + 5],$$
$$A_2 = [2, 1 + \lceil (z - 1)/2 \rceil], \qquad B_2 = [x + 3 + \lceil (z - 1)/2 \rceil, x + z + 2].$$

Thus,

$$\begin{aligned} \{0, 1\} < A_2 < B_2 \leq x + z + 2 < A_1 < B_1 \\ < \{2x + 2z + 5, 2x + 2z + 6\} \leq 2n - 2. \end{aligned} \tag{3.2}$$

Note that the only intersections of the vertex sets are at the endpoints of $\hat{e}$, which coincide with the endvertices of P_y and an endvertex of each of P_x and P_z.

Next, let E_r denote the set of edge labels in G_r for $1 \leq r \leq 2$. By (P3) and (Q3), we have edge labels

$$E_1 = [4, x+1],$$
$$E_2 = [x+2, x+z],$$

yielding edge lengths of the same values. Moreover, the edge $\hat{e}$ has length $(x+z+4)^* = x+z+3$; the edge in the subpath $(2x+2z+6, x+z+4)$ has length $x+z+2$; and the edges in P_y have lengths $(2x+2z+6)^* = 1$, $(2x+2z+5)^* = 2$, $(2x+2z+4)^* = 3$, and $x+z+1$. Thus the subgraph G' has one edge of each length $\ell \in [1, x+z+3] = [1, n-1]$, and the defined labeling is a ρ-labeling of G'.

Finally, let $A = \{0,1\} \cup A_1 \cup A_2$, $B = B_1 \cup B_2$, and $C = \{2x+2z+5, 2x+2z+6\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G')$ where only edge $\hat{e}$ has both endvertices in $B \cup C$. Condition (r$'$2) of a 1-rotational ρ-tripartite labeling is clear from inequality (3.2). Note that

$$|(2x+2z+6)-(x+z+2)| + |(2x+2z+6)-(x+z+2)| = 2x+2z+8 = 2n,$$

twice the number of edges of G. Thus, condition (r$'$3) also holds.

Subcase 2b. $j \geq 2$.
Without loss of generality, we assume x is odd. Let $x = 2i+1$ for some integer $i \geq 0$. Now let $P_x = G_1$, $P_y = (i, i+1, 3i+4j+z+2) + G_2 + G_3$, $P_z = G_4$, and $\hat{e} = \{i, 3i+2j+z+2\}$, where

$$G_1 = P(0, 2i+8j+2z-1, 2i),$$
$$G_2 = Q(i+3, 3i+2j+z+5, 2j-3),$$
$$G_3 = P(i+j+1, 3i+j+z+1, 2j+1),$$
$$G_4 = Q(i+2j+2, 3i+2j+3, z-1).$$

An example of the induced vertex labeling on G can be seen in Figure 7.

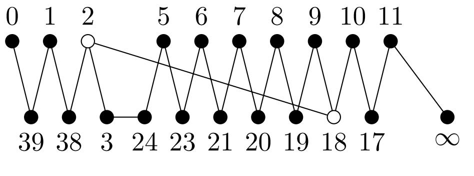

Fig. 7. A 1-rotational ρ-tripartite labeling of $P_5+P_{13}+P_4+(u,w)$ with the shared endvertices of $\hat{e}$ shown in white

Note that by (P1) and (Q1), the last vertex of G_1 is i; the first vertex of G_2 is $3i+4j+z+2$, and the last is $i+j+1$; the first vertex of G_3 is $i+j+1$, and the last is $3i+2j+z+2$; and the first vertex of G_4 is $3i+2j+z+2$. For $1 \leq r \leq 4$, let

A_r and B_r denote the sets labeled A' and B' in (P2) and (Q2), corresponding to the path G_r. We then compute

$$\begin{aligned}
A_1 &= [0, i], & B_1 &= [3i + 8j + 2z, 4i + 8j + 2z - 1],\\
A_2 &= [i + 3, i + j + 1], & B_2 &= [3i + 3j + z + 4, 3i + 4j + z + 2],\\
A_3 &= [i + j + 1, i + 2j + 1], & B_3 &= [3i + 2j + z + 2, 3i + 3j + z + 2],\\
A_4 &= [i + 2j + 2, i + 2j + 1 + \lceil (z - 1)/2 \rceil], & &\\
& & B_4 &= [3i + 2j + 3 + \lceil (z - 1)/2 \rceil, 3i + 2j + z + 2].
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &\le A_1 \le \{i, i + 1\} < A_2 \le A_3 < A_4\\
&< B_4 \le 3i + 2j + z + 2 \le B_3 < B_2 < B_1\\
&< 4i + 8j + 2z = 2n - 2.
\end{aligned} \tag{3.3}$$

Note that besides $V(G_2) \cap V(G_3) = \{i + j + 1\}$, the only intersections of the vertex sets are at the endpoints of $\hat{e}$, which coincide with the endvertices of P_y and an endvertex of each of P_x and P_z.

Next, let E_r denote the set of edge labels in G_r for $1 \le r \le 4$. By (P3) and (Q3), we have edge labels

$$\begin{aligned}
E_1 &= [2i + 8j + 2z, 4i + 8j + 2z - 1],\\
E_2 &= [2i + 2j + z + 3, 2i + 4j + z - 1],\\
E_3 &= [2i + z + 1, 2i + 2j + z + 1],\\
E_4 &= [2i + 2, 2i + z],
\end{aligned}$$

yielding edge lengths

$$\begin{aligned}
E_1^* &= [2, 2i + 1],\\
E_2^* &= [2i + 2j + z + 3, 2i + 4j + z - 1],\\
E_3^* &= [2i + z + 1, 2i + 2j + z + 1],\\
E_4^* &= [2i + 2, 2i + z].
\end{aligned}$$

Moreover, the edge $\hat{e}$ has length $2i + 2j + z + 2$, and the edges in the subpath $(i, i + 1, 3i + 4j + z + 2)$ have lengths 1 and $(2i + 4j + z + 1)^* = 2i + 4j + z$. Thus the subgraph G' has one edge of each length $\ell \in [1, 2i + 4j + z] = [1, n - 1]$, and the defined labeling is a ρ-labeling of G'.

Finally, let $A = \bigcup_{r=1}^{4} A_r$, $B = \bigcup_{r=1}^{4} B_r$, and $C = \{i + 1\}$. Then, $\{A, B, C\}$ is a tripartition of $V(G')$ where only edge $(i + 1, 3i + 4j + z + 2)$ has both endvertices in $B \cup C$. Condition (r′2) of a 1-rotational ρ-tripartite labeling is clear from inequality (3.3). Note that

$$|(3i + 4j + z + 2) - (i + 1)| + |(3i + 4j + z + 2) - (i + 1)| = 4i + 8j + 2z + 2 = 2n,$$

twice the number of edges of G. Thus, condition (r′3) also holds.

Case 3. $y \equiv 2 \pmod 4$.
Without loss of generality, we assume x is odd. Let $x = 2i + 1$ and let $y = 4j + 2$ for some integers $i \geq 0$ and $j \geq 1$. If $i > 0$, then let $P_x = (4i + 4j + z + 3, 1) + G_1$, $P_y = (3i + 4j + z + 2, 3i) + G_2 + G_3$, $P_z = G_4$, and $\hat{e} = \{3i + 4j + z + 2, 3i + 2j\}$, where

$$\begin{aligned} G_1 &= P(1, 2i + 4j + z + 2, 2i - 1), \\ G_2 &= P(3i, 3i + 2j + z + 2, 2j - 2), \\ G_3 &= P(3i + j - 1, 3i + j + z - 2, 2j + 2), \\ G_4 &= P(3i + 2j, 3i + 2j, z - 1). \end{aligned}$$

In the case where $i = 0$, we define P_y, P_z, and $\hat{e}$ all the same as when $i > 0$, but P_x would instead be a single vertex that identifies with the first vertex of P_y rendering G_1 undefined (see Figure 8). As such, all references to the vertices and edges in G_1 that follow below assume $i > 0$. Examples of the induced vertex labeling on G can be seen in Figures 8 and 9.

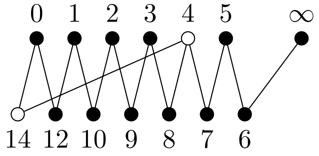

Fig. 8. A 1-rotational ρ-tripartite labeling of $P_1 + P_{10} + P_4 + (u, w)$ with the shared endvertices of $\hat{e}$ shown in white

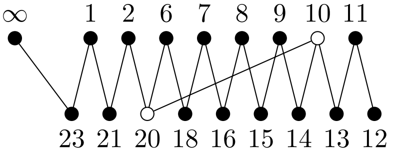

Fig. 9. A 1-rotational ρ-tripartite labeling of $(w, u) + P_5 + P_{10} + P_4$ with the shared endvertices of $\hat{e}$ shown in white

Note that by (P1), the last vertex of G_1 is $3i + 4j + z + 2$; the first vertex of G_2 is $3i$, and the last is $3i + j - 1$; the first vertex of G_3 is $3i + j - 1$, and the last is $3i + 2j$; and the first vertex of G_4 is $3i + 2j$. For $1 \leq r \leq 4$, let A_r and B_r denote the sets labeled A' and B' in (P2), corresponding to the path G_r.

We then compute

$$\begin{aligned} A_1 &= [1, i], & B_1 &= [3i + 4j + z + 2, 4i + 4j + z + 1],\\ A_2 &= [3i, 3i + j - 1], & B_2 &= [3i + 3j + z + 2, 3i + 4j + z],\\ A_3 &= [3i + j - 1, 3i + 2j], & B_3 &= [3i + 2j + z, 3i + 3j + z],\\ A_4 &= [3i + 2j, 3i + 2j + \lfloor (z-1)/2 \rfloor], & &\\ & & B_4 &= [3i + 2j + 1 + \lfloor (z-1)/2 \rfloor, 3i + 2j + z - 1]. \end{aligned}$$

Thus,

$$A_1 < A_2 \leq A_3 \leq 3i + 2j \leq A_4 < B_4 < B_3 < B_2 < 3i + 4j + z + 2 \leq B_1. \tag{3.4}$$

Note that the vertex sets in inequality (3.4) are contained in $[0, 2n - 2]$ for any $i \geq 0$. Also note that besides $V(G_2) \cap V(G_3) = \{3i + j - 1\}$, the only intersections of the vertex sets are at the endpoints of $\hat{e}$, which coincide with the endvertices of P_y and an endvertex of each of P_x and P_z.

Next, let E_r denote the set of edge labels in G_r for $1 \leq r \leq 4$. By (P3), we have edge labels

$$\begin{aligned} E_1 &= [2i + 4j + z + 2, 4i + 4j + z],\\ E_2 &= [2j + z + 3, 4j + z],\\ E_3 &= [z, 2j + z + 1],\\ E_4 &= [1, z - 1], \end{aligned}$$

yielding edge lengths

$$\begin{aligned} E_1^* &= [4j + z + 3, 2i + 4j + z + 1],\\ E_2^* &= [2j + z + 3, 4j + z],\\ E_3^* &= [z, 2j + z + 1],\\ E_4^* &= [1, z - 1]. \end{aligned}$$

Moreover, the edge $\hat{e}$ has length $2j + z + 2$. If $i = 0$, then the edge in $(3i + 4j + z + 2, 3i)$ has length $(4j + z + 2)^* = 4j + z + 1$, while the subpath $(4i + 4j + z + 3, 1)$ is not defined; otherwise, if $i > 0$, then the edges in $(4i + 4j + z + 3, 1)$ and $(3i + 4j + z + 2, 3i)$ have lengths $(4i + 4j + z + 2)^* = 4j + z + 1$ and $4j + z + 2$, respectively. Thus the subgraph G' has one edge of each length $\ell \in [1, 2i + 4j + z + 1] = [1, n - 1]$, and the defined labeling is a ρ-labeling of G'.

Finally, let $A = \bigcup_{r=1}^{4} A_r$ and $B = \{4i + 4j + z + 3\} \cup \bigcup_{r=1}^{4} B_r$, where the inclusion of the vertex $4i + 4j + z + 3$ in B is only when such a vertex is defined (i.e., when $i > 0$). Then, $\{A, B\}$ is a bipartition of $V(G')$. Condition (r′2) of a 1-rotational ρ-tripartite labeling is clear from inequality (3.4), and since G' is bipartite (i.e., $C = \varnothing$), condition (r′3) also holds. (In fact, we have a ρ^+-labeling of G'.)

Case 4. $y \equiv 3 \pmod 4$.
Without loss of generality, we assume $x \le z$. Let $y = 4j + 3$ for some integer $j \ge 0$. Now let $P_x = G_1$, $P_y = (8j + 2x + 2z + 2, 0) + G_2 + G_3$, $P_z = G_4$, and $\hat{e} = \{8j + 2x + 2z + 2, 4j + x + z\}$, where

$$\begin{aligned} G_1 &= Q(8j + x + 2z + 2, 8j + x + 2z + 3, x - 1), \\ G_2 &= P(0, 6j + x + z + 3, 2j), \\ G_3 &= P(j, 3j + x + z - 1, 2j + 1), \\ G_4 &= Q(4j + 1, 4j + x + 1, z - 1). \end{aligned}$$

See Figure 10 for the induced labeling on such a graph G.

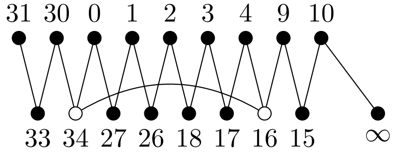

Fig. 10. A 1-rotational ρ-tripartite labeling of $P_4 + P_{11} + P_4 + (u, w)$ with the shared endvertices of $\hat{e}$ shown in white

Note that by (P1) and (Q1), the first vertex of G_1 is $8j + 2x + 2z + 2$; the first vertex of G_2 is 0, and the last is j; the first vertex of G_3 is j, and the last is $4j + x + z$; and the first vertex of G_4 is $4j + x + z$. For $1 \le r \le 4$, let A_r and B_r denote the sets labeled A' and B' in (P2) and (Q2), corresponding to the path G_r. We then compute

$$A_1 = [8j + x + 2z + 2, 8j + x + 2z + 1 + \lceil (x-1)/2 \rceil],$$
$$B_1 = [8j + x + 2z + 3 + \lceil (x-1)/2 \rceil, 8j + 2x + 2z + 2],$$
$$A_2 = [0, j], \qquad B_2 = [7j + x + z + 4, 8j + x + z + 3],$$
$$A_3 = [j, 2j], \qquad B_3 = [4j + x + z, 5j + x + z],$$
$$A_4 = [4j + 1, 4j + \lceil (z-1)/2 \rceil],$$
$$B_4 = [4j + x + 1 + \lceil (z-1)/2 \rceil, 4j + x + z].$$

Thus,

$$\begin{aligned} 0 \le A_2 \le A_3 < A_4 < B_4 \le 4j + x + z \le B_3 < B_2 \\ < A_1 < B_1 \le 8j + 2x + 2z + 2 = 2n - 2. \end{aligned} \tag{3.5}$$

Note that besides $V(G_2) \cap V(G_3) = \{j\}$, the only intersections of the vertex sets are at the endpoints of $\hat{e}$, which coincide with the endvertices of P_y and an endvertex of each of P_x and P_z.

Next, let E_r denote the set of edge labels in G_r for $1 \leq r \leq 4$. By (P3) and (Q3), we have edge labels

$$\begin{aligned} E_1 &= [2, x], \\ E_2 &= [6j + x + z + 4, 8j + x + z + 3], \\ E_3 &= [2j + x + z, 4j + x + z], \\ E_4 &= [x + 1, x + z - 1], \end{aligned}$$

yielding edge lengths

$$\begin{aligned} E_1^* &= [2, x], \\ E_2^* &= [x + z, 2j + x + z - 1], \\ E_3^* &= [2j + x + z, 4j + x + z], \\ E_4^* &= [x + 1, x + z - 1]. \end{aligned}$$

Moreover, the edge $\hat{e}$ has length $(4j + x + z + 2)^* = 4j + x + z + 1$, and the edge in the subpath $(8j + 2x + 2z + 2, 0)$ has length $(8j + 2x + 2z + 2)^* = 1$. Thus the subgraph G' has one edge of each length $\ell \in [1, 4j + x + z + 1] = [1, n - 1]$, and the defined labeling is a ρ-labeling of G'.

Finally, let $A = \bigcup_{r=1}^{4} A_r$, $B = \bigcup_{r=1}^{3} B_r$, and $C = B_4$. Then, $\{A, B, C\}$ is a tripartition of $V(G')$ where only edge $\hat{e}$ has both endvertices in $B \cup C$. Condition (r′2) of a 1-rotational ρ-tripartite labeling is clear from inequality (3.5). Note that

$$|(8j+2x+2z+2)-(4j+x+z)|+|(8j+2x+2z+2)-(4j+x+z)| = 8j+2x+2z+2 = 2n,$$

twice the number of edges of G. Thus, condition (r′3) also holds. □

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