Optimality conditions in multiobjective programming problems with interval valued objective functions[*]

by

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Abstract: We devote this paper to study of multiobjective programming problems with interval valued objective functions. For this, we consider two order relations LU and LS on the set of all closed intervals and propose several concepts of Pareto optimal solutions and generalized convexity. Based on generalized convexity (viz. LU and LS-pseudoconvexity) and generalized differentiability (viz. gH-differentiablity) of interval valued functions, the KKT optimality conditions for aforesaid problems are obtained. The theoretical development is illustrated by suitable examples.

Keywords: interval valued functions, gH-differentiablity, Pareto optimal solutions, pseudoconvexity, KKT optimality conditions

1. Introduction

The study of uncertain programming problems has been of considerable interest in the recent past. Due to inexactness in the data of real world problems, sometimes coefficients of objective functions and/or constraints are taken as intervals. This technique has been termed interval-valued programming and has been studied by many scholars in the past. Some of the recent results can be seen in Wu (2007, 2008, 2009), Inuiguchi and Mizoshita (2012), Bhurjee and Panda (2012), Chalco-Cano et al. (2013), Zhang (2013), Zhang et al. (2012), Hosseinzade and Hassanpour (2011), Jayswal et al. (2011), Singh et al. (2014), and in the references therein.

The field of vector optimization, also called multiobjective programming, has grown remarkably in different directions regarding the settings of optimality conditions and duality theory. With and without differentiability assumptions, it has been enriched by the applications of different types of generalizations of convexity theory. In this paper we are concerned with interval valued multiobjective programming, therefore it is necessary to introduce a concept of derivative for interval valued functions. A variety of notions for the derivative of set valued functions have been defined and studied in Hukuhara (1967), Banks and Jacobs (1970), De Blasi (1976), Aubin and Cellina (1984), Aubin and Frankowska (1990, 2000), Ibrahim (1996).

Recently, the concept of H-derivative was used to study interval valued nonlinear programming problems in Wu (2007, 2009), Zhang et al. (2012). However, this definition of differentiability is having certain limitations, since H-differentiable functions (say f) should satisfy the condition that the diameter diam (f) is nondecreasing in its domain (see Banks and Jacobs, 1970; Bede and Gal, 2005). To deal with this, some alternative concepts of derivatives of interval valued functions have been introduced in Bede and Gal (2005), Chalco-Cano and Roman-Flores (2008), Stefanini (2010), Chalco-Cano et al. (2011). In Stefanini and Bede (2009), the authors have introduced the concept of generalized Hukuhara derivative of interval valued functions, which is more general than the H-derivative and the weak derivative of interval valued functions (see Chalco-Cano et al., 2013).

On the other hand, convexity also plays an important role in the study of optimization and many approaches have been developed and applied to define convexity of interval valued functions. The concepts of LU, WC convexity and LU, WC pseudoconvexity of interval valued functions were proposed in Wu (2007, 2009), and the concepts of preinvexity and invexity were extended to interval valued functions in Zhang et al. (2012). In Ahmad et al. (2014), the authors derived KKT optimality conditions in order to obtain (LS and LU) optimal solutions for invex interval-valued programming problems by considering generalized Hukuhara differentiability and generalized convexity (viz. η-preinvexity, η-invexity etc.). In this paper, we study the KKT optimality conditions for multiobjective programming problems with interval valued objective function by considering pseudoconvexity and gH-differentiability.

The paper is organised as follows: in Section 2 we give some arithmetic of intervals and then give the concept of gH-differentiability of interval valued functions. In Section 3 we propose some solution concepts following from Wu (2009) and Chalco-Cano et al. (2013) respectively. Further, in Section 4 we derive KKT optimality conditions for (interval) multiobjective programming problems by considering objective functions to be gH-differentiable and LU and LS-pseudoconvex. Moreover, by using the gradient of interval valued functions the same are obtained. The illustrating examples are presented where necessary. Finally we conclude in Section 5.

2.   Preliminaries

Let $\mathcal{K}_c$ denote the class of all closed and bounded intervals in R, i.e.,

$$\mathcal{K}_c = \{[a, b] : a, b \in R \text{ and } a \leq b\}$$

with $b - a$ being the width of the interval $[a, b] \in \mathcal{K}_c$.

2.1.   Arithmetic of intervals

Let $A \in \mathcal{K}_c$, then we adopt the notation $A = [a^L, a^U]$, where a^L and a^U mean the lower and upper bounds, respectively. Assume that $A = [a^L, a^U], B = [b^L, b^U] \in \mathcal{K}_c$ and $\lambda \in R$, then by definition we have

$$A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U] \tag{2.1}$$

$$\lambda A = \lambda[a^L, a^U] = \begin{cases} [\lambda a^L, \lambda a^U], \text{if } \lambda \geq 0 \\ [\lambda a^U, \lambda a^L], \text{if } \lambda < 0 \end{cases} . \tag{2.2}$$

Therefore we have

$$-A = -[a^L, a^U] = [-a^U, -a^L]$$

and

$$A - B = A + (-B) = [a^L - b^U, a^U - b^L].$$

Aubin and Cellina (1984) and Assev (1986) have shown that the space $\mathcal{K}_c$ is not a linear space with operations (2.1) and (2.2), since it does not contain inverse element and therefore subtraction is not well defined.

Now, if $A = B + C$, then the Hukuhara difference (H-difference) or geometrical or Pontryagin (Tolstonogov 2000) difference of A and B, denoted by $A \ominus_H B$ (Chalco-Cano et al. 2013), is equal to C. If $A = [a^L, a^U]$, $B = [b^L, b^U]$, $A \ominus_H B = C = [c^L, c^U]$ exists if $a^L - b^L \leq a^U - b^U$, where $c^L = a^L - b^L$ and $c^U = a^U - b^U$ (Wu, 2007, 2009)

Next, in Stefanini and Bede (2009), the concept of the generalization of H-difference of two intervals has been introduced as follows.

DEFINITION 1 (Stefanini and Bede, 2009) Let $A, B \in \mathcal{K}_c$. The generalized Hukuhara difference (gH-difference) is defined as

$$A \ominus_g B = C \Longleftrightarrow \begin{cases} (i) \ A = B + C \\ or \ (ii) \ B = A + (-1)C \end{cases} .$$

Also for any two intervals $A = [a, b]$, $B = [c, d] \in \mathcal{K}_c$, $A \ominus_g B$ always exists and

$$A \ominus_g B = [\min\{a - c, b - d\}, \max\{a - c, b - d\}].$$

2.2. Differentiation of interval valued functions

The function $f : R^n \longrightarrow \mathcal{K}_c$, defined on Euclidean space R^n, is said to be the interval valued function. That is, $f(\mathbf{x}) = f(x_1, ..., x_n)$ is a closed interval in R for each $\mathbf{x} \in R^n$. The interval valued function $f(\mathbf{x})$ can also be written as $f(\mathbf{x}) = [f^L(\mathbf{x}), f^U(\mathbf{x})]$, where f^L and f^U are real valued functions and $f^L(\mathbf{x}) \leq f^U(\mathbf{x})$ for every $\mathbf{x} \in R^n$, and are known as lower and upper (end point) functions of f.

A straightforward concept of differentiability of interval valued functions was introduced in Wu (2007).

DEFINITION 2 Consider $f(\mathbf{x}) = [f^L(\mathbf{x}), f^U(\mathbf{x})]$ to be an interval valued function defined on $X \subset R^n$. We say that f is weakly continuously differentiable at $\mathbf{x}_0$, if the real valued functions f^L and f^U are continuously differentiable at $\mathbf{x}_0$ (i.e., all partial derivatives of f^L and f^U exist in some neighborhood of $\mathbf{x}_0$ and are continuous at $\mathbf{x}_0$).

Next, in the papers of Wu (2007, 2009), the author used the concept of H–differentiability for interval valued functions to study KKT optimality conditions of programming problems with interval valued objective functions. However, this definition of differentiability is restrictive; e.g., consider a simple interval valued function $f(x) = [ax^5 + x^3 - 1, a - ax^3 - a^2x^5]$, where $-1 < a \in R$. The H-derivative of f does not exist since H-difference $f(0 + h) \ominus_H f(0)$ does not exist as $h \longrightarrow 0^+$. In fact, if $f(\mathbf{x}) = Ph(\mathbf{x})$, where P is an interval and $h(\mathbf{x})$ is a real valued function with $h'(x) < 0$, then f is not differentiable at $\mathbf{x} = \mathbf{x}_0$ (Bede and Gal, 2005).

REMARK 1 From the above we see that H-differentiablity of interval valued functions is restrictive and, further, the simple interval valued function $f(x) = [-1, 1]|x|$, where $x \in R$, is not weakly continuously differentiable at $x = 0$. In order to overcome this problem, Chalco-Cano et al. (2013) considered the concept of gH–differentiability of interval valued functions introduced in Stefanini and Bede (2009) to investigate interval valued programming problems. Note that in this paper T denotes the interval $T = (t_1, t_2)$.

DEFINITION 3 (Stefanini and Bede, 2009) Let $f : T \longrightarrow \mathcal{K}_c$ be an interval valued function. Then f is said to be gH-differentiable at $t_0 \in T$ if

$$f'(t_0) = \lim_{h \longrightarrow 0} \frac{f(t_0 + h) \ominus_g f(t_0)}{h}$$

exists in $\mathcal{K}_c$. Also we say that f is gH-differentiable on T if f is gH-differentiable at each $t_0 \in T$.

THEOREM 1 (Chalco-Cano et al., 2011) Let $f(t) = [f^L(t), f^U(t)]$ be an interval valued function. If f^L and f^U are differentiable at $t_0 \in T$ then f is gH-differentable at t_0 and

$$f'(t_0) = [\min\{(f^L)'(t_0), (f^U)'(t_0)\}, \max\{(f^L)'(t_0), (f^U)'(t_0)\}].$$

The converse of above theorem is not true (see Chalco-Cano et al., 2011). However, we have the following result.

THEOREM 2 *(Chalco-Cano et al., 2011) Let $f(t) = [f^L(t), f^U(t)]$ be an interval valued function. Then f is gH-differentiable at $t_0 \in T$ if and only if one of the following cases holds:*
  (i) f^L and f^U are differentiable at t_0.
  (ii) The derivatives $(f^L)'_-(t_0), (f^L)'_+(t_0), (f^U)'_-(t_0)$ and $(f^U)'_+(t_0)$ exist and satisfy $(f^L)'_-(t_0) = (f^U)'_+(t_0)$ and $(f^L)'_+(t_0) = (f^U)'_-(t_0)$.

PROPOSITION 1 *(Aubin and Cellina, 1984) Let $f(t) = [f^L(t), f^U(t)]$ be an interval valued function defined on $X \subseteq R^n$ and $\boldsymbol{x}_0 \in X$. Then f is continuous at $\boldsymbol{x}_0$ if and only if f^L and f^U are continuous at x_0.*

DEFINITION 4 (Chalco-Cano et al., 2013) Let $f(t) = [f^L(t), f^U(t)]$ be an interval valued function defined on $X \subseteq R^n$ and let $\mathbf{x}_0 = (x_1^{(0)}, ..., x_n^{(0)})$ be fixed in X.
  (i) We consider the interval valued function $h_i(x_i) = f(x_1^{(0)}, ..., x_{i-1}^{(0)}, x_i^{(0)},$
       $x_{i+1}^{(0)}, ..., x_n^{(0)})$. If h_i is gH-differentable at $x_i^{(0)}$, then we say that f has the ith
       partial gH-derivative at $\mathbf{x}_0$ $\left(\text{denoted by } \left(\frac{\partial f}{\partial x_i} \right)_g (\mathbf{x}_0) \right)$ and $\left(\frac{\partial f}{\partial x_i} \right)_g (\mathbf{x}_0) =$
       $(h_i)'(x_i^{(0)})$.
  (ii) We say that f is continuously gH-differentiable at $\mathbf{x}_0$ if all the partial gH-
       derivatives of $\left(\frac{\partial f}{\partial x_i} \right)(\mathbf{x}_0), i = 1, ..., n$ exist in some neighbourhood of $\mathbf{x}_0$ and
       are continuous at $\mathbf{x}_0$ (in the sense of interval valued function).

REMARK 2 We remark that the continuous gH-differentiablity is more general than the weakly continuously differentiability of interval valued function. For example the function $f(t) = [-|t|, |t|], t \in R$, which is not weakly continuous differentiable at $t = 0$, is continuously gH-differentiable at $t = 0$ and $f'(t) = [-1, 1]$, for all $t \in R$.

Next we consider the (interval) multivalued function $F(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_r(\mathbf{x}))$ defined on $X \subseteq R^n$, where f_k is the interval valued function for $k = 1, ..., r$. Therefore, we have $f_k(\mathbf{x}) = [f_k^L(\mathbf{x}), f_k^U(\mathbf{x})], k = 1, ..., r$. Now we introduce the following:

DEFINITION 5 Let $F(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_r(\mathbf{x}))$ be (interval) multivalued function. We say that F is
  (i) (weakly) continuously differentiable at $\mathbf{x}_0 \in X$ if $f_k, k = 1, ..., r$, are weakly continuously differentiable at $\mathbf{x}_0$.
  (ii) continuously gH-differentiable at $\mathbf{x}_0 \in X$ if $f_k, k = 1, ..., r$, are continuously gH-differentiable at $\mathbf{x}_0$.

Note that from Definitions 2 and 5(i), we see that the (interval) multivalued function $F = (f_1(\mathbf{x}), ..., f_r(\mathbf{x}))$ is (weakly) continuously differentiable at $\mathbf{x}_0$ if the real valued functions f_k^L and $f_k^U, k = 1, ..., r$ are differentable at $\mathbf{x}_0$.

3. Solution concepts

Consider the following (interval) multiobjective programing problem:
$(MIP1)$

Minimize $F(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_r(\mathbf{x}))$

Subject to $\mathbf{x} = (x_1, ..., x_n) \in X \subseteq R^n$.

Here, $f_k(\mathbf{x}) = [f_k^L(\mathbf{x}), f_k^U(\mathbf{x})], k = 1, ..., r$, are interval valued functions and the feasible set X is assumed to be a convex subset of R^n. Since each f_k is a closed interval in R, we may follow the similar solution concept as that proposed in Wu (2007). In Wu (2007), a partial ordering $"\preceq_{LU}"$ was invoked between two closed intervals as follows:

Let $A, B \in \mathcal{K}_c$, then we say that $A \preceq_{LU} B$ iff $a^L \le b^L$ and $a^U \le b^U$ and $A \prec_{LU} B$ iff $A \preceq_{LU} B$ and $A \ne B$ or, equivalently, $A \prec_{LU} B$ if and only if

$$\left\{ \begin{array}{l} a^L < b^L \\ a^U \le b^U \end{array} \right. \quad or \quad \left\{ \begin{array}{l} a^L \le b^L \\ a^U < b^U \end{array} \right. \quad or \quad \left\{ \begin{array}{l} a^L < b^L \\ a^U < b^U \end{array} \right. . \tag{3.1}$$

Next a vector $\mathbf{A} = (A_1, ..., A_r)$ is said to be an interval valued vector if each component $A_k = [a_k^L, a_k^U]$ is a closed interval for $k = 1, ..., r$. Also for any two interval valued vectors $\mathbf{A} = (A_1, ..., A_r)$ and $\mathbf{B} = (B_1, ..., B_r)$ we write $\mathbf{A} \preceq_{LU} \mathbf{B}$ if and only if $A_k \preceq_{LU} B_k$ for each $k = 1, ..., r$, and $\mathbf{A} \prec_{LU} \mathbf{B}$ if and only if $A_k \preceq_{LU} B_k$ for each $k = 1, ..., r$, and $A_h \prec_{LU} B_h$ for at least one index h. Based on the above, since $F(\mathbf{x})$ is also interval valued vector, Wu (2009) proposed the concept of Pareto optimal solutions as follows.

DEFINITION 6 (Wu, 2009) Let $\mathbf{x}^*$ be feasible solution of (MIP1). We say that $\mathbf{x}^*$ is
 (i) LU-Pareto optimal solution of $(MIP1)$ if there exists no $\bar{\mathbf{x}} \in X$, s.t. $F(\bar{\mathbf{x}}) \prec_{LU} F(\mathbf{x}^*)$.
 (ii) strongly LU-Pareto optimal solution of $(MIP1)$ if these exists no $\bar{x} \in X$, s.t. $F(\bar{x}) \preceq_{LU} F(x^*)$.
 (iii) weakly LU-Pareto optimal solution of $(MIP1)$ if these exists no $\bar{\mathbf{x}} \in X$, s.t. $f_k(\bar{\mathbf{x}}) \prec_{LU} f_k(\mathbf{x}^*)$ for $k = 1, \cdots, r$.

REMARK 3 (Wu, 2009) Let us denote by $X_{WP}^{LU}, X_P^{LU}, X_{SP}^{LU}$ the set of weakly LU-Pareto optimal solutions, LU-Pareto optimal solutions and strongly LU-Pareto optimal solutions, respectively. Then $X_{SP}^{LU} \subseteq X_P^{LU} \subseteq X_{WP}^{LU}$.

EXAMPLE 1 Consider the following interval valued functions.

$$f_1 = \left\{ \begin{array}{l} [0, x], if\ x \ge 0 \\ [x, 0], if\ x \le 0 \end{array} \right., \quad f_2 = \left\{ \begin{array}{l} [0, x(x^2 + 1)], if\ x \ge 0 \\ [x(x^2 + 1), 0], if\ x \le 0 \end{array} \right.,$$

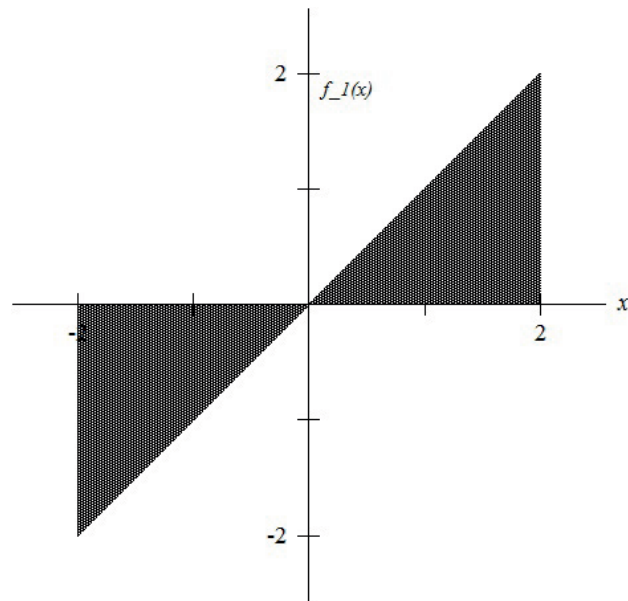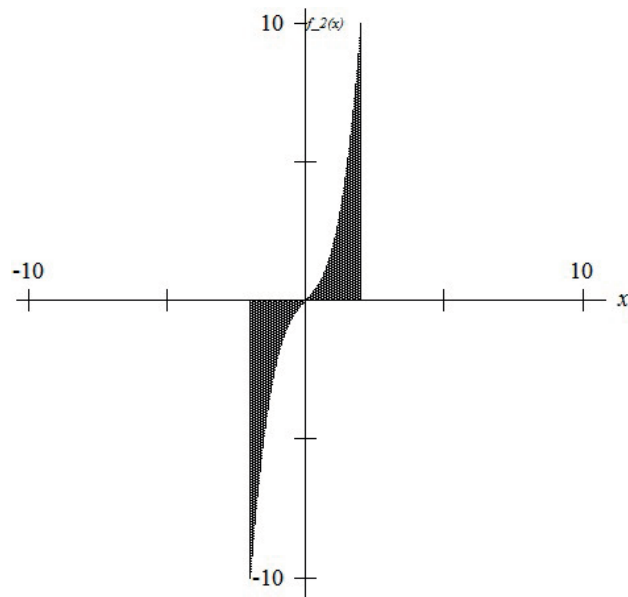$f_3 = [\sin x^2, \sin x^2 + 1]$, $f_4 = [\sin x^3, \sin x^3 + 1]$, $f_5 = [\sin (x-1)^3, \sin (x-1)^3 + 1]$.
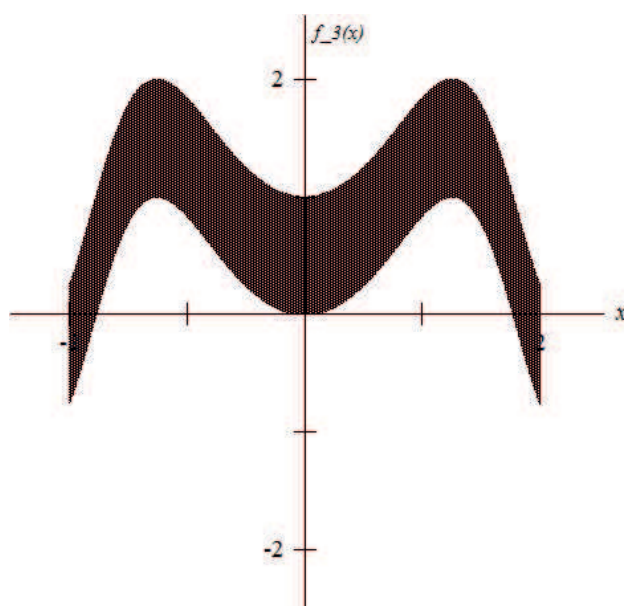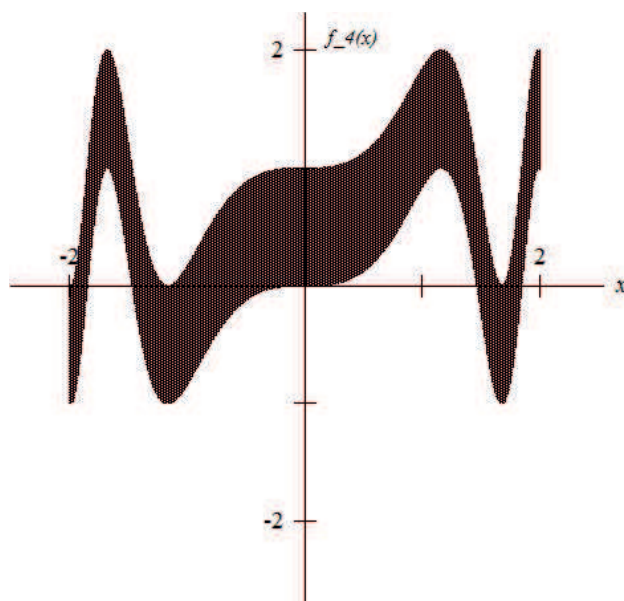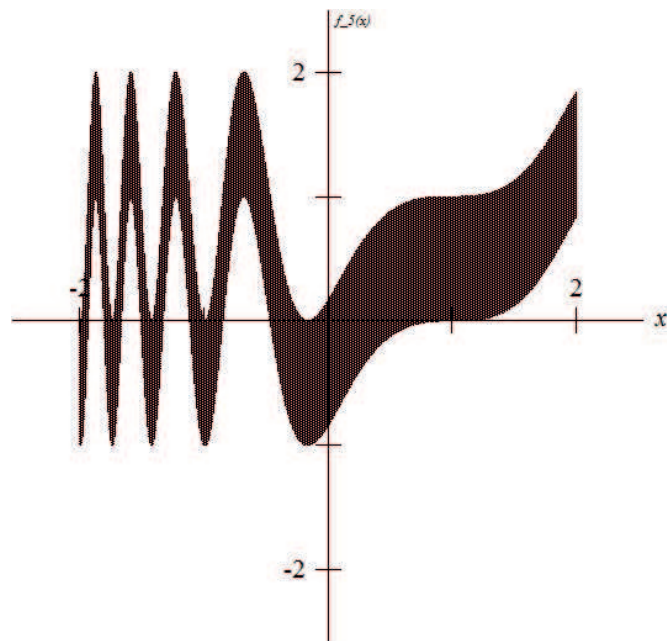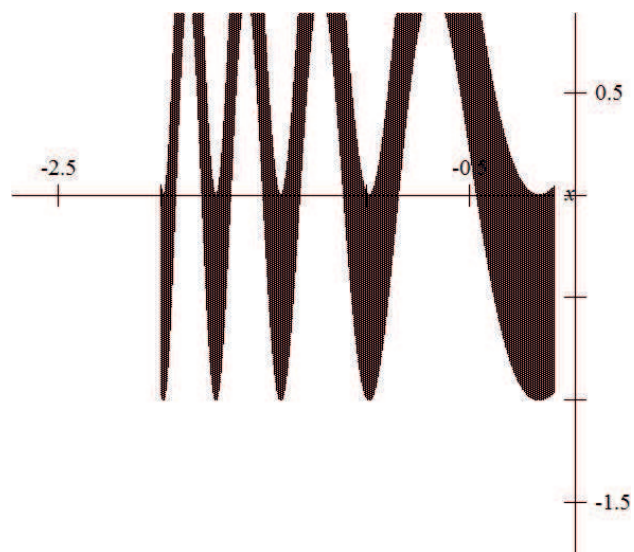
Figure 1.

Figure 2.

Figure 3.

Figure 4.

Figure 5.

Figure 6.

Now, consider the problem (P_1) as follows

$$\min F(x)$$

subject to $x - 2 \leq 0$

$$-x - 2 \leq 0.$$

Consider $F(x) = (f_1(x), f_2(x))$, then it is clear that there exist no $-2 \leq \bar{x} \leq 2$ such that $f_1(\bar{x}) \preceq_{LU} f_1(-2)$ and $f_2(\bar{x}) \preceq_{LU} f_2(-2)$ (see Figs. 1 and 2). Therefore, by Definition 6 (ii) and Remark 3, we say that $-2 \in X_{SP}^{LU} \cap X_P^{LU} \cap X_{WP}^{LU}$ for problem (P_1).

Next, if we assume that $F(x) = (f_1(x), f_2(x), f_3(x))$, then there exists $\bar{x}_{31} = 2$, such that $f_3(2) = f_3(-2)$ (see Fig. 3). Therefore, $-2 \in X_P^{LU} \cap X_{WP}^{LU}$, but $-2 \notin X_{SP}^{LU}$ for problem (P_1).

Again, if we assume that $F(x) = (f_3(x), f_4(x), f_5(x))$, then $-2 \leq \bar{x}_{41} \approx -1.1656$, $\bar{x}_{42} \approx -1.6833 \leq 2$, such that $f_4(\bar{x}_{41}) = f_4(\bar{x}_{42}) = f_3(-2)$ (see Fig. 4) and $-2 \leq \bar{x}_{51} \approx -1.7325$, $\bar{x}_{52} \approx -1.418$, $\bar{x}_{53} \approx -0.9849$, $\bar{x}_{54} \approx -0.1599 \leq 2$, such that $f_5(\bar{x}_{51}) = f_4(\bar{x}_{52}) = f_4(\bar{x}_{53}) = f_4(\bar{x}_{54}) = f_3(-2)$ (see Figs. 5 and 6). Therefore, $-2 \in X_{WP}^{LU}$, but $-2 \notin X_P^{LU}, X_{SP}^{LU}$ for problem (P_1).
Note that the values are determined and graphs are plotted by using GrafEq 2.13, available at www.peda.com/grafeq/

Next, we consider another solution concept, following from Chalco-Cano et al. (2013):
Let $A = [a^L, a^U]$, the width (spread) of A is defined by $w(A) = a^S = a^U - a^L$.
Let $A = [a^L, a^U], B = [b^L, b^U]$ be two closed intervals. Chalco-Cano et al. (2013) proposed the ordering relation between A and B by considering the minimization and maximization problems separately.

(i) For maximization, we write $A \succeq_{LS} B$ if and only if $a^U \geq b^U$ and $a^S \leq b^S$, the width of the interval can be regarded as uncertainty (noise, risk or a type variance). Therefore, the interval with smaller width (i.e., smaller uncertainty) and higher upper bound is considered better.

(ii) For minimization, we write $A \preceq_{LS} B$ if and only if $a^L \leq b^L$ and $a^S \leq b^S$. In this case, the interval with smaller width (i.e., smaller uncertainty) and smaller lower bound is considered better.

We write $A \prec_{LS} B$ if and only if $A \succeq_{LS} B$ and $A \neq B$, i.e., $A \prec_{LS} B$ if and only if

$$\begin{cases} a^L < b^L \\ a^S \leq b^S \end{cases} or \begin{cases} a^L \leq b^L \\ a^S < b^S \end{cases} or \begin{cases} a^L < b^L \\ a^S < b^S \end{cases}. \tag{3.2}$$

Next, consider $\mathbf{A} = (A_1, ..., A_r)$ and $\mathbf{B} = (B_1, ..., B_r)$ to be two interval valued vectors. We write $\mathbf{A} \preceq_{LS} \mathbf{B}$ if and only if $A_k \preceq_{LS} B_k$ for each $k = 1, ..., r$, and $\mathbf{A} \prec_{LS} \mathbf{B}$ if and only if $A_k \preceq_{LS} B_k$ for $k = 1, ..., r$, and $A_h \prec_{LS} B_h$ for at least one index h.

DEFINITION 7 Let $\mathbf{x}^*$ be a feasible solution of $(MIP1)$. We say that $\mathbf{x}^*$ is

(i) LS-Pareto optimal solution of $(MIP1)$ if there exists no $\bar{x} \in X$, s.t. $F(\bar{x}) \prec_{LS} F(\mathbf{x}^*)$.

(ii) strongly LS-Pareto optimal solution of $(MIP1)$ if these exists no $\bar{\mathbf{x}} \in X$. s.t. $F(\bar{\mathbf{x}}) \preceq_{LS} F(\mathbf{x}^*)$.

(iii) weakly LS-Pareto optimal solution of $(MIP1)$ if these exists no $\bar{\mathbf{x}} \in X$. s.t. $f_k(\bar{\mathbf{x}}) \prec_{LS} f_k(\mathbf{x}^*)$ for $k = 1, \cdots, r$.

REMARK 4 Let us denote by $X_{WP}^{LS}, X_P^{LS}, X_{SP}^{LS}$ the set of weakly LS-Pareto optimal solutions, LS-Pareto optimal solutions, and strongly LS-Pareto optimal solutions, respectively. Then it is easy to see that $X_{SP}^{LS} \subseteq X_P^{LS} \subseteq X_{WP}^{LS}$.

EXAMPLE 2 Consider the following interval valued functions.

$$f_6 = \begin{cases} [\min\{x^3, 2\sin x\}, 0] \ if \ x \leq 0 \\ [0, \max\{x^2, 2\sin x\}] \ if \ x \geq 0 \end{cases}, \ f_7 = [\cos x^3 - 1, \cos x^3],$$

$$f_8 = [\cos x^4 - 1, \cos x^4], f_9 = [\cos x^5 - 1, \cos x^5].$$

Consider $F(x) = (f_1(x), f_2(x), f_6(x))$, then it is easy to see that there exist no $-2 \leq \bar{x} \leq 2$ such that $f_1(\bar{x}) \preceq_{LS} f_1(0), f_2(\bar{x}) \preceq_{LS} f_2(0)$ and $f_6(\bar{x}) \preceq_{LS} f_6(0)$ (see Figs. 1, 2 and 7). Therefore, by Definition 7 (ii) and Remark 4, we say that $0 \in X_{SP}^{LS} \cap X_P^{LS} \cap X_{WP}^{LS}$ for problem (P_1).

Next, if we assume that $F(x) = (f_1(x), f_2(x), f_6(x), f_7(x))$, then there exist $\bar{x}_{7i} \in (0 - \epsilon_7, 0 + \epsilon_7), \epsilon_7 \approx 0.31798 > 0$, such that $f_7(\bar{x}_{7i}) = f_7(0)$ (see Fig. 8). Therefore $0 \in X_P^{LS} \cap X_{WP}^{LS}$, but $0 \notin X_{SP}^{LS}$ for problem (P_1).

Again, if we assume that $F(x) = (f_7(x), f_8(x), f_9(x))$, then there exist $\bar{x}_{8i} \in (0 - \epsilon_8, 0 + \epsilon_8), \epsilon_8 \approx 0.396315 > 0$, such that $f_8(\bar{x}_{8i}) = f_8(0)$ (see Fig. 9) and $\bar{x}_{9i} \in (0 - \epsilon_9, 0 + \epsilon_9), \epsilon_9 \approx 0.458054 > 0$, such that $f_9(x_{9i}) = f_9(0)$ (see Figs. 10 and 11). Therefore, $0 \in X_{WP}^{LS}$, but $-2 \notin X_P^{LS}, X_{SP}^{LS}$ for problem (P_1).

PROPOSITION 2 Let $A, B \in \mathcal{K}_c$.
  (i) If $A \preceq_{LS} B$ then $A \preceq_{LU} B$. (Chalco-Cano et al., 2013).
  (ii) If $A \prec_{LS} B$ then $A \prec_{LU} B$.

PROOF For (ii) we have for $A \prec_{LS} B$:
Case I. $a^L < b^L, a^S \leq b^S$. This implies $a^L < b^L, a^U - a^L \leq b^U - b^L$. Then we have $a^U < a^U + (b^L - a^L) \leq b^L + (b^U - b^L) = b^U$. Therefore, we have $A \prec_{LU} B$.
Case II. $a^L \leq b^L, a^S < b^S$ and Case III. $a^L < b^L, a^S < b^S$ follow, similarly. □

Note that the converse of Proposition 2 is not valid.

PROPOSITION 3 Let $\boldsymbol{A} = (A_1, ..., A_r)$ and $\boldsymbol{B} = (B_1, ..., B_r)$ be interval valued vectors.
  (i) If $\boldsymbol{A} \preceq_{LS} \boldsymbol{B}$ then $\boldsymbol{A} \preceq_{LU} \boldsymbol{B}$.
  (ii) If $\boldsymbol{A} \prec_{LS} \boldsymbol{B}$ then $\boldsymbol{A} \prec_{LU} \boldsymbol{B}$.

PROOF (i) Since $\mathbf{A}$ and $\mathbf{B}$ are interval valued vectors and $\mathbf{A} \preceq_{LS} \mathbf{B}$, then $A_k \preceq_{LS} B_k$ for all $k = 1, ..., r$. Therefore, result follows from (i) of Proposition 2 and (ii) follows from above and (ii) of Proposition 2 immediately. □
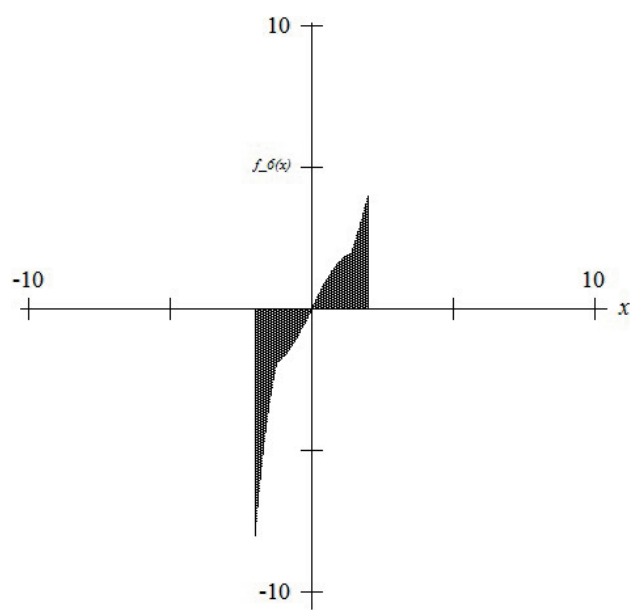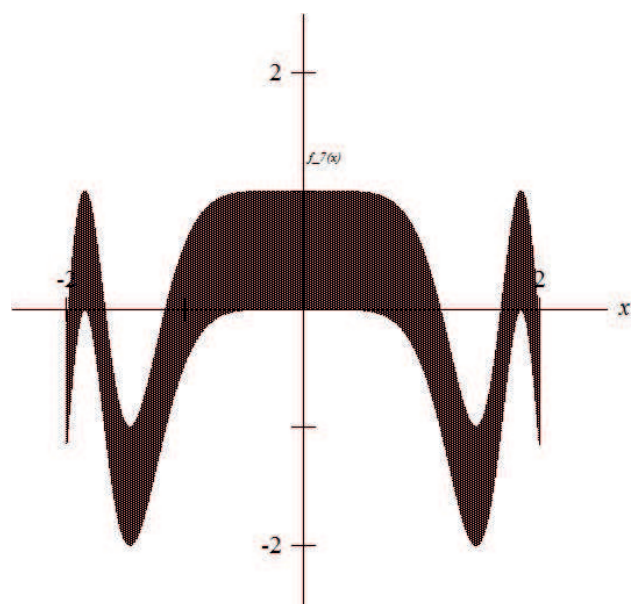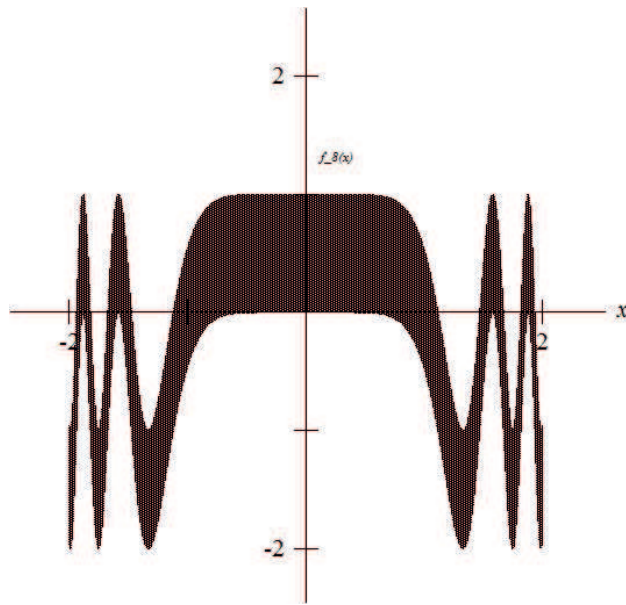
Figure 7.

Figure 8.

Figure 9.
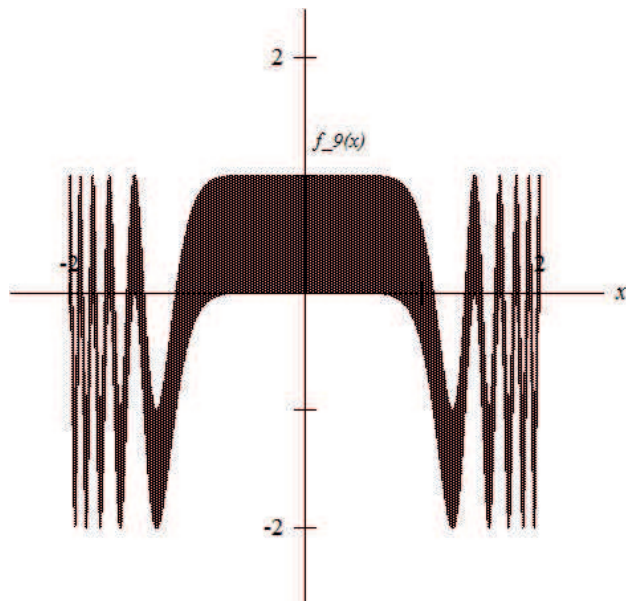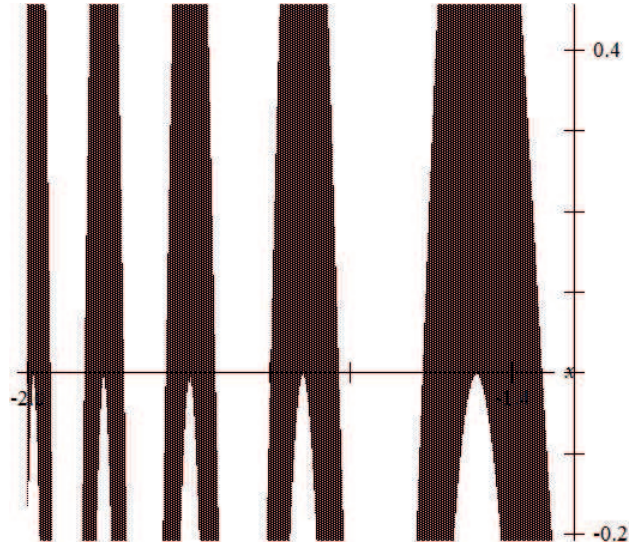

Figure 10.

Figure 11.

Note that the converse of Proposition 3 is not valid, for example let $\mathbf{A} = ([-x, y], [-x, y])$ and $\mathbf{B} = \left(\left[\frac{-x}{2}, y\right], \left[\frac{-x}{2}, y\right]\right)$, $x, y \in R$, then $\mathbf{A} \prec_{LU} \mathbf{B}$, but $\mathbf{A} \npreceq_{LS} \mathbf{B}$.

The following theorem gives the relation between two solution concepts.

THEOREM 3 *Let X be a feasible set of* $(MIP1)$. *Then*

    (i) $X_{SP}^{LU} \subseteq X_{SP}^{LS}$
    (ii) $X_{P}^{LU} \subseteq X_{P}^{LS}$
    (iii) $X_{WP}^{LU} \subseteq X_{WP}^{LS}$.

PROOF Let $\mathbf{x}$ be the feasible solution of $(MIP1)$.
For (i) Let $\mathbf{x} \in X_{SP}^{LU}$. If it is possible that $\mathbf{x} \notin X_{SP}^{LS}$, then by Definition 7 there exist $\hat{\mathbf{x}} \in X$, s.t., $F(\hat{\mathbf{x}}) \preceq_{LS} F(\mathbf{x})$. From Proposition 3, we see that $F(\hat{\mathbf{x}}) \preceq_{LU} F(\mathbf{x})$. This is a contradiction. Hence, we see that $X_{SP}^{LU} \subseteq X_{SP}^{LS}$.
(ii) follows along similar lines.
For (iii) let $\mathbf{x} \in X_{WP}^{LU}$ and consider $x \notin X_{WP}^{LS}$; then by Definition 7 there exists $\hat{\mathbf{x}} \in X$ s.t., $f_k(\bar{\mathbf{x}}) \prec_{LS} f_k(\mathbf{x})$ for all $k = 1, ..., r$. From Proposition 2 $f_k(\hat{\mathbf{x}}) \prec_{LU} f_k(\mathbf{x})$ for all $k = 1, ..., r$. This, however, is a contradiction, because $\mathbf{x} \in X_{WP}^{LU}$. Hence, $X_{WP}^{LU} \subseteq X_{WP}^{LS}$. $\qquad\square$

Note that the converse of above theorem is not valid as we show in the following example.

EXAMPLE 3 Consider the following optimization problem

$$\min \ F(x) = \left([-x, 0], \left[\frac{-x}{2}, 0 \right] \right) \tag{3.3}$$

subject to $x \in R^+$.

(i) We show $x^* = 0 \in X_{SP}^{LS}$. Since, if we suppose that $x^* = 0 \notin X_{SP}^{LS}$, then by Definition 7, there exist $x \neq 0$ in R^+ s.t. $F(x) \preceq_{LS} F(0)$, i.e.,

$$\left([-x, 0], \left[\frac{-x}{2}, 0 \right] \right) \preceq_{LS} ([0,0], [0,0]),$$

i.e.,

$$f_1^S(x) = x \leq 0 = f_1^S(0) \text{ and } f_2^S(x) = \frac{x}{2} \leq 0 = f_2^S(0),$$

which is a contradiction, because $x > 0$. Hence, $x^* = 0 \in X_{SP}^{LS}$. But $x^* = 0 \notin X_{SP}^{LU}$, since there exists $1 \in R^+$ s.t. $F(1) = \left([-1,0], \left[\frac{-1}{2}, 0 \right] \right) \prec_{LU} F(0) = ([0,0], [0,0])$. Also, since $x^* = 0 \in X_{SP}^{LS}$, from Remark 4, we have $x^* = 0 \in X_P^{LS}$ and hence (ii) follows similarly. Also from Remark 4, we have $x^* = 0 \in X_{WP}^{LS}$, but $x^* = 0 \notin X_{WP}^{LU}$, since there exist $1 \in R^+$, s.t. $f_1(1) \prec_{LU} f_1(0)$ and $f_2(1) \prec_{LU} f_2(0)$.

4.   Karush-Kuhn-Tucker type optimality conditions

Consider (interval) multiobjective programming problem
($MIP2$)

   Minimize $F(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_r(\mathbf{x}))$

   Subject to $g_i(\mathbf{x}) \leq 0, i = 1, ..., m,$

where $X = \{\mathbf{x} \in R^n : g_i(\mathbf{x}) \leq 0, i = 1, ..., m\}$ is a feasible set.

   In this section we shall obtain KKT type optimality conditions for the optimization problem ($MIP2$) by using gH-differentiability of interval valued functions. Firstly we define the concept of pseudoconvexity for interval valued functions.

DEFINITION 8 (Bazarra et al., 1993) Let f be a differentiable real valued function defined on non-empty convex subset X of R^n, then f is said to be pseudoconvex at $\mathbf{x}^*$ if for $f(\mathbf{x}) < f(\mathbf{x}^*)$ there is $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) < 0$ for $\mathbf{x} \in X$ and f is strictly pseudoconvex at $\mathbf{x}^*$ if for $f(\mathbf{x}) \leq f(\mathbf{x}^*)$ there is $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) < 0$ for $\mathbf{x} \in X$.

   Wu (2009) extended the concept of pseudoconvexity to interval valued functions as follows.

DEFINITION 9 (Wu, 2009) Consider an interval valued function f defined on nonempty convex subset $X \subseteq R^n$. We say that f is LU-pseudoconvex (respectively strictly LU-pseudoconvex) at $\mathbf{x}^* \in X$ if and only if f^L and f^U are pseudoconvex (respectively strictly pseudoconvex) at $\mathbf{x}^*$.

Note that if interval valued function f is strictly LU-peudoconvex at $\mathbf{x}^*$ then f is also LU-pseudoconvex at $\mathbf{x}^*$ (Wu, 2009). Similarly, we may extend the concept of pseudoconvexity to interval valued function in the LS-sense as follows.

DEFINITION 10 Consider an interval valued function f defined on nonempty convex subset X of R^n and let $\mathbf{x}^* \in X$. We say that f is LS-pseudoconvex (respectively strictly LS-pseudoconvex) at $\mathbf{x}^*$ if and only if f^L and f^S are pseudoconvex (respectively strictly pseudoconvex) at $\mathbf{x}^*$.

The above definitions can be extend to (interval) multivalued functions as follows:

DEFINITION 11 Let X be a nonempty convex subset of R^n and let $\mathbf{x}^* \in X$. We say that the (interval) multivalued function $F(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_r(\mathbf{x}))$ is
   (i) LU-pseudoconvex (respectively strictly LU-pseudoconvex) at $\mathbf{x}^*$ if and only if $f_k, k = 1, ..., r$ are LU-pseudoconvex (respectively strictly LU-pseudoconvex) at $\mathbf{x}^*$.
   (ii) LS-pseudoconvex (respectively strictly LS-pseudoconvex) at $\mathbf{x}^*$ if and only if $f_k, k = 1, ..., r$ are LS-pseudoconvex (respectively strictly LS-pseudoconvex) at $\mathbf{x}^*$.

PROPOSITION 4 *Let F be (interval) multivalued function defined on convex subset X of R^n and let $\boldsymbol{x}^* \in X$. Then*
   (i) *F is LU-pseudoconvex (respectively strictly LU-pseudoconvex) at $\boldsymbol{x}^*$ if and only if f_k^L and $f_k^U, k = 1, ..., r$ are pseudoconvex (respectively strictly pseudoconvex) at $\boldsymbol{x}^*$.*
   (ii) *F is LS-pseudoconvex (respectively strictly LS-pseudoconvex) at $\boldsymbol{x}^*$ if and only if f_k^L and $f_k^S, k = 1, ..., r$ are pseudoconvex (respectively strictly pseudoconvex) at $\boldsymbol{x}^*$.*

PROOF From Definitions 9, 10 and 11 the result follows immediately.   □

DEFINITION 12 (Bazarra et al., 1993) The cone of feasible directions of non-empty set $X \in R^n$ at $\mathbf{x}^*$ is defined as

$$\mathcal{D} = \{\mathbf{d} \in R^n : \mathbf{d} \neq 0, \text{ there exist } \delta > 0, \text{ such that } \mathbf{x}^* + \tau\mathbf{d} \in X, \ \forall \ \tau \in (0, \delta)\}$$

and $\mathbf{d} \in D$ is called feasible direction of X.

PROPOSITION 5 *(Bazarra et al., 1993) Let $X = \{\boldsymbol{x} \in R^n : g_i(\boldsymbol{x}) \leq 0, i = 1, ..., m\}$ be a feasible set and a point $\boldsymbol{x}^* \in X$. Let g_i be differentiable at $\boldsymbol{x}^*$ for all $i = 1, ..., m$. Let $J(\boldsymbol{x}^*) = \{i : g_i(\boldsymbol{x}^*) = 0\}$ be the index set for the active constraints. Then*

$$\mathcal{D} \subseteq \{\boldsymbol{d} \in R^n : \nabla g_i(\boldsymbol{x}^*)^T \boldsymbol{d} \leq 0 \text{ for each } i \in J(\boldsymbol{x}^*)\}.$$

(Note that this proposition still holds true if we just assume that g_i are continuous at $\mathbf{x}^*$ instead of differentiable at $\mathbf{x}^*$ for $i \notin J$).

Next, the Tucker's theorem of alternative states that, given the matrices P and Q, exactly one of the following systems has a solution:

System 1: $P\mathbf{x} \leq 0, P\mathrm{x} \neq \mathbf{0}$, $Q\mathrm{x} \leq 0$ for some $\mathbf{x} \in R^n$;

System 2: $P^T \lambda + Q^T \mu = \mathbf{0}$ for some $\lambda > 0$ and $\mu \geq 0$.

We also say that the constraint functions $g_i, i = 1, ..., m$, satisfy KKT-assumptions at $\mathbf{x}^*$ if g_i are continuous on R^n and are continuously differentiable at $\mathbf{x}^* \in X$ (Wu, 2007).

In the rest of this paper, we shall assume that the feasible set X of problem $(MIP2)$ is a convex subset of R^n and the real valued constraint functions $g_i, i = 1, ..., m$, satisfy KKT-assumptions at $\mathbf{x}^* \in X$.

THEOREM 4 *Assume that the (interval) multiobjective function F is strictly LU-pseudoconvex and continuously gH-differentiable at $\boldsymbol{x}^*$. If there exist (Lagrange) multipliers $0 < \lambda_k^L, \lambda_k^U \in R, k = 1, ..., r$ and $0 \leq \mu_i^L, \mu_i^U \in R, i = 1, ..., m$ such that the following KKT conditions hold:*

(i) $\sum_{k=1}^r \lambda_k^L \nabla f_k^L(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}$;

(ii) $\sum_{k=1}^r \lambda_k^U \nabla f_k^U(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(x^*) = \boldsymbol{0}$;

(iii) $\mu_i^L g_i(\boldsymbol{x}^*) = 0 = \mu_i^U g_i(\boldsymbol{x}^*), i = 1, ..., m$,

then $\boldsymbol{x}^* \in X_P^{LU} \cap X_P^{LS}$ for $(MIP2)$.

PROOF Since F is strictly LU-pseudoconvex at $\mathbf{x}^*$, we see by Proposition 4 f_k^L and $f_k^U, k = 1, ..., r$, are strictly pseudoconvex at $\mathbf{x}^*$. We shall prove the result by contradiction. Suppose that $\mathbf{x}^* \notin X_P^{LU}$, then by Definition 6 there exists $\hat{\mathbf{x}} (\neq \mathbf{x}^*) \in X$ such that

$$F(\hat{\mathbf{x}}) \prec_{LU} F(\mathbf{x}^*)$$

i.e. there exists $h, 1 \leq h \leq r$ such that

$$f_h(\hat{\mathbf{x}}) \prec_{LU} f_h(\mathbf{x}^*)$$

or, equivalently,

$$f_h^L(\hat{\mathbf{x}}) < f_h^L(\mathbf{x}^*) \text{ or } f_h^U(\hat{\mathbf{x}}) < f_h^U(\mathbf{x}^*).$$

Case I. Consider the case $f_h^L(\hat{\mathbf{x}}) < f_h^L(\mathbf{x}^*)$. Since f_h^L is strictly pseudoconvex, we have

$$\nabla f_h^L(\mathbf{x}^*)^T(\hat{\mathbf{x}} - \mathbf{x}^*) < 0. \tag{4.1}$$

Also for $k \neq h, k = 1, ..., r$, we have either $f_k^L(\hat{\mathbf{x}}) < f_k^L(\mathbf{x}^*)$ or $f_k^L(\hat{\mathbf{x}}) \leq f_k^L(\mathbf{x}^*)$. Therefore, we have

$$\nabla f_h^L(\mathbf{x}^*)^T(\hat{\mathbf{x}} - \mathbf{x}^*) < 0, \text{ for } k \neq h. \tag{4.2}$$

Now, let $\mathbf{d} = \hat{\mathbf{x}} - \mathbf{x}^*$. Then $\mathbf{y} = \mathbf{x}^* + \tau \mathbf{d} = \mathbf{x}^* + \tau(\hat{\mathbf{x}} - \mathbf{x}^*) = \tau\hat{\mathbf{x}} + (1-\tau)\mathbf{x}^*$.

Therefore, $\mathbf{y} \in X$ for $\tau \in (0,1)$, since X is a convex set and $\hat{\mathbf{x}}, \mathbf{x}^* \in X$. This shows that $\mathbf{d} \in \mathcal{D}$ is a feasible direction of X. From Proposition 5, we have

$$\nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leq 0, i \in J(\mathbf{x}^*). \tag{4.3}$$

Further, let P be the matrix whose rows are $\nabla f_k^L(\mathbf{x}^*)^T$ for $k = 1, ..., r$, and Q be the matrix whose rows are $\nabla g_i(\mathbf{x}^*)^T$ for $i \in J$. From (4.1) - (4.3) we conclude that $\mathbf{d}$ is the solution of system 1 of Tucker's theorem. Hence, there exist no multipliers $0 < \lambda_k^L, k = 1, ..., r$, and $0 \leq \mu_i^L, i \in J$, such that

$$\sum_{k=1}^{r} \lambda_k^L \nabla f_k^L(\mathbf{x}^*) + \sum_{i \in J} \nabla \mu_i^L g_i(\mathbf{x}^*) = \mathbf{0}.$$

Now, by taking $\mu_i^L = 0$ for $i \notin J, i = 1, ..., m$, we get a contradiction with respect to (i) and (iii) of the theorem.

Case (II). In this case consider $f_h^U(\mathbf{x}) < f_h^U(\mathbf{x}^*)$; then, by proceeding similarly as before, we get a contradiction with respect to (ii) and (iii) of the theorem. This contradiction shows that $\mathbf{x}^* \in X_P^{LU}$. Hence, the result follows from Theorem 3. $\square$

EXAMPLE 4 Consider the following programming problem:

$$\min\ F = \left([4x_1 - x_2 - 1, 4x_1 - x_2 + 1], \left[\frac{-x_1}{2} + x_2 - 1, \frac{-x_1}{2} + x_2 + 1\right] \right)$$

$$\text{subject to } -x_1 + 1 \leq 0;$$

$$2x_1 + x_2 - 8 \leq 0;$$

$$x_2 - 5 \leq 0;$$

$$x_1 - x_2 - 4 \leq 0;$$

$$x_1, x_2 \geq 0.$$

It is easy to see that the above problem satisfies the assumptions of Theorem 4. Now, according to conditions (i), (ii) and (iii) of the theorem we consider the following expression.

$$\lambda_1^L \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \lambda_2^L \begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix} + \mu_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mu_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mu_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

and

$$\lambda_1^U \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \lambda_2^U \begin{bmatrix} \frac{-1}{2} \\ 1 \end{bmatrix} + \mu_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mu_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mu_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with

$$\mu_1(-x_1 + 1) = 0;$$
$$\mu_2(2x_1 + x_2 - 8) = 0;$$
$$\mu_3(x_2 - 5) = 0$$
$$\mu_4(x_1 - x_2 - 4) = 0.$$

That is, we have to solve the following simultaneous equations:

$$4\lambda_1^L - \frac{\lambda_2^L}{2} - \mu_1 + 2\mu_2 + \mu_4 = 0;$$
$$-\lambda_1^L + \lambda_2^L + \mu_2 + \mu_3 - \mu_4 = 0.$$

and

$$4\lambda_1^U - \frac{\lambda_2^U}{2} - \mu_1 + 2\mu_2 + \mu_4 = 0;$$
$$-\lambda_1^U + \lambda_2^U + \mu_2 + \mu_3 - \mu_4 = 0.$$

Upon solving them, we obtain

$$\mathbf{x}^{*T} = (0, 1), \lambda_1^L = \lambda_2^L = \frac{1}{7}, \lambda_1^U = \lambda_2^U = \frac{1}{7}$$

and

$$(\mu_1, \mu_2, \mu_3, \mu_4) = \left(\frac{1}{2}, 0, 0, 0\right)$$

.

Therefore, we have $\mathbf{x}^{*T} = (0, 1) \in X_P^{LU} \cap X_P^{LS}$ for the above problem.

THEOREM 5 *Assume that the (interval) multiobjective function F is strictly LS-pseudoconvex and (weakly) continuously differentiable at $\boldsymbol{x}^*$. If there exist (Lagrange) multipliers $0 < \lambda_k^L, \lambda_k^S \in R, k = 1, ..., r$, and $0 \le \mu_i^L, \mu_i^S \in R, i = 1, ..., m$, such that the following KKT conditions hold:*
(i) $\sum_{k=1}^r \lambda_k^L \nabla f_k^L(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i^L \nabla g_i(x^*) = 0;$
(ii) $\sum_{k=1}^r \lambda_k^S \nabla f_k^S(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i^S \nabla g_i(x^*) = 0;$
(iii) $\mu_i^L g_i(\boldsymbol{x}^*) = 0 = \mu_i^S g_i(\boldsymbol{x}^*), i = 1, ..., m,$
then $x^* \in X_P^{LS}$ for (MIP2).

PROOF The proof is same as that of Theorem 4.

REMARK 5 We remark that in Theorem 4 and Theorem 5, the objective function F has been taken strictly LU-pseudoconvex and strictly LS-pseudoconvex at $\mathbf{x}^*$, respectively. However, it is interesting to know that these results still hold true if we assume the (interval) multiobjective function F to be LU-pseudoconvex and LS-pseudoconvex at $\mathbf{x}^*$. That is, we have the following interesting results.

Theorem 6 *(A) Assume that the (interval) multiobjective function F is LU-pseudoconvex and continuously gH-differentiable at $\boldsymbol{x}^*$. If there exist (Lagrange) multipliers $0 < \lambda_k^L, \lambda_k^U \in R, k = 1, ..., r$, and $0 \leq \mu_i^L, \mu_i^U \in R, i = 1, ..., m$, such that the following KKT conditions hold:*

(i) $\sum_{k=1}^{r} \lambda_k^L \nabla f_k^L(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i^L \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0};$
(ii) $\sum_{k=1}^{r} \lambda_k^U \nabla f_k^U(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i^U \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0};$
(iii) $\mu_i^L g_i(\boldsymbol{x}^*) = 0 = \mu_i^U g_i(\boldsymbol{x}^*), i = 1, ..., m,$
then $\boldsymbol{x}^* \in X_P^{LU} \cap X_P^{LS}$ for $(MIP2)$.
(B) Assume that the (interval) multiobjective function F is LS-pseudoconvex and (weakly) continuously differentiable at $\boldsymbol{x}^*$. If there exist (Lagrange) multipliers $0 < \lambda_k^L, \lambda_k^S, k = 1, ..., r$, and $0 \leq \mu_i^L, \mu_i^S \in R, i = 1, ..., m$, such that the following KKT conditions hold:

(i) $\sum_{k=1}^{r} \lambda_k^L \nabla f_k^L(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i^L \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0};$
(ii) $\sum_{k=1}^{r} \lambda_k^S \nabla f_k^S(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i^S \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0};$
(iii) $\mu_i^L g_i(\boldsymbol{x}^*) = 0 = \mu_i^S g_i(\boldsymbol{x}^*), i = 1, ..., m,$
then $\boldsymbol{x}^* \in X_P^{LS}$ for $(MIP2)$.

Proof The proof is same as that of Theorem 4.

Next we shall present some results for weakly LU-Pareto optimal solutions and weakly LS-Pareto optimal solutions.

Theorem 7 *Assume that there is an interval valued objective function, say hth interval valued function $f_h, h \in \{1, ..., r\}$, such that it is LU-pseudoconvex and continuously gH-differentiable at $\boldsymbol{x}^*$. If there exist (Lagrange) multipliers $0 \leq \mu_i^L, \mu_i^U \in R, i = 1, ..., m$, such that*

(i) $\nabla f_h^L(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i^L \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0};$
(ii) $\nabla f_h^U(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i^U \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0};$
(iii) $\mu_i^L g_i(\boldsymbol{x}^*) = 0 = \mu_i^U g_i(\boldsymbol{x}^*), i = 1, ..., m,$
then $\boldsymbol{x}^* \in X_{WP}^{LU} \cap X_{WP}^{LS}$ for $(MIP2)$.

Proof Since for any h we have that f_h is LU-pseudoconvex at $\mathbf{x}^*$, then we see by Definition 9, f_h^L and f_h^U are pseudoconvex at $\mathbf{x}^*$. We shall prove this result by contradiction. Suppose that $\mathbf{x}^* \notin X_{WP}^{LU}$, then by Definition 6 there exists $\hat{\mathbf{x}} \in X$ such that $f_h(\hat{\mathbf{x}}) \prec_{LU} f_h(\mathbf{x}^*)$. That is, we have either $f_h^L(\hat{\mathbf{x}}) < f_h^L(\mathbf{x}^*)$ or $f_h^U(\hat{\mathbf{x}}) < f_h^U(\mathbf{x}^*)$.
Case I. Consider the case $f_h^L(\hat{\mathbf{x}}) < f_h^L(\mathbf{x}^*)$. Since f_h^L is psedoconvex at $\mathbf{x}^*$, therefore we have

$$\nabla f_h^L(\mathbf{x}^*)^T(\hat{\mathbf{x}} - \mathbf{x}^*) < 0.$$

Let $\mathbf{d} = \hat{\mathbf{x}} - \mathbf{x}^*$. Then $\mathbf{y} = \mathbf{x}^* + \tau \mathbf{d} \in X$ for $\tau \in (0, 1)$, since X is convex and $\hat{\mathbf{x}}, \mathbf{x}^* \in X$. This shows that $\mathbf{d} \in \mathcal{D}$, is a feasible direction of X. From Proposition 5, we see that

$$\nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leq 0 \text{ for } i \in J(\mathbf{x}^*).$$

Further, let P be the matrix whose rows are $\nabla f_h^L(\mathbf{x}^*)^T$, and Q be a matrix whose rows are $\nabla g_i(\mathbf{x}^*)^T$ for $i \in J$. Then the result follows from similar arguments to those for Theorem 4. $\qquad\square$

THEOREM 8 *Assume that there is an interval valued objective function, say hth interval valued function $f_h, h \in \{1, ..., r\}$, such that it is LS-pseudoconvex and (weakly) continuously differentiable at $\boldsymbol{x}^*$. If there exist (Lagrange) multipliers $0 \leq \mu_i^L, \mu_i^S \in R, i = 1, ..., m$, such that the following KKT conditions hold*
(i) $\nabla f_h^L(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i^L \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}$;
(ii) $\nabla f_h^S(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i^S \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}$;
(iii) $\mu_i^L g_i(\boldsymbol{x}^*) = \boldsymbol{0} = \mu_i^S g_i(\boldsymbol{x}^*), i = 1, ..., m$,
then $\boldsymbol{x}^* \in X_{WP}^{LS}$ for $(MIP2)$.

PROOF The proof is same as that of Theorem 7.

Next we present some results for strongly LU-Pareto optimal solutions and strongly LS-Pareto optimal solutions.

Further, let f be an interval valued function defined on a non-empty convex subset $X \in R^n$ then we say that f is strictly L-pseudoconvex (respectively strictly U-pseudoconvex, strictly S-pseudoconvex) at $\mathbf{x}^*$ if f^L (respectively f^U, f^S) is strictly psedoconvex at $\mathbf{x}^*$, Wu (2009).

Note that f is strictly LU-pseudoconvex (respectively LS-psedoconvex) at $\mathbf{x}^*$ if f is strictly L-psedoconvex and strictly U-psedoconvex (respectively strictly L-psedoconvex and strictly S-psedoconvex) at $\mathbf{x}^*$ simultaneously.

THEOREM 9 *Assume that there is an interval valued objective function say $f_h, h \in \{1, ..., r\}$ such that it is continuously gH-differentiable and strictly L-psedoconvex (respectively strictly U-psedoconvex) at $\boldsymbol{x}^*$. If there exist (Lagrange) multipliers $0 \leq \mu_i \in R, i = 1, ..., m$, such that the following KKT conditions hold*
(i) $\nabla f_h^L(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}$,
$\left(respectively \ \nabla f_h^U(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}\right)$.
(ii) $\mu_i g_i(\boldsymbol{x}^*) = 0, i = 1, ..., m$,
then $\boldsymbol{x}^* \in X_{SP}^{LU} \cap X_{SP}^{LS}$ for $(MIP2)$.

PROOF Suppose $\mathbf{x}^* \notin X_{SP}^{LU}$, then by Definition 6 there exists $\hat{\mathbf{x}} \in X$ such that $F(\hat{\mathbf{x}}) \preceq_{LU} F(\mathbf{x}^*)$. That is $f_k(\hat{\mathbf{x}}) \preceq_{LU} f_k(\mathbf{x}^*)$ for $k = 1, ..., r$. In particular, we have

$$f_h^L(\hat{\mathbf{x}}) \leq f_h^L(\mathbf{x}^*) \ \left(respectively \ f_h^U(\hat{\mathbf{x}}) \leq f_h^U(\mathbf{x}^*)\right).$$

Since f_h^L (respectively f_h^U) is strictly psedoconvex at $\mathbf{x}^*$, therefore we have

$$\nabla f_h^L(\mathbf{x}^*)^T(\hat{\mathbf{x}} - \mathbf{x}^*) < 0 \ \left(resp. \ \nabla f_h^U(\mathbf{x}^*)^T(\hat{\mathbf{x}} - \mathbf{x}^*) < 0\right).$$

Then the result follows from similar arguments as those discussed regarding Theorem 4. $\qquad\square$

Next, we present some KKT conditions for $(MIP2)$ using the gradient of interval valued objective functions via gH-derivative. Consider an interval valued function f, then the gradient of f at $\mathbf{x}_0$ is defined as

$$\nabla_g f(\mathbf{x}_0) = \left(\left(\frac{\partial f}{\partial x_1} \right)_g (\mathbf{x}_0), ..., \left(\frac{\partial f}{\partial x_n} \right)_g (\mathbf{x}_0) \right),$$

where $\left(\frac{\partial f}{\partial x_j} \right)_g (\mathbf{x}_0)$ is the jth partial gH-derivative of f at x_0 (see Definition 5). From Theorem 1, we see that if f^L and f^U are differentiable functions, then f is gH-differentiable and in this case,

$$\left(\frac{\partial f}{\partial x_j} \right)_g (\mathbf{x}_0) = \left[\min \left\{ \left(\frac{\partial f^L}{\partial x_j} \right) (\mathbf{x}_0), \left(\frac{\partial f^U}{\partial x_j} \right) (x_0) \right\}, \right.$$

$$\left. \max \left\{ \left(\frac{\partial f^L}{\partial x_j} \right) (x_0), \left(\frac{\partial f^U}{\partial x_j} \right) (x_0) \right\} \right]$$

is a closed interval.

EXAMPLE 5 Consider the interval valued function

$$f(x) = [2x_1^2 + 3x_2^2, x_1^3 + 3x_2 + 1].$$

Then we have

$$\left(\frac{\partial f}{\partial x_1} \right)_g (x) = [\min \{4x_1, 3x_1^2\}, \max \{4x_1, 3x_1^2\}]$$

and

$$\left(\frac{\partial f}{\partial x_2} \right)_g (x) = [\min \{6x_2, 3\}, \max \{6x_2, 3\}].$$

So, the gradient of f is given by

$$\nabla_g f(x) = ([\min \{4x_1, 3x_1^2\}, \max \{4x_1, 3x_1^2\}], [\min \{6x_2, 3\}, \max \{6x_2, 3\}).$$

REMARK 6 Now, if we consider the H-derivative of f, then there is no partial derivative $\left(\frac{\partial f}{\partial x_1} \right)_H (0, 1)$ and so there is no gradient of f. Thus, the gradient of f defined using H-derivative is restrictive. Further, if we assume f to be weakly continuously differentiable, then clearly we cannot talk about gradient as we cannot define the partial derivative of f. Therefore, the gradient of f defined using gH-derivative is more general and it is more robust for optimization.

Consider the following equation

$$\sum_{k=1}^{r} \lambda_k \nabla_g f_k(\mathbf{x}_0) + \sum_{i=1}^{m} \mu_i \nabla g_i(\mathbf{x}_0) = \mathbf{0}; \tag{4.4}$$

where the letters have their usual meaning. Since $\sum_{i=1}^{m} \mu_i \frac{\partial g_i}{\partial x_j}(\mathbf{x}_0), \left(\frac{\partial F}{\partial x_j}\right)_g (\mathbf{x}_0) \in R$, therefore from Theorem 2, $f^L, f^U, k = 1, ..., r$, are continuously differentiable at $\mathbf{x}_0$. Therefore, (4.4) is equivalent to

$$\sum_{k=1}^{r} \lambda_k \frac{\partial f_k^L}{\partial x_j}(\mathbf{x}_0) + \sum_{i=1}^{m} \mu_i \frac{\partial g_i}{\partial x_j}(\mathbf{x}_0) = \mathbf{0} = \sum_{k=1}^{r} \lambda_k \frac{\partial f_k^U}{\partial x_j}(\mathbf{x}_0) + \sum_{i=1}^{m} \mu_i \frac{\partial g_i}{\partial x_j}(\mathbf{x}_0). \tag{4.5}$$

For all $j = 1, ..., n$, (4.5) can be equivalently written as

$$\begin{cases} \sum_{k=1}^{r} \lambda_k \nabla f_k^L(\mathbf{x}_0) + \sum_{i=1}^{m} \mu_i \nabla g_i(\mathbf{x}_0) = \mathbf{0} \\ \sum_{k=1}^{r} \lambda_k \nabla f_k^U(\mathbf{x}_0) + \sum_{i=1}^{m} \mu_i \nabla g_i(\mathbf{x}_0) = \mathbf{0} \end{cases} . \tag{4.6}$$

THEOREM 10 *Assume that the (interval) multiobjective function F is strictly LU-psedoconvex and continuously gH-differentiable at $\boldsymbol{x}^*$. If there exist (Lagrange) multipliers $0 \leq \lambda_k \in R, k = 1, ..., r$ and $0 \leq \mu_i \in R, i = 1, ..., m$, such that the following KKT conditions hold:*
(i) $\sum_{k=1}^{r} \lambda_k \nabla_g f_k(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}$;
(ii) $\mu_i g_i(\boldsymbol{x}^*) = 0, i = 1, ..., m$,
then $\boldsymbol{x}^* \in X_P^{LU} \cap X_P^{LS}$ for (MIP2).

PROOF   Since hypothesis (i) is equation (4.4) for $\mathbf{x}_0 = \mathbf{x}^*$, which is equivalent to (4.6), we get
(i) $\sum_{k=1}^{r} \lambda_k \nabla f_k^L(\mathbf{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0}$,
(ii) $\sum_{k=1}^{r} \lambda_k \nabla f_k^U(\mathbf{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0}$.
Then the result follows from Theorem 4.

THEOREM 11 *Assume that the (interval) multiobjective function F is strictly LS-psedoconvex and continuously gH-differentiable at $\boldsymbol{x}^*$. If there exist (Lagrange) multipliers $0 \leq \lambda_k \in R, k = 1, ..., r$, and $0 \leq \mu_i \in R, i = 1, ..., m$, such that the following KKT conditions hold*
(i) $\sum_{k=1}^{r} \lambda_k \nabla_g f_k(\boldsymbol{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}$
(ii) $\mu_i g_i(\boldsymbol{x}^*) = 0, i = 1, ..., m$,
then $\boldsymbol{x}^* \in X_P^{LS}$ for (MIP2).

PROOF Since hypothesis (i) is equation (4.4) for $\mathbf{x}_0 = \mathbf{x}^*$, which means that we obtain from (4.6)
(i) $\sum_{k=1}^{r} \lambda_k \nabla f_k^L(\mathbf{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0}$,
(ii) $\sum_{k=1}^{r} \lambda_k \nabla f_k^S(\mathbf{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0}$,
then the result follows from Theorem 4. □

Definition 13 (Wu, 2009) Let $f(\mathbf{x})=[f^L(\mathbf{x}), f^U(\mathbf{x})]$ be an interval valued function defined on $X \subseteq R^n$. We say that f is LU-nonincreasing at $\mathbf{x}^*$ if $\mathbf{x} \geq \mathbf{x}^*$ if and only if $f(\mathbf{x}) \preceq_{LU} f(\mathbf{x}^*)$.

We can similarly define the LS-nonincreasing properly by considering the " $\preceq_{LS}$ " order relation.

Theorem 12 *Assume that there is an interval valued function, say $f_h, h \in \{1, ..., r\}$, such that it is LU-nonincreasing and it is also strictly U-psedoconvex and continuously gH-differentiable at $\boldsymbol{x}^*$. Further assume that $\nabla f_h^L(\boldsymbol{x}^*) \neq \nabla f_h^U(\boldsymbol{x}^*)$. If there exist (Lagrange) multipliers $0 \leq \mu_i \in R, i = 1, ..., m$, such that the KKT conditions (i) and (iii) or (ii) and (iii) hold simultaneously:*
  (i) $\nabla f_h^L(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}$;
  (ii) $\nabla f_h^U(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}$;
  (iii) $\mu_i g_i(\boldsymbol{x}) = 0, i = 1, ..., m$,
then $x^* \in X_{SP}^{LS}$ for (MIP2).

Proof Suppose that $\mathbf{x}^* \notin X_{SP}^{LU}$. Then, by Definition 6, there exists $\hat{\mathbf{x}}(\neq \mathbf{x}^*) \in X$ such that $\nabla f_h^U(\mathbf{x}^*)^T(\hat{\mathbf{x}} - \mathbf{x}^*) < 0$, since f_h is strictly U-pseudoconvex. By using similar arguments to those for Theorem 4, we see that $\mathbf{x}^* \in X_{SP}^{LU}$ for (MIP2) if conditions (ii) and (iii) are satisfied.
Further, since f_h is gH-differentiable at $\mathbf{x}^*$, then

$$\left(\frac{\partial f_h^L}{\partial x_i}\right)(\mathbf{x}^*) \leq \left(\frac{\partial f_h^U}{\partial x_i}\right)(\mathbf{x}^*), \quad \text{for all} \quad i = 1, ..., n.$$

Therefore, we have

$$\nabla f_h^L(\mathbf{x}^*) \leq \nabla f_h^U(\mathbf{x}^*).$$

Also, since f_h is LU-nonincreasing and $\nabla f_h^L(\mathbf{x}^*) \neq \nabla f_h^U(\mathbf{x}^*)$, we have

$$\nabla f_h^L(\mathbf{x}^*)^T(\hat{\mathbf{x}} - \mathbf{x}^*) < \nabla f_h^U(\mathbf{x}^*)^T(\hat{\mathbf{x}} - \mathbf{x}^*) = 0,$$

i.e.,

$$\nabla f_h^L(\mathbf{x}^*)^T(\hat{\mathbf{x}} - \mathbf{x}^*) < 0.$$

Now by using similar arguments to those of Theorem 4 the result follows if conditions (i) and (iii) are satisfied.   □

Theorem 13 *Suppose there is an (interval) multiobjective function, say $f_h, h \in \{1, ..., r\}$, such that it is LS-nonincreasing and it is strictly L-psedoconvex (respectively strictly S-psedoconvex) and continuously gH-differentiable at $\boldsymbol{x}^*$. Further assume that $\nabla f_h^S(\boldsymbol{x}^*) \leq \nabla f_h^L(\boldsymbol{x}^*)$ (respectively $\nabla f_h^L(\boldsymbol{x}^*) \leq \nabla f_h^S(\boldsymbol{x}^*)$). If there exist (Lagrange) multipliers $0 \leq \mu_i \in R, i = 1, ..., m$, such that the KKT conditions (i) and (iii) or KKT conditions (ii) and (iii) hold simultaneously:*

(i) $\nabla f_h^L(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}$, $(respectively \ \nabla f_h^S(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0})$;

(ii) $\nabla f_h^S(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0}$, $(respectively \ \nabla f_h^L(\boldsymbol{x}^*) + \sum_{i=1}^m \mu_i \nabla g_i(\boldsymbol{x}^*) = \boldsymbol{0})$;

(iii) $\mu_i g_i(\boldsymbol{x}) = 0, i = 1, ..., m,$

then $\boldsymbol{x}^* \in X_{SP}^{LS}$ for $(MIP2)$.

PROOF Suppose that $\mathbf{x}^* \notin X_{SP}^{LU}$. Then, by Definition 6, there exists $\mathbf{x}(\neq \mathbf{x}^*) \in X$ such that $\nabla f_h^L(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) < 0$ (respectively $\nabla f_h^S(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) < 0$), since f_h is strictly L-pseudoconvex (respectively strictly S-pseudoconvex). By using similar arguments to those for Theorem 4, we see that $\mathbf{x}^* \in X_{SP}^{LU}$ for $(MIP2)$ if conditions (i) and (iii) are satisfied. On the other hand, since $\nabla f_h^S(\mathbf{x}^*) \leq \nabla f_h^L(\mathbf{x}^*)$ (respectively $\nabla f_h^L(\mathbf{x}^*) \leq \nabla f_h^S(\mathbf{x}^*)$), by using similar arguments to those for Theorem 12 the result follows if condition (ii) and (iii) are satisfied. □

5. Conclusions

In this paper we have considered two order relations on interval space, namely the relation LU and the relation LS which incorporate the quantitative properties of width (noise, risk, etc.). Also, following Wu (2009) and Stefanini and Bede (2009), respectively, by considering pseudoconvexity and gH-derivative for interval valued functions, we have obtained KKT conditions for multiobjective optimization problems with interval valued objective functions considering LU and LS order relations. For the case of order relation LU the results obtained are more general than those obtained in Wu (2009), and for the order relation LS, the results obtained are novel. Moreover, we have considered the gradient for interval valued functions using gH-derivative and we have used it to obtain the KKT optimality conditions. These results are more general than other similar results obtained using H-derivative and, consequently, the gradient of the interval valued function is more general when defined using gH-derivative.

Although the equality constraints are not considered in this paper, we can use a similar methodology to that proposed in this paper to handle equality constraints. The constraint functions in this paper are still real valued, in future research, one may consider the extension to the constraint functions being the interval valued functions.

Acknowledgements

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