Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XV, Częstochowa 2010

ON SOME GENERALIZATIONS OF GOŁĄB–SCHINZEL FUNCTIONAL EQUATION

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Abstract. Composite functional equations in several variables generalizing the Gołąb–Schinzel equation are considerd and some simple methods allowing us to determine their one-to-one solutions, bijective solutions or the solutions having exactly one zero are presented. For an arbitrarily fixed real p, the functional equation

$$\phi\left([p\,\phi(y)+(1-p)]x+[(1-p)\phi(x)+p\,]y\right)=\phi(x)\phi(y),\quad x,y\in\mathbb{R},$$

being a special generalization of the Gołąb–Schinzel equation, is considered.

1. Introduction

Composite functional equations in several variables, i.e. equations involving the superpositions of unknown functions, represent an important class of equations. The translation equation (cf. Aczél [1], p. 245),

$$\phi(\phi(x,s),t)=\phi(x,s+t),$$

the Gołąb–Schinzel equation ([2], see also [1], pp. 311–312)

$$\phi(x+y\phi(x))=\phi(x)\phi(y), \tag{1}$$

or the equation [3]

$$\phi(x + y\phi(x)) + \phi(x - y\phi(x)) = 2\phi(x)\phi(y), \tag{2}$$

are the examples. In section 1, we consider more general functional equations than (1) and (2) and give some conditions allowing us to determine their one-to-one solutions, bijective solutions or the solutions having exactly one zero. In section 2, for an arbitrarily fixed real p, we deal with the functional equation

$$\phi\left([p\,\phi(y) + (1-p)]x + [(1-p)\phi(x) + p\,]y\right) = \phi(x)\phi(y), \quad x, y \in \mathbb{R},$$

being a special generalization of equation (1).

2. Main result

Let X be a set. For a function $\phi : X \to X$ and a positive integer number k, by the symbol ϕ^k we denote the kth iteration of the function ϕ.

The following result reduces the problem of determining the solutions of a functional equation of a composite type to an application of the implicit function theorem.

Theorem 1. *Let $m, n \in \mathbb{N}$ be fixed. Let $I,\ I_1 \subseteq \mathbb{R}$ be intervals such that $0 \in I_1$ and $I_1 \subset I$. Let $G : (I \times I_1)^2 \mapsto I$ and $H : (I \times I_1^n) \times (I \times I_1^m) \longmapsto I_1$. Suppose that for all $x, y \in I;\ x_1, \ldots, x_n, y_2, \ldots, y_m \in I_1$,*

$$H(x, x_1, x_2, \ldots, x_n, y, 0, y_2, \ldots, y_m) = 0. \tag{3}$$

If a function $\phi : I \longmapsto I_1$ satisfies the functional equation

$$\phi(G(x, \phi(x), y, \phi(y))) = H(x, \phi(x), \phi^2(x), \ldots, \phi^n(x), y, \phi(y), \phi^2(y) \ldots, \phi^m(y)) \tag{4}$$

for all $x,\ y \in I$ and there exists exactly one $z_0 \in I$ such that $\phi(z_0) = 0$, then

$$G(x, \phi(x), z_0, 0) = z_0, \qquad x \in I.$$

Proof. Taking $y = z_0$ in equation (4) and applying condition (3), we get

$$\phi(G(x, \phi(x), z_0, 0)) = 0, \qquad x \in I.$$

Since ϕ has exactly one zero, we obtain $G(x, \phi(x), z_0, 0) = z_0$ for all $x \in I$. This completes the proof. □

Remark 1. *Equation (4) generalizes the Gołąb–Schinzel equation (1).*

In what follows, for $p \in \mathbb{R}$ and $\phi : X \to (0, \infty)$ the symbol $X \ni x \to [\phi(x)]^p$ stands for the superposition of the power function $(0, \infty) \ni u \to u^p$ and ϕ.

Now we present some applications of Theorem 1.

Corollary 1. *Let $k,\ l \in \mathbb{N}$ be fixed and let $\phi : \mathbb{R} \longmapsto \mathbb{R}$ be a function with exactly one zero point. Then ϕ satisfies the functional equation*

$$\phi\left(x + y[\phi(x)]^{\frac{2k-1}{2l-1}}\right) = \phi(x)\phi(y), \qquad x, y \in \mathbb{R}, \tag{5}$$

if and only if for some $c \in \mathbb{R}$, $c \neq 0$,

$$\phi(x) = (cx+1)^{\frac{2l-1}{2k-1}}, \qquad x \in \mathbb{R}\,. \tag{6}$$

Proof. In Theorem 1 take $I = I_1 = \mathbb{R}$, $n = m = 1$ and define $G : \mathbb{R}^4 \longmapsto \mathbb{R}$ by

$$G(x, x_1, y, y_1) := x + y(x_1)^{\frac{2k-1}{2l-1}}, \qquad x,\ x_1,\ y\ , y_1 \in \mathbb{R},$$

and $H : \mathbb{R}^4 \longmapsto \mathbb{R}$ by

$$H(x, x_1, y, y_1) := x_1 y_1, \qquad x,\ x_1,\ y,\ y_1 \in \mathbb{R}.$$

Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ satisfies equation (5) and has exactly one zero $z_0 \in \mathbb{R}$. Since $H(x, x_1, z_0, 0) = 0$ for all $x,\ x_1 \in \mathbb{R}$, the assumptions of Theorem 1 are fulfilled. From (5), applying Theorem 1, we get

$$G(x, \phi(x), z_0, 0) = z_0, \qquad x \in \mathbb{R},$$

that is

$$x + z_0[\phi(x)]^{\frac{2k-1}{2l-1}} = z_0, \qquad x \in \mathbb{R},$$

whence $z_0 \neq 0$ and

$$\phi(x) = \left(1 - \frac{x}{z_0}\right)^{\frac{2l-1}{2k-1}}, \qquad x \in \mathbb{R}.$$

Putting here $c := -\frac{1}{z_0}$, we obtain (6). Since ϕ given by (6) satisfies equation (5), the proof is completed. □

Remark 2. *It is known that (cf. [1], pp. 132-133) if $\phi : \mathbb{R} \longmapsto \mathbb{R}$ is a continuous solution of the Gołąb–Schinzel equation*

$$\phi(x + y\phi(x)) = \phi(x)\phi(y), \qquad x, y \in \mathbb{R},$$

then there exists $c \in \mathbb{R} \setminus \{0\}$ *such that either*

$$\phi(x) = \sup\{cx+1, 0\}, \qquad x \in \mathbb{R},$$

or there exists $c \in \mathbb{R}$ *such that*

$$\phi(x) = cx + 1, \qquad x \in \mathbb{R}, \tag{7}$$

or

$$\phi(x) = 0, \qquad x \in \mathbb{R}.$$

The second solution can be obtained from Corollary 1 in a different way. Taking $k = l$ in the equation (5) and applying Corollary 1, we obtain (7) as a only solution having only zero in $\mathbb{R}$.

Corollary 2. *Let* $a < 0$ *and* $p \in \mathbb{R}$, $p > 0$, *be fixed. Suppose that* $\phi : [a, \infty) \longmapsto [0, \infty)$ *has exactly one zero in* $[a, \infty)$. *A function* ϕ *satisfies the functional equation*

$$\phi\left(x + y[\phi(x)]^p\right) = \phi(x)\phi(y), \qquad x \geq a,\ y \geq 0, \tag{8}$$

if and only if

$$\phi(x) = \left(1 - \frac{x}{a}\right)^{\frac{1}{p}}, \qquad x \geq a. \tag{9}$$

Proof. Suppose that $\phi : [a, \infty) \longmapsto [0, \infty)$ satisfies equation (8) and $z_0 \geq a$ is the only zero of ϕ. In Theorem 1 take $n = m = 1$, $I := [a, \infty)$, $I_1 := [0, \infty)$, the function $G : (I \times I_1)^2 \longmapsto I$ defined by

$$G(x, x_1, y, y_1) := x + y(x_1)^p, \qquad x, y \in I,\ \ x_1, y_1 \in I_1,$$

and the function $H : (I \times I_1)^2 \longmapsto I_1$ defined by

$$H(x, x_1, y, y_1) := x_1 y_1, \qquad x, y \in I,\ \ x_1, y_1 \in I_1.$$

Since $H(x, x_1, y, 0) = 0$, for all $x,\ y \in I,\ x_1 \in I_1$, the assumptions of Theorem 1 are satisfied. Therefore

$$G(x, \phi(x), z_0, 0) = z_0, \qquad x \in I,$$

so $x + z_0[\phi(x)]^p = z_0$ for all $x \geq a$. It follows that $z_0 \neq 0$ and, consequently,

$$\phi(x) = (1 - \frac{x}{z_0})^{\frac{1}{p}}, \qquad x \geq a.$$

Since ϕ is non–negative, we have $1 - \frac{x}{z_0} \geq 0$ for all $x \in [a, \infty)$. Thus $z_0 = a$. Since the converse implication is easy to verify, the proof is completed. □

Remark 3. *Note that for* $p = 0$ *equation (8) in Corollary 2 becomes the Cauchy functional equation.*

Theorem 2. *Let* $n \in \mathbb{N}$ *be fixed. Let* I, I_1 *be intervals such that* $I_1 \subset I \subseteq \mathbb{R}$. *Let* $G : (I \times I_1)^2 \longmapsto I$ *and* $H : (I \times I_1^n)^2 \longmapsto I_1$ *be given functions. Suppose that* H *is symmetric, that is*

$$H(x, x_1, x_2, \dots, x_n, y, y_1, y_2, \dots, y_n) = H(y, y_1, y_2, \dots, y_n, x, x_1, x_2, \dots, x_n) \tag{10}$$

for all $x, y \in I$, $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in I_1$.

If $\phi : I \longmapsto I_1$ *is a solution of the functional equation*

$$\phi(G(x, \phi(x), y, \phi(y))) = H(x, \phi(x), \phi^2(x), \dots, \phi^n(x), y, \phi(y), \phi^2(y) \dots, \phi^n(y)) \tag{11}$$

for all $x, y \in I$, *then*

$$\phi(G(x, \phi(x), y, \phi(y))) = \phi(G(y, \phi(y), x, \phi(x))), \qquad x, y \in I.$$

If, moreover ϕ *is one-to-one function, then*

$$G(x, \phi(x), y, \phi(y)) = G(y, \phi(y), x, \phi(x)), \qquad x, y \in I. \tag{12}$$

Proof. Suppose that $\phi : I \longmapsto I_1$ satisfies Eq. (11) and $H : (I \times I_1^n)^2 \longmapsto I_1$ satisfies condition (10). Then for all x, $y \in I$ we have

$$\begin{aligned} \phi(G(x, \phi(x), y, \phi(y))) &= H(x, \phi(x), \phi^2(x), \dots, \phi^n(x), y, \phi(y), \phi^2(y) \dots, \phi^n(y)) \\ &= H(y, \phi(y), \phi^2(y), \dots, \phi^n(y), x, \phi(x), \phi^2(x) \dots, \phi^n(x)) \\ &= \phi(G(y, \phi(y), x, \phi(x))), \end{aligned}$$

so,

$$\phi(G(x, \phi(x), y, \phi(y))) = \phi(G(y, \phi(y), x, \phi(x))), \qquad x, y \in I.$$

If ϕ is one-to-one, then obviously equality (12) holds true.

Remark 4. *If the function* G *in Theorem 2 is not symmetric, then in general equality (12) allows us to obtain the one-to-one solutions of (11).*

Applying Theorem 2 we obtain

Corollary 3. *Let a, $p \in \mathbb{R}$ be fixed and such that $a < 0$, $p \neq 0$. A one-to-one function $\phi : (a, \infty) \longmapsto (0, \infty)$ satisfies the functional equation*

$$\phi\left(x + y[\phi(x)]^p\right) = \phi(x)\phi(y)\,, \qquad x > a,\ y \geq 0, \tag{13}$$

if, and only if,

$$\phi(x) = \left(1 - \frac{x}{a}\right)^{\frac{1}{p}}, \qquad x > a. \tag{14}$$

Proof. In Theorem 2 take $n = 1$, $I = (a, \infty)$, $I_1 = (0, \infty)$, the function $G : (I \times I_1)^2 \longmapsto I$ defined by

$$G(x, x_1, y, y_1) := x + y(x_1)^p, \qquad x,\ y \in I,\ \ x_1,\ y_1 \in I_1,$$

and $H : (I \times I_1)^2 \longmapsto I_1$ defined by

$$H(x, x_1, y, y_1) := x_1 y_1 \qquad x,\ y \in I,\ \ x_1,\ y_1 \in I_1,$$

Since

$$H(x, x_1, y, y_1) = H(y, y_1, x, x_1), \qquad x,\ y \in I,\ \ x_1,\ y_1 \in I_1,$$

the assumptions of Theorem 2 are satisfied. Applying Theorem 2, we have from (12):

$$x + y[\phi(x)]^p = y + x[\phi(y)]^p, \qquad x,\ y \in I,$$

whence

$$\frac{[\phi(x)]^p - 1}{x} = \frac{[\phi(y)]^p - 1}{y}, \qquad x,\ y \in I, \qquad x,\ y \neq 0.$$

So, there exists a constant $c \in \mathbb{R} \setminus \{0\}$ such that

$$x^{-1}([\phi(x)]^p - 1) = c$$

for all $x \in I,\ x \neq 0$. Hence

$$\phi(x) = (cx + 1)^{\frac{1}{p}}, \qquad x > a,\ \ x \neq 0.$$

Equation (13) implies that

$$cx + 1 > 0, \qquad x > a,$$

and, consequently, $ca+1 \geq 0$. On the other hand, if ϕ satisfies equation (13), then obviously the following inequality

$$x+y[(cx+1)^{\frac{1}{p}}]^p > a, \qquad x,\, y > a,$$

is true, which means that

$$x+y[(cx+1)] > a, \qquad x,\, y > a.$$

It follows that $a+ca^2+a \geq a$, so $ca+1 \leq 0$. Both inequalities imply that $ca+1=0$, whence $c=-\frac{1}{a}$, and ϕ has to be of the form (14).

To show that the function ϕ given by (14) satisfies equation (13), let us note that

$$x+y[\phi(x)]^p > a, \qquad x, y > a.$$

In fact, this inequality is equivalent to $(x-a)(y-a)>0$. Now, it is easy to verify that (14) satisfies equation (13). This completes the proof. $\square$

Remark 5. *Taking* $a, p \in \mathbb{R}$, $a<0$, *and* $p>0$, *we can show in the same way that the one–to–one function* $\phi : [a, +\infty) \longmapsto [0, +\infty)$ *satisfies the functional equation*

$$\phi\left(x+y[\phi(x)]^p\right) = \phi(x)\phi(y), \qquad x,\, y \geq a,$$

if and only if

$$\phi(x) = \left(1-\frac{x}{a}\right)^{\frac{1}{p}}, \qquad x \geq a.$$

Remark 6. *Let* $I,\ I_1 \subseteq \mathbb{R}$ *be intervals. Let* $G : (I \times I_1)^2 \longmapsto I$ *and* $H : I_1 \times I_1 \longmapsto I_1$ *be the given functions. Assume that* $\phi : I \longmapsto I_1$, $\phi(I) = I_1$ *is a bijective solution of the functional equation*

$$\phi(G(x,\phi(x),y,\phi(y))) = H(\phi(x),\phi(y)), \qquad x,y \in I. \tag{15}$$

Then the function $\phi^{-1} : I_1 \longmapsto I$ *satisfies the (non-composite) functional equation*

$$G(\phi^{-1}(x), x, \phi^{-1}(y), y) = \phi^{-1}(H(x,y)), \qquad x,y \in I_1. \tag{16}$$

In fact, putting $\phi^{-1}(x)$ in place of x and $\phi^{-1}(y)$ in place of y in equation (15), we obtain (16).

Sometimes the above remark allows us to determine effectively the bijective solutions for functional equations of form (15). We have the following

Corollary 4. *Let $k,\, l \in \mathbb{N}$ be fixed and let $I = I_1 = \mathbb{R}$. The bijection function $\phi : \mathbb{R} \longmapsto \mathbb{R}$ satisfies functional equation*

$$\phi\left(x + y[\phi(x)]^{\frac{2k-1}{2l-1}}\right) = \phi(x)\phi(y), \qquad x, y \in \mathbb{R}, \tag{17}$$

if and only if

$$\phi(x) = (cx+1)^{\frac{2l-1}{2k-1}}, \qquad x \in \mathbb{R}, \tag{18}$$

for some $c \in \mathbb{R},\ c \neq 0$.

Proof. According to Remark 6, a bijection $\phi : \mathbb{R} \mapsto \mathbb{R}$ satisfies equation (17) if and only if $\phi^{-1} : \mathbb{R} \mapsto \mathbb{R}$ satisfies the equation

$$\phi^{-1}(x) + \phi^{-1}(y)x^{\frac{2k-1}{2l-1}} = \phi^{-1}(xy), \qquad x, y \in \mathbb{R}.$$

Putting here $y = 0$, we obtain

$$\phi^{-1}(x) = \phi^{-1}(0)\left(1 - x^{\frac{2k-1}{2l-1}}\right), \qquad x \in \mathbb{R},$$

which implies (18). □

3. A special generalization of Gołąb–Schinzel functional equation

In this section we examine the functional equation

$$\phi\left([p\,\phi(y) + (1-p)]x + [(1-p)\phi(x) + p\,]y\right) = \phi(x)\phi(y), \quad x, y \in \mathbb{R}, \tag{19}$$

where $p \in \mathbb{R}$ is an arbitrarily fixed parameter. For $p = 0$ or $p = 1$ it reduces to the classical Gołąb–Schinzel equation.

Theorem 3. *Let $p \in \mathbb{R}$ be fixed.*

1. *If $p \neq \frac{1}{2}$, then the one-to-one function $\phi : \mathbb{R} \longmapsto \mathbb{R}$ satisfies (19) if and only if*

   $$\phi(x) = cx + 1, \qquad x \in \mathbb{R},$$

   for some $c \in \mathbb{R} \setminus \{0\}$.

2. *If $p = \frac{1}{2}$, then bijection $\phi : \mathbb{R} \longmapsto \mathbb{R}$ satisfies (19) if and only if*

   $$\phi(x) = cx + 1, \qquad x \in \mathbb{R},$$

   for some $c \in \mathbb{R} \setminus \{0\}$.

Proof. Take $n = 1$, $I = \mathbb{R}$ and define $G : (\mathbb{R} \times \mathbb{R})^2 \longmapsto \mathbb{R}$ by

$$G(x, x_1, y, y_1) := [py_1 + (1 - p)]x + [(1 - p)x_1 + p]y, \qquad x,\ y,\ x_1,\ y_1 \in \mathbb{R},$$

and $H : (\mathbb{R} \times \mathbb{R})^2 \longmapsto \mathbb{R}$ by

$$H(x, x_1, y, y_1) := x_1 y_1, \qquad x,\ y,\ x_1,\ y_1 \in \mathbb{R}.$$

Note, that

$$H(x, x_1, y, y_1) = H(y, y_1, x, x_1), \qquad x,\ y,\ x_1,\ y_1 \in \mathbb{R}.$$

Applying Theorem 2, we obtain

$$\begin{aligned}&[p\phi(y) + (1 - p)]x + [(1 - p)\phi(x) + p]y\\ &= [p\phi(x) + (1 - p)]y + [(1 - p)\phi(y) + p]x\end{aligned}$$

for all $x,\ y \in \mathbb{R}$, whence

$$(2p - 1)[x(\phi(y) - 1)] = (2p - 1)[y(\phi(x) - 1)]\,, \qquad x,\ y \in \mathbb{R}.$$

If $p \neq \frac{1}{2}$, hence we get

$$\frac{\phi(x) - 1}{x} = \frac{\phi(y) - 1}{y}, \qquad x,\ y \in \mathbb{R} \setminus \{0\}.$$

Therefore, there exists a constant $c \in \mathbb{R} \setminus \{0\}$ such that $\phi(x) = cx + 1$ for all $x \in \mathbb{R} \setminus \{0\}$. Putting $x = y = 0$ in equation (19), we get $[\phi(0)] = [\phi(0)]^2$, consequently we obtain either $\phi(0) = 0$ or $\phi(0) = 1$. Since ϕ is one–to–one and $\phi(-\frac{1}{c}) = 0$, the case $\phi(0) = 0$ cannot occur. Thus $\phi(x) = cx + 1$ for all $x \in \mathbb{R}$.

For $p = \frac{1}{2}$ equation (19) has the form:

$$\phi\left(\frac{1}{2}[x(\phi(y) + 1) + y(\phi(x) + 1)]\right) = \phi(x)\phi(y), \qquad x,\ y \in \mathbb{R}. \tag{20}$$

If a bijection $\phi : \mathbb{R} \longmapsto \mathbb{R}$ satisfies (20), then according to the Remark 2 the function $\phi^{-1} : \mathbb{R} \longmapsto \mathbb{R}$ satisfies the equation

$$2\phi^{-1}(xy) = (y + 1)\phi^{-1}(x) + (x + 1)\phi^{-1}(y), \qquad x,\ y \in \mathbb{R}.$$

Putting here $y = 0$, we get

$$2\phi^{-1}(0) = \phi^{-1}(x) + (x + 1)\phi^{-1}(0), \qquad x \in \mathbb{R}.$$

Hence, as $\phi^{-1}(0) \neq 0$,

$$\phi^{-1}(x) = \phi^{-1}(0)(1-x), \qquad x \in \mathbb{R},$$

whence

$$\phi(x) = 1 - \frac{1}{\phi^{-1}(0)}\, x, \qquad x \in \mathbb{R}. \qquad \square$$

Theorem 4. *Let $p \in \mathbb{R}$ be fixed. A function $\phi : \mathbb{R} \longmapsto \mathbb{R}$ satisfies equation (19) and has exactly one zero if and only if there exists a constant $c \in \mathbb{R} \setminus \{0\}$ such that*

$$\phi(x) = cx + 1, \qquad x \in \mathbb{R}.$$

Proof. Note that substitution of $(1-p)$ for p in equation (19) gives the same equation. Thus, without any loss of generality, we can assume that $p \neq 1$. Take $n = m = 1$, $I = I_1 = \mathbb{R}$, and define $H : (\mathbb{R} \times \mathbb{R})^2 \mapsto \mathbb{R}$ by

$$H(x, x_1, y, y_1) := x_1 y_1, \qquad x,\, x_1,\, y,\, y_1 \in \mathbb{R}, \tag{21}$$

and $G : (\mathbb{R} \times \mathbb{R})^2 \mapsto \mathbb{R}$ by

$$G(x, x_1, y, y_1) := [py_1 + (1-p)]x + [(1-p)x_1 + p]y, \qquad x, x_1, y, y_1 \in \mathbb{R}.$$

Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ satisfies equation (19) and $z_0 \neq 0$ is a unique zero of ϕ. Note that if $y = z_0$, then $H(x, x_1, z_0, 0) = 0$ for all x, $x_1 \in \mathbb{R}$, so the function (21) satisfies the condition (3) of Theorem 1. Therefore, if ϕ satisfies equation (19), then

$$(1-p)x + [(1-p)\phi(x) + p]z_0 = z_0, \qquad x \in \mathbb{R}.$$

Hence we obtain $\phi(x) = 1 - \frac{x}{z_0}$ for all $x \in \mathbb{R}$. $\square$

References

[1] J. Aczél. *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966.

[2] S. Gołąb, A. Schinzel. Sur l'équation fonctionnelle $f[x + yf(x)] = f(x)f(y)$. *Publ. Math. Debrecen*, **6**, 113–125, 1959.

[3] P. Kahlig, J. Matkowski. On some extension of Gołąb–Schinzel functional equation. *Ann. Math. Siles.*, **8**, 13–31, 1994.